International Journal of Difference Equations
ISSN 0973-5321, Volume 3, Number 2, pp. 277–288 (2008)
http://campus.mst.edu/ijde

Second Order Linear Difference Equations and Karamata Sequences

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Abstract

We establish necessary and sufficient conditions for all positive solutions of a linear second order difference equation to be Karamata sequences, i.e., slowly varying or regularly varying or rapidly varying. Moreover, we discuss relations with the standard classification of nonoscillatory solutions and with the notion of recessive solutions. Our results lead to a complete characterization of positive solutions with respect to their regularly or rapidly varying behavior.

AMS Subject Classifications: 26A12, 39A11, 39A12.

Keywords: Regularly varying sequence, rapidly varying sequence, linear difference equation, nonoscillatory solution.

Received February 28, 2008; Accepted July 3, 2008
Communicated by Lynn Erbe

*Supported by the grant KJB100190701 of the Grant Agency of the Czech Academy of Sciences, by the grant 201/07/0145 of the Czech Grant Agency, and by the Institutional Research Plan No. AV0Z10190503.

1 Introduction

We consider the second order linear difference equation

$$\Delta^2 y_k - p_k y_{k+1} = 0 \tag{1.1}$$

on $\mathbb{N}$, where $p_k > 0$. Equation (1.1) has been the object of a wide investigation, and there exists an extensive literature about its qualitative theory, see e.g., [1] and the references given therein. In this paper, we present a complete characterization of solutions to equation (1.1) with respect to their regularly or rapidly varying behavior. Moreover, we show how such a characterization is related to certain standard classification of nonoscillatory solutions of (1.1). The main statements are obtained using some previous results by the authors [12, 13] concerning decreasing solutions and new observations about increasing solutions.

In the continuous case, many results concerning the relations between regularly or rapidly varying functions and behavior of solutions of linear second order differential equations are known. The works in this field most relevant for our investigation are [8, 11] and, in particular, the monograph by Marić [10]. A comprehensive treatment of regular variation and related topics can be found in [2]. A part of the results presented in this paper (especially, Theorems 3.1–3.3 and Corollary 3.5) can be understood as discrete counterparts of some of the results from the quoted works. However, the key steps of the proofs differ from the continuous case, due to peculiarities of the discrete case.

In Section 2 we recall the definition of regularly and rapidly varying sequences and we present some of their properties which are useful in our theory. In the third section, we establish necessary and sufficient conditions for all positive solutions of (1.1) to be regularly varying or rapidly varying, where the index of regular variation is closely related to the limit behavior of the coefficient p. We also compare properties of the series conditions that are contained in the theorems. The paper is concluded by observations which arise out of a combination of the classification of solutions with respect to regular and rapid variation with the standard "$\mathbb{M}$-classification" of nonoscillatory solutions in the sense of [4, 5]. A relation to the notion of recessive solutions is also discussed.

2 Preliminaries

First we recall basic properties of regularly and slowly varying sequences. The following definition is due to Karamata [9].

Definition 2.1. A positive sequence $y = \{y_k\}$, $k \in \mathbb{N}$, is said to be *regularly varying of index* ϱ, $\varrho \in \mathbb{R}$, if

$$\lim_k \frac{y_{[\lambda k]}}{y_k} = \lambda^\varrho \text{ for all } \lambda > 0,$$

where $[u]$ denotes the integer part of u.

In [3], a very useful imbedding theorem was established, which enables us to use the "continuous" theory in developing a theory of regularly varying sequences. In [12], we derived an equivalent definition (see Definition 2.2 below), which is a slight modification of a definition given by Galambos and Seneta in [7], equivalent to Definition 2.1, and which is more suitable for some of our applications.

Definition 2.2. A positive sequence $y = \{y_k\}$, $k \in \mathbb{N}$, is said to be *regularly varying of index* ϱ, $\varrho \in \mathbb{R}$, if there exists $C > 0$ and a positive sequence $\{\alpha_k\}$ such that

$$\lim_k \frac{y_k}{\alpha_k} = C, \quad \lim_k k\frac{\Delta\alpha_k}{\alpha_k} = \varrho.$$

If $\varrho = 0$ in Definition 2.1 or in Definition 2.2, then $\{y_k\}$ is said to be *slowly varying*. Let us denote by $\mathcal{RV}(\varrho)$ the totality of regularly varying sequences of index ϱ and by $\mathcal{SV}$ the totality of slowly varying sequences. A positive sequence $\{y_k\}$ is said to be *normalized regularly varying of index* ϱ if it satisfies $\lim_k k\Delta y_k/y_k = \varrho$. If $\varrho = 0$, then y is called a *normalized slowly varying sequence*. In the sequel, $\mathcal{NRV}(\varrho)$ and $\mathcal{NSV}$ will denote, respectively, the set of all normalized regularly varying sequences of index ϱ, and the set of all normalized slowly varying sequences. For instance, the sequence $\{y_k\} = \{\log k\} \in \mathcal{NSV}$, and the sequence $\{y_k\} = \{k^\varrho \log k\} \in \mathcal{NRV}(\varrho)$, for every $\varrho \in \mathbb{R}$; on the other hand, the sequence $\{y_k\} = \{1 + (-1)^k/k\} \in \mathcal{SV} \setminus \mathcal{NSV}$.

In the following lemma we list the properties of regularly varying sequences which are useful in our theory.

Lemma 2.3. *Regularly varying sequences have the following properties:*

(i) A sequence $y \in \mathcal{RV}(\varrho)$ *iff* $y_k = k^\varrho \varphi_k \exp\left\{\sum_{j=1}^{k-1} \psi_j/j\right\}$, *where* $\{\varphi_k\}$ *tends to a positive constant and* $\{\psi_k\}$ *tends to 0 as* $k \to \infty$. *Moreover,* $y \in \mathcal{RV}(\varrho)$ *iff* $y_k = k^\varrho L_k$, *where* $L \in \mathcal{SV}$.

(ii) A sequence $y \in \mathcal{RV}(\varrho)$ *iff* $y_k = \varphi_k \prod_{j=1}^{k-1}(1 + \delta_j/j)$, *where* $\{\varphi_k\}$ *tends to a positive constant and* $\{\delta_k\}$ *tends to* ϱ *as* $k \to \infty$.

(iii) If a sequence $y \in \mathcal{NRV}(\varrho)$, *then in the representation formulae given in (i) and (ii), it holds* $\varphi_k \equiv$ *const* > 0, *and the representation is unique. Moreover,* $y \in \mathcal{NRV}(\varrho)$ *iff* $y_k = k^\varrho S_k$, *where* $S \in \mathcal{NSV}$.

(iv) Let $y \in \mathcal{RV}(\varrho)$. *If one of the following conditions holds (a)* $\Delta y_k \le 0$ *and* $\Delta^2 y_k \ge 0$, *or (b)* $\Delta y_k \ge 0$ *and* $\Delta^2 y_k \le 0$, *or (c)* $\Delta y_k \ge 0$ *and* $\Delta^2 y_k \ge 0$, *then* $y \in \mathcal{NRV}(\varrho)$.

(v) Let $y \in \mathcal{RV}(\varrho)$. *Then* $\lim_k y_k/k^{\varrho-\varepsilon} = \infty$ *and* $\lim_k y_k/k^{\varrho+\varepsilon} = 0$ *for every* $\varepsilon > 0$.

(vi) Let $u \in \mathcal{RV}(\varrho_1)$ *and* $v \in \mathcal{RV}(\varrho_2)$*. Then* $uv \in \mathcal{RV}(\varrho_1+\varrho_2)$ *and* $1/u \in \mathcal{RV}(-\varrho_1)$*. The same holds if* $\mathcal{RV}$ *is replaced by* $\mathcal{NRV}$*.*

(vii) Let $y \in \mathcal{RV}(\varrho)$*,* $\varrho \in \mathbb{R}$*, be strictly convex, i.e.,* $\Delta^2 y_k > 0$ *for every* $k \in \mathbb{N}$*. Then* y *is decreasing provided* $\varrho \le 0$*, and it is increasing provided* $\varrho > 0$*.*

(viii) If $y \in \mathcal{RV}(\varrho)$*, then* $\lim_k y_{k+1}/y_k = 1$*.*

Proof. We prove only statements (iv), (vi), and (vii), since the proof of properties (i), (v), and (viii) can be found in [3], and the proof of (ii) and (iii) is in [12].

(iv) For (a) and (b) see [12]. To prove (c), we proceed as follows. Let $y \in \mathcal{RV}(\varrho)$. First we prove that $\limsup_k k\Delta y_k/y_k \le \varrho$. Take $\lambda > 1$. Then

$$y_{[\lambda k]} - y_k = \sum_{j=k}^{[\lambda k]-1} \Delta y_j \ge \Delta y_k([\lambda k] - k) \ge \Delta y_k[k(\lambda-1)-1]. \tag{2.1}$$

Hence

$$\Delta y_k \le \frac{y_{[\lambda k]} - y_k}{k(\lambda-1)-1}$$

for k sufficiently large. Hence, using the definition, we get

$$\begin{aligned}\limsup_k \frac{k\Delta y_k}{y_k} &\le \limsup_k \frac{k}{k(\lambda-1)-1}\cdot\frac{y_{[\lambda k]} - y_k}{y_k}\\ &= \lim_k \frac{k}{k(\lambda-1)-1}\left(\frac{y_{[\lambda k]}}{y_k} - 1\right)\\ &= \frac{\lambda^\varrho - 1}{\lambda - 1}\end{aligned}$$

for every $\lambda > 1$. If $\varrho = 0$, the proof is complete. For $\varrho > 0$, taking the limit as $\lambda \to 1^+$, we obtain

$$\limsup_k \frac{k\Delta y_k}{y_k} \le \lim_{\lambda\to 1^+} \frac{\lambda^\varrho - 1}{\lambda - 1} = \varrho. \tag{2.2}$$

Now we prove that $\liminf_k k\Delta y_k/y_k \ge \varrho$. Take $\lambda \in (0,1)$. Then, similarly as in the previous case, we have

$$\Delta y_k \ge \frac{y_k - y_{[\lambda k]}}{k(1-\lambda)+1}.$$

This implies

$$\liminf_k \frac{k\Delta y_k}{y_k} \ge \lim_{\lambda\to 1^-} \frac{1-\lambda^\varrho}{1-\lambda} = \varrho.$$

(vi) Follows from the representation formula in (i), where for the “normalized” version, $C > 0$ is instead of $\{\varphi_k\}$. Alternatively, we can immediately use Definition 2.1 for a “general” version and Definition 2.2 and property (i) for a “normalized” version.

(vii) First note that $\Delta^2 y_k > 0$ implies eventual monotonicity. Indeed, either we have $\Delta y_k < 0$ for all k, or $\Delta y_k > 0$ for all $k > m$ suitably large. Let $y \in \mathcal{SV}$; we prove that is decreasing. By a contradiction assume that $\Delta y_k \geq 0$. Thanks to convexity, we have $\Delta y_k \geq M > 0$ for large k and for some $M > 0$. Summing from s to $k-1$ we obtain $y_k \geq y_s + (k-s)M$, and y cannot be slowly varying by (v). Similarly we proceed when $y \in \mathcal{RV}(\varrho)$ with $\varrho < 0$. If $\varrho > 0$, then $y_k \to \infty$ as $k \to \infty$ by (v) since $y_k = k^{\varrho} L_k$ with $L \in \mathcal{SV}$, and in view of the convexity, y is increasing. □

The set of Karamata sequences includes also the so called rapidly varying sequences. We refer to [6] for basic properties and useful applications.

Definition 2.4. A positive sequence $\{y_k\}$ is said to be *rapidly varying of index* ∞ if

$$\lim_{k\to\infty} \frac{y_{[\lambda k]}}{y_k} = \begin{cases} \infty & \text{for } \lambda > 1 \\ 0 & \text{for } 0 < \lambda < 1. \end{cases}$$

A positive sequence $\{y_k\}$ is said to be *rapidly varying of index* $-\infty$ if

$$\lim_{k\to\infty} \frac{y_{[\lambda k]}}{y_k} = \begin{cases} 0 & \text{for } \lambda > 1 \\ \infty & \text{for } 0 < \lambda < 1. \end{cases}$$

In the sequel, $\mathcal{RPV}(\infty)$ and $\mathcal{RPV}(-\infty)$ will denote the set of all rapidly varying sequences of index ∞ and of index $-\infty$, respectively. In [6], the authors proved an imbedding type theorem, which, similarly as in the case of regular variation, enables to use the "continuous" theory in developing a theory of rapidly varying sequences.

In the next lemma, some properties of rapidly varying sequences are presented. Observe that (i) leads to a simplification of the definition.

Lemma 2.5. *Rapidly varying sequences have the following properties:*

(i) *We have* $\lim_{k\to\infty} y_{[\lambda k]}/y_k = 0$ *for* $0 < \lambda < 1$ *iff* $\lim_{k\to\infty} y_{[\lambda k]}/y_k = \infty$ *for* $\lambda > 1$.

(ii) *If* $y \in \mathcal{RPV}(\infty)$, *then* $\lim_k y_k/k^{\varrho} = \infty$ *for all* $\varrho \geq 0$.

(iii) *Let* $u = 1/y$. *Then* $y \in \mathcal{RPV}(\infty)$ *iff* $u \in \mathcal{RPV}(-\infty)$.

(iv) *Let* $y \in \mathcal{RPV}(-\infty)$ *and* $\{\Delta y_k\}$ *increase. Then* $\lim_k k\Delta y_k/y_k = -\infty$.

(v) *If* $y_k > 0$ *and* $\lim_k k\Delta y_k/y_k = -\infty$, *then* $y \in \mathcal{RPV}(-\infty)$.

(vi) *Let* $y \in \mathcal{RPV}(\tau)$, $\tau \in \{-\infty, \infty\}$, *be convex. Then* y *is increasing provided* $\tau = \infty$, *and it is decreasing provided* $\tau = -\infty$.

Proof. The proof of statements (i), (ii) can be found in [6], while for (iv) and (v) see [13]. Property (iii) is immediate from the definition, and (vi) follows from (ii) and (iii). □

3 Solutions of Second Order Linear Difference Equations and Karamata Sequences

In this section we establish necessary and sufficient conditions for solutions of (1.1) to be regularly or rapidly varying. First note that all nontrivial solutions of (1.1) are nonoscillatory (i.e., eventually of one sign) and eventually monotone. Because of linearity, without loss of generality, we may consider just positive solutions of (1.1); we denote this set as $\mathbb{M}$. The set $\mathbb{M}$ can be further divided into the two classes $\mathbb{M}^+$ and $\mathbb{M}^-$, where

$$\mathbb{M}^+ = \{y \in \mathbb{M} : \exists k_y \in \mathbb{N} \text{ such that } y_k > 0, \Delta y_k > 0 \text{ for } k \geq k_y\},$$
$$\mathbb{M}^- = \{y \in \mathbb{M} : y_k > 0, \Delta y_k < 0\},$$

and these classes are always nonempty, see e.g., [4, 5].

The main results are presented in Theorems 3.1, 3.2, and 3.3. The statements for decreasing solutions follow from the results in [12, 13] as a particular case. The other statements are new. Note that in addition to the properties of regularly and rapidly varying sequences and various inequalities used in these papers, the so called Riccati technique plays an important role in deriving the results in [12].

Theorem 3.1. *Equation* (1.1) *has a fundamental set of solutions*

$$u_k = L_k \in \mathcal{SV}, \quad v_k = k\tilde{L}_k \in \mathcal{RV}(1)$$

if and only if

$$\lim_k k \sum_{j=k}^{\infty} p_j = 0. \tag{3.1}$$

Moreover, $L, \tilde{L} \in \mathcal{NSV}$ *with* $\tilde{L}_k \sim L_k^{-1}$. *All positive decreasing solutions of* (1.1) *belong to* $\mathcal{NSV}$ *and all positive increasing solutions of* (1.1) *belong to* $\mathcal{NRV}(1)$.

Theorem 3.2. *Equation* (1.1) *has a fundamental set of solutions*

$$u_k = k^{\varrho_1} L_k \in \mathcal{RV}(\varrho_1), \quad u_k = k^{\varrho_2} \tilde{L}_k \in \mathcal{RV}(\varrho_2)$$

if and only if

$$\lim_k k \sum_{j=k}^{\infty} p_j = A > 0, \tag{3.2}$$

where $\varrho_1 < 0$, $\varrho_2 = 1 - \varrho_1$ *are the roots of the equation* $\varrho^2 - \varrho - A = 0$. *Moreover* L, $\tilde{L} \in \mathcal{NSV}$ *with* $\tilde{L}_k \sim 1/((1-2\varrho_1)L_k)$. *All positive decreasing solutions of* (1.1) *belong to* $\mathcal{NRV}(\varrho_1)$ *and all positive increasing solutions of* (1.1) *belong to* $\mathcal{NRV}(\varrho_2)$.

Theorems 3.1 and 3.2 could be stated as one theorem, assuming $A \geq 0$ in (3.2) and, consequently, $\varrho_1 \leq 0$. However, for better understanding and easier reading we prefer to distinguish the two cases and to present two theorems. This categorization will be shown to be useful also with respect to the subsequent observations, where a trichotomy character of our results will be revealed and relations with the $\mathbb{M}$-classification will be shown.

Proof of Theorems 3.1 and 3.2. Theorem 3.1 and Theorem 3.2 will be proved simultaneously, assuming $A \geq 0$ and, consequently, $\varrho_1 \leq 0$.

Sufficiency. Let $\lim_k k \sum_{j=k}^{\infty} p_j = A$. From [12], if u is a positive decreasing solution, then $u \in \mathcal{NRV}(\varrho_1)$. Put $z_k = 1/(u_k u_{k+1})$, then $z \in \mathcal{NRV}(-2\varrho_1)$ by Lemma 2.3. Hence, $\lim_k k\Delta z_k / z_k = -2\varrho_1$. A second linearly independent solution v of (1.1) is given by $v_k = u_k \sum_{j=a}^{k-1} z_j$. Taking into account that u is decreasing (recessive), it holds $\sum^{\infty} z_j = \infty$, hence the discrete L'Hospital rule yields

$$\lim_k \frac{k/u_{k+1}}{v_k} = \lim_k \frac{k z_k}{\sum_{j=a}^{k-1} z_j} = \lim_k \frac{z_k + (k+1)\Delta z_k}{z_k} = 1 - 2\varrho_1. \tag{3.3}$$

By Lemma 2.3, since $u_k = k^{\varrho_1} L_k$ with $L \in \mathcal{NSV}$, from (3.3) we have $(1-2\varrho_1)v_k \sim k^{1-\varrho_1}/L_k$, i.e., $v_k = k^{1-\varrho_1}\tilde{L}_k = k^{\varrho_2}\tilde{L}_k$, where $\tilde{L}_k \sim 1/((1-2\varrho_1)L_k)$. By Lemma 2.3, $\tilde{L}$ is slowly varying, which implies $v \in \mathcal{RV}(\varrho_2)$. Further, v is increasing and convex, thus it is normalized by Lemma 2.3, and therefore $\tilde{L}$ is normalized too.

Necessity. See the proof in [12]. Just note that in view of the convexity, a solution $u \in \mathcal{RV}(\varrho_1)$ necessarily decreases and a solution $v \in \mathcal{RV}(\varrho_2)$ necessarily increases by Lemma 2.3. $\square$

Theorem 3.3. *Equation* (1.1) *has a fundamental set of solutions*

$$u \in \mathcal{RPV}(-\infty), \quad v \in \mathcal{RPV}(\infty)$$

if and only if for all $\lambda > 1$

$$\lim_k k \sum_{j=k}^{[\lambda k]-1} p_j = \infty. \tag{3.4}$$

Moreover, all positive decreasing solutions of (1.1) *belong to* $\mathcal{RPV}(-\infty)$ *and all positive increasing solutions of* (1.1) *belong to* $\mathcal{RPV}(\infty)$.

Proof. Sufficiency. Let (3.4) hold. If u is a decreasing solution of (1.1), then from [13] it follows that $u \in \mathcal{RPV}(-\infty)$. Let v be an increasing solution of (1.1). By summing equation (1.1) from k to $\left[\sqrt{\lambda}k\right]-1$ we obtain

$$\Delta v_{\left[\sqrt{\lambda}k\right]} \geq \Delta v_{\left[\sqrt{\lambda}k\right]} - \Delta v_k = \sum_{j=k}^{\left[\sqrt{\lambda}k\right]-1} p_j v_{j+1}.$$

Summation of this inequality from k to $\left[\sqrt{\lambda}k\right]-1$ yields

$$\begin{aligned} &v_{\left[\sqrt{\lambda}k\right]+1} - v_{\left[\sqrt{\lambda}k\right]} + v_{\left[\sqrt{\lambda}(k+1)\right]+1} - v_{\left[\sqrt{\lambda}(k+1)\right]} + \\ &\quad + \cdots + v_{\left[\sqrt{\lambda}\left(\left[\sqrt{\lambda}k\right]-2\right)\right]+1} - v_{\left[\sqrt{\lambda}\left(\left[\sqrt{\lambda}k\right]-2\right)\right]} + \\ &\qquad + v_{\left[\sqrt{\lambda}\left(\left[\sqrt{\lambda}k\right]-1\right)\right]+1} - v_{\left[\sqrt{\lambda}\left(\left[\sqrt{\lambda}k\right]-1\right)\right]} \geq \sum_{j=k}^{\left[\sqrt{\lambda}k\right]-1} \sum_{i=j}^{\left[\sqrt{\lambda}j\right]-1} p_i v_{i+1}. \end{aligned} \tag{3.5}$$

Since $[z]+1=[z+1]$ and $\sqrt{\lambda}>1$, we have

$$\begin{aligned} \left[\sqrt{\lambda}k\right]+1 &\leq \left[\sqrt{\lambda}(k+1)\right] \\ \left[\sqrt{\lambda}(k+1)\right]+1 &\leq \left[\sqrt{\lambda}(k+2)\right] \\ &\cdots \\ \left[\sqrt{\lambda}\left(\left[\sqrt{\lambda}k\right]-2\right)\right]+1 &\leq \left[\sqrt{\lambda}\left(\left[\sqrt{\lambda}k\right]-1\right)\right]. \end{aligned}$$

Hence, since v is increasing, from (3.5) we get

$$v_{\left[\sqrt{\lambda}\left(\left[\sqrt{\lambda}k\right]-1\right)\right]+1} - v_{\left[\sqrt{\lambda}k\right]} \geq \sum_{j=k}^{\left[\sqrt{\lambda}k\right]-1} \sum_{i=j}^{\left[\sqrt{\lambda}j\right]-1} p_i v_{i+1}.$$

This implies

$$v_{[\lambda k]} \geq v_{\left[\sqrt{\lambda}\left[\sqrt{\lambda}k\right]\right]} \geq v_{\left[\sqrt{\lambda}\left(\left[\sqrt{\lambda}k\right]-1\right)\right]+1} \geq v_k \sum_{j=k}^{\left[\sqrt{\lambda}k\right]-1} \sum_{i=j}^{\left[\sqrt{\lambda}j\right]-1} p_i. \tag{3.6}$$

In view of (3.4), for any arbitrarily large constant $M>0$ there exists k_0 sufficiently large such that

$$\sum_{j=k}^{\left[\sqrt{\lambda}k\right]-1} p_j \geq \frac{M}{k}$$

for $k \geq k_0$. Then, since v is positive, from (3.6) we have

$$\begin{aligned}\frac{v_{[\lambda k]}}{v_k} &\geq M \sum_{j=k}^{[\sqrt{\lambda}k]-1} \frac{1}{j} \geq M \int_k^{[\sqrt{\lambda}k]} \frac{ds}{s} \\ &= M \ln \frac{\left[\sqrt{\lambda}k\right]}{k} \geq M \ln \frac{\sqrt{\lambda}k - 1}{k} \\ &= M \ln \left(\sqrt{\lambda} - \frac{1}{k} \right).\end{aligned}$$

Since M was arbitrarily large, this implies $\lim_k v_{[\lambda k]}/v_k = \infty$, and hence v is rapidly varying.

Necessity. The proof of the necessity of (3.4) follows from the results in [13]. Just note that in view of $\mathbb{M} = \mathbb{M}^+ \cup \mathbb{M}^-$, a solution $u \in \mathcal{RPV}(-\infty)$ necessarily decreases and a solution $v \in \mathcal{RPV}(\infty)$ necessarily increases by Lemma 2.5. □

Let us introduce the following notation, for $\lambda > 1$

$$P = \lim_k k^{\alpha-1} \sum_{j=k}^{\infty} p_j, \quad P_\lambda = \lim_k k^{\alpha-1} \sum_{j=k}^{[\lambda k]-1} p_j.$$

A relation between P and P_λ is given in the following theorem, whose proof can be found in [13].

Theorem 3.4 ([13]). *We have*

$$P = A \geq 0, \ \ A \in \mathbb{R}, \quad \textit{iff} \quad P_\lambda = \frac{A(\lambda - 1)}{\lambda}, \ \ \forall \lambda > 1.$$

However, in general, $P = \infty$ does not imply $P_\lambda = \infty$; an example of a positive sequence $\{p_k\}$ for which $P = \infty$ and $P_\lambda \neq 0$ with $\lambda = 2$ is the following

$$p_k = \tilde{p}_k + \frac{1}{k(k+1)}, \quad k \in \mathbb{N}$$

where

$$\tilde{p}_k = \begin{cases} 0 & \text{for } \ k \in [2^{2n}, 2^{2n+1} - 1] \cup [2^{2n+1} + 1, 2^{2n+2} - 1] \\ 2^{-n} & \text{for } \ k = 2^{2n+1} \end{cases}.$$

The reader can find the details in [13]. Theorems 3.1–3.4 now yield the following statement.

Corollary 3.5. *All positive solutions of* (1.1) *are Karamata sequences (i.e., either regularly or rapidly varying sequences) iff the limit*

$$\lim_k k \sum_{j=k}^{[\lambda k]-1} p_j$$

exists (finite or infinite), for every $\lambda > 1$.

4 Relations of Karamata Solutions to $\mathbb{M}$-classification

In this section we show how a classical classification of solutions according to their asymptotic behavior can be connected to the results from the previous section.

We have already introduced the classes $\mathbb{M}$, $\mathbb{M}^+$, and $\mathbb{M}^-$ with their basic properties. Here we recall some more detailed information, see e.g., [4,5]. The classes $\mathbb{M}^+$ and $\mathbb{M}^-$ can be *a-priori* divided into the following subclasses

$$\begin{aligned} \mathbb{M}_A^+ &= \left\{ y \in \mathbb{M}^+ : \lim_k \Delta y_k \in (0,\infty) \right\}, \\ \mathbb{M}_\infty^+ &= \left\{ y \in \mathbb{M}^+ : \lim_k \Delta y_k = \infty \right\}, \\ \mathbb{M}_B^- &= \left\{ y \in \mathbb{M}^- : \lim_k y_k \in (0,\infty) \right\}, \\ \mathbb{M}_0^- &= \left\{ y \in \mathbb{M}^- : \lim_k y_k = 0 \right\}, \end{aligned}$$

i.e., $\mathbb{M}^+ = \mathbb{M}_A^+ \cup \mathbb{M}_\infty^+$ and $\mathbb{M}^- = \mathbb{M}_B^- \cup \mathbb{M}_0^-$. It is easy to see that every $y \in \mathbb{M}^+$ satisfies $\lim_k y_k = \infty$ and every $y \in \mathbb{M}^-$ satisfies $\lim_k \Delta y_k = 0$. Denote

$$I = \lim_k \sum_{j=1}^{k} j p_j.$$

The existence in each subclass can be described in terms of convergence or divergence of the series in I; consequently, the question of which subclasses of $\mathbb{M}^+$ and $\mathbb{M}^-$ may coexist is answered. The following relations hold:

$$\begin{aligned} \mathbb{M}^+ = \mathbb{M}_\infty^+ &\iff I = \infty \iff \mathbb{M}^- = \mathbb{M}_0^-, \\ \mathbb{M}^+ = \mathbb{M}_A^+ &\iff I < \infty \iff \mathbb{M}^- = \mathbb{M}_B^-. \end{aligned}$$

Introduce the notation:

$$\begin{aligned} \mathbb{M}_{SV}^- &= \mathbb{M}^- \cap \mathcal{NSV}, \\ \mathbb{M}_{RV}^-(\varrho_1) &= \mathbb{M}^- \cap \mathcal{NRV}(\varrho_1), \varrho_1 < 0, \\ \mathbb{M}_{RPV}^- &= \mathbb{M}^- \cap \mathcal{RPV}(-\infty), \\ \mathbb{M}_{RV}^+(\varrho_2) &= \mathbb{M}^+ \cap \mathcal{NRV}(\varrho_2), \varrho_2 > 0, \\ \mathbb{M}_{RPV}^+ &= \mathbb{M}^+ \cap \mathcal{RPV}(\infty). \end{aligned}$$

The results proved in the previous sections can be summarized as follows:

$$\begin{aligned} \mathbb{M}^- = \mathbb{M}_{SV}^- \text{ and } \mathbb{M}^+ = \mathbb{M}_{RV}^+(1) &\iff P = 0, \\ \mathbb{M}^- = \mathbb{M}_{RV}^-(\varrho_1) \text{ and } \mathbb{M}^+ = \mathbb{M}_{RV}^+(\varrho_2) &\iff P = C > 0, \\ \mathbb{M}^- = \mathbb{M}_{RPV}^- \text{ and } \mathbb{M}^+ = \mathbb{M}_{RPV}^+ &\iff P_\lambda = \infty, \forall \lambda > 1, \end{aligned}$$

where $\varrho_1 < \varrho_2$ are the roots of $\varrho^2 - \varrho - C = 0$. Since

$$k \sum_{j=k}^{[\lambda k]-1} p_j < k \sum_{j=k}^{\infty} p_j < \sum_{j=k}^{\infty} j p_j,$$

the following relations between I and P (resp., P_λ) hold: If $I < \infty$, then $P = P_\lambda = 0$ for all $\lambda > 1$; if $P > 0$ or $P_\lambda > 0$ for all $\lambda > 1$, then $I = \infty$. We already know that $P < \infty$ iff $P_\lambda < \infty$ and $P_\lambda = \infty$ implies $P = \infty$, while $P = \infty$ does not imply $P_\lambda = \infty$.

These observations lead to the next result.

Theorem 4.1. *For equation* (1.1) *the following hold:*

(i) If $P = 0$ and $I < \infty$, then $\mathbb{M}_A^+ = \mathbb{M}^+ = \mathbb{M}_{RV}^+(1)$ and $\mathbb{M}_B^- = \mathbb{M}^- = \mathbb{M}_{SV}^-$. If $P = 0$ and $I = \infty$, then $\mathbb{M}_\infty^+ = \mathbb{M}^+ = \mathbb{M}_{RV}^+(1)$ and $\mathbb{M}_0^- = \mathbb{M}^- = \mathbb{M}_{SV}^-$.

(ii) If $P = C > 0$, then $\mathbb{M}_\infty^+ = \mathbb{M}^+ = \mathbb{M}_{RV}^+(\varrho_2)$ and $\mathbb{M}_0^- = \mathbb{M}^- = \mathbb{M}_{RV}^-(\varrho_1)$.

(iii) If $P_\lambda = \infty$ for all $\lambda > 1$, then $\mathbb{M}_\infty^+ = \mathbb{M}^+ = \mathbb{M}_{RPV}^+$ and $\mathbb{M}_0^- = \mathbb{M}^- = \mathbb{M}_{RPV}^-$.

Notice that the previous theorem can be reformulated in terms of I:

(i) If $I < \infty$, then $\mathbb{M}_A^+ = \mathbb{M}^+ = \mathbb{M}_{RV}^+(1)$ and $\mathbb{M}_B^- = \mathbb{M}^- = \mathbb{M}_{SV}^-$.

(ii) If $I = \infty$ and $P = 0$, then $\mathbb{M}_\infty^+ = \mathbb{M}^+ = \mathbb{M}_{RV}^+(1)$ and $\mathbb{M}_0^- = \mathbb{M}^- = \mathbb{M}_{SV}^-$. If $I = \infty$ and $P = C > 0$, then $\mathbb{M}_\infty^+ = \mathbb{M}^+ = \mathbb{M}_{RV}^+(\varrho_2)$ and $\mathbb{M}_0^- = \mathbb{M}^- = \mathbb{M}_{RV}^-(\varrho_1)$. If $I = \infty$ and $P_\lambda = \infty$ for all $\lambda > 1$, then $\mathbb{M}_\infty^+ = \mathbb{M}^+ = \mathbb{M}_{RPV}^+$ and $\mathbb{M}_0^- = \mathbb{M}^- = \mathbb{M}_{RPV}^-$.

We end this section by illustrating relations between recessive solutions and Karamata solutions of (1.1). Recall that there exists a solution y of (1.1) (with $p_k > 0$), uniquely determined up to a constant factor, such that $\lim_k y_k/x_k = 0$, for any other solution x of (1.1) satisfying $x \neq \lambda y$, $\lambda \in \mathbb{R}$. The solution y is called a *recessive solution* and x a *dominant solution*, see e.g., [1]. Let $\mathcal{M}$ denote the set of recessive solutions of (1.1). It is known that $\mathcal{M} = \mathbb{M}_0^-$ if $I = \infty$ and $\mathcal{M} = \mathbb{M}_B^-$ if $I < \infty$. Combining these results with the above ones we have the following:

(i) If $P = 0$, then $\mathcal{M} = \mathbb{M}_{SV}^-$.

(ii) If $P = C > 0$, then $\mathcal{M} = \mathbb{M}_{RV}^-(\varrho_1)$.

(iii) If $P_\lambda = \infty$ for all $\lambda > 1$, then $\mathcal{M} = \mathbb{M}_{RPV}^-$.

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