International Journal of Difference Equations
ISSN 0973-5321, Volume 3, Number 2, pp. 317–330 (2008)
http://campus.mst.edu/ijde

The Quasilinearization Method for Dynamic Equations with m-point Boundary Value Problems on Time Scales

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Abstract

The method of quasilinearization is applied to the nonlinear second order dynamic equations with m-point boundary conditions on time scales, and two sequences could be constructed which converge uniformly to the unique solution of the m-point boundary value problems from above and below with high rate of convergence.

AMS Subject Classifications: 34B15, 39A12.
Keywords: Time scales, quasilinearization method, boundary value problems, lower and upper solutions, convergence.

1 Introduction

In this paper, we consider the nonlinear dynamic equations on time scales

$$x^{\Delta\nabla}(t) + f(t, x(t)) = 0, \quad t \in [0, T] \subset \mathbb{T}, \tag{1.1}$$

Received July 10, 2007; Accepted October 17, 2008
Communicated by Johnny Henderson
*Supported by the Key Project of Chinese Ministry of Education (207014) and the Natural Science Foundation of Hebei Province of China (A2006000941)

subject to

$$\beta x(0) - \gamma x^{\Delta}(0) = 0, \quad x(T) - \sum_{i=1}^{m-2} a_i x(\xi_i) = 0, \ \ m \geq 3, \tag{1.2}$$

where T is left dense and the following assumptions are satisfied:

(H$_1$) $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(t,\cdot)$ is ld-continuous on $[0,T]$;

(H$_2$) $\beta > 0$, $\gamma \geq 0$, $a_i \geq 0$ for $i = 1, 2, \cdots, m-2$, and $a_{m-2} > 0$, $\xi_i \in [0,T]$ for $i = 1, 2, \cdots, m-2$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < T$, $0 < \sum_{i=1}^{m-2} a_i \xi_i < T$ and $d = \beta(T - \sum_{i=1}^{m-2} a_i \xi_i) + \gamma(1 - \sum_{i=1}^{m-2} a_i) > 0$.

Throughout this paper, $\mathbb{T}$ is a time scale (nonempty closed subset of $\mathbb{R}$) and $[0,T]$ be a compact subset of $\mathbb{T}$ such that $[0,T] = \{t \in \mathbb{T} \mid 0 \leq t \leq T\}$. The sets $[0,T]^{\kappa}$, $[0,T]_{\kappa}$ and $[0,T]_{\kappa}^{\kappa}$ are defined by

(1) If T is a left-scattered point, then $[0,T]^{\kappa} = [0,T)$, otherwise $[0,T]^{\kappa} = [0,T]$;

(2) If 0 is a right-scattered point, then $[0,T]_{\kappa} = (0,T]$, otherwise $[0,T]_{\kappa} = [0,T]$;

(3) $[0,T]_{\kappa}^{\kappa} = [0,T]^{\kappa} \cap [0,T]_{\kappa}$.

The more basic notions about time scales could be found in [3, 4, 7–9, 11, 14]. The following two lemmas which are derived from [6] will be used in the proof of the later comparison theorem.

Lemma 1.1. *Let $g : [0,T] \to \mathbb{R}$ be a function which is Δ-differentiable on $[0,T]^{\kappa}$. If g has a local extremum at $t_0 \in [0,T]^{\kappa}$, then*

$$g^{\Delta}(t_0) \leq 0, \quad g^{\Delta}(\rho(t_0)) \geq 0,$$

where $g^{\Delta}(s)$ is nonnegative or nonpositive according as $s < t_0$ or $s > t_0$.

Lemma 1.2. *For $g : [0,T] \to \mathbb{R}$ and $t \in [0,T]_{\kappa}$, the following conditions hold:*

(i) *If g is continuous at t and t is left-scattered, then g is ∇-differentiable and*

$$g^{\nabla}(t) = \frac{g(\rho(t)) - g(t)}{\rho(t) - t};$$

(ii) *If t is left-dense, then g is ∇-differentiable at t iff $\lim_{s \to t} \frac{g(t) - g(s)}{t - s}$ exists, and*

$$g^{\nabla}(t) = \lim_{s \to t} \frac{g(t) - g(s)}{t - s}.$$

Recently, the study of dynamic systems on time scales has received a great deal of attention. For example, the existence of solutions for boundary value problem of dynamic equations were initiated and extended by P. Wang and Y. Wang (see [13, 14]). It is well known that the method of quasilinearization has been extensively developed and applied to a wide range of problems on $\mathbb{R}$, it combined with the method of upper and lower solutions is an effective and fruitful technique, which obtains a sequence of approximate solutions and converges rapidly to the solutions of the given systems. At present, the method has also been successfully applied to boundary value problems of dynamic equations on time scales (see [1, 2, 5, 10, 12]). However, to the best of our knowledge, there are few results on the high order convergence for boundary value problems of dynamic equations, especially, on the case of m-point boundary value problems on time scales. In this paper, by using the method of quasilinearization, we construct the two sequences which converge uniformly to the unique solution of the m-point boundary value problems from above and below, and the order of convergence is $k+1$.

2 Preliminaries

We define the set

$$\mathbb{D} := \{x \in \mathbb{B} : \ x^{\Delta} \text{ is continuous on } [0,T]^{\kappa}, \ x^{\Delta} \text{ is } \nabla \text{ differentiable and } x^{\Delta\nabla} \text{ is continuous on } [0,T]^{\kappa}_{\kappa}\},$$

where the Banach space $\mathbb{B} = C([0,T],\mathbb{R})$ is the set of real-valued continuous functions x defined on $[0,T]$ with the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

Definition 2.1. A real-valued function $v \in \mathbb{D}$ on $[0,T]$ is said to be a lower solution of the BVP (1.1), (1.2), if

$$v^{\Delta\nabla}(t) + f(t, v(t)) \geq 0, \quad t \in [0,T], \tag{2.1}$$

$$\beta v(0) - \gamma v^{\Delta}(0) \leq 0, \quad v(T) - \sum_{i=1}^{m-2} a_i v(\xi_i) \leq 0. \tag{2.2}$$

Definition 2.2. A real-valued function $w \in \mathbb{D}$ on $[0,T]$ is said to be an upper solution of the BVP (1.1), (1.2), if

$$w^{\Delta\nabla}(t) + f(t, w(t)) \leq 0, \quad t \in [0,T], \tag{2.3}$$

$$\beta w(0) - \gamma w^{\Delta}(0) \geq 0, \quad w(T) - \sum_{i=1}^{m-2} a_i w(\xi_i) \geq 0. \tag{2.4}$$

Theorem 2.3. *Assume that*

(i) v *and* w *are lower and upper solutions of the BVP* (1.1), (1.2), *respectively;*

(ii) $f(t,x)$ *is strictly decreasing in* x *for each* $t \in [0,T]$.

Then $v(t) \le w(t)$ *on* $[0,T]$.

Proof. Set $m(t) = v(t) - w(t)$ on $[0,T]$. If the conclusion is false, then there exist an $\varepsilon > 0$ and a $t_0 \in [0,T]$ such that

$$m(t_0) = \varepsilon \quad \text{and} \quad m(t) \le \varepsilon, \ \ t \in [0,T].$$

If $t_0 \in [0,T]$ and $t_0 \neq 0$, then by Lemma 1.1 we have

$$m^{\Delta}(t_0) \le 0 \ \text{ and } \ m^{\Delta}(\rho(t_0)) \ge 0,$$

where $m^{\Delta}(s)$ is nonnegative or nonpositive according as $s < t_0$ or $s > t_0$. If t_0 is left-scattered, then it follows from Lemma 1.2 that

$$m^{\Delta\nabla}(t_0) = \frac{m^{\Delta}(t_0) - m^{\Delta}(\rho(t_0))}{t_0 - \rho(t_0)} \le 0.$$

On the other hand, in view of the assumptions, we have

$$\begin{aligned} m^{\Delta\nabla}(t_0) &= v^{\Delta\nabla}(t_0) - w^{\Delta\nabla}(t_0) \\ &= v^{\Delta\nabla}(t_0) + f(t, v(t_0)) - (w^{\Delta\nabla}(t_0) + f(t, w(t_0))) \\ &\quad + (f(t, w(t_0)) - f(t, v(t_0))) \\ &> 0, \end{aligned}$$

which is a contradiction. If t_0 is left-dense, it follows from Lemma 1.2 that

$$m^{\Delta\nabla}(t_0) = \lim_{s \to t_0^-} \frac{m^{\Delta}(s) - m^{\Delta}(t_0)}{s - t_0} \le 0.$$

which leads to a contradiction as above. If $t_0 = 0$, then we get

$$m(t_0) > 0 \ \text{ and } \ m^{\Delta}(t_0) \le 0.$$

Hence, by (2.2) and (2.4), we have

$$0 < \beta(v(0) - w(0)) \le \gamma(v^{\Delta}(0) - w^{\Delta}(0)) \le 0,$$

which is also a contradiction. The proof is therefore complete. □

To state the main result of this paper, we need the following lemmas.

Lemma 2.4 (see [13]). *If* $d = \beta\left(T - \sum_{i=1}^{m-2} a_i\xi_i\right) + \gamma\left(1 - \sum_{i=1}^{m-2} a_i\right) > 0$, *then for* $h \in C_{\mathrm{ld}}[0,T]$, *the BVP*

$$x^{\Delta\nabla}(t) + h(t) = 0, \;\; t \in [0,T], \tag{2.5}$$

subject to

$$\beta x(0) - \gamma x^{\Delta}(0)) = 0, \quad x(T) - \sum_{i=1}^{m-2} a_i x(\xi_i) = 0, \;\; m \geq 3 \tag{2.6}$$

has the unique solution

$$\begin{aligned} x(t) = &-\int_0^t (t-s)h(s)\nabla s \\ &+ \frac{\beta t + \gamma}{d}\left(\int_0^T (T-s)h(s)\nabla s - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)h(s)\nabla s\right). \end{aligned}$$

Similar to Lemma 2.4, we can easily obtain the following one.

Lemma 2.5. *If* $d = \beta\left(T - \sum_{i=1}^{m-2} a_i\xi_i\right) + \gamma\left(1 - \sum_{i=1}^{m-2} a_i\right) > 0$, *then for* $f \in C_{\mathrm{ld}}[0,T]$, x *is the solution of the BVP* (1.1), (1.2) *if and only if* x *is the solution of the integral equation*

$$\begin{aligned} x(t) = &\frac{\beta t + \gamma}{d}\left(\int_0^T (T-s)f(s,x(s))\nabla s - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)f(s,x(s))\nabla s\right) \\ &- \int_0^t (t-s)f(s,x(s))\nabla s. \end{aligned}$$

We define the set K by

$$K := \{x \in \mathbb{B} : \; x(t) \in \mathbb{D}, \; v(t) \leq x(t) \leq w(t), \; t \in [0,T]\}.$$

Clearly, K is a closed convex set. Meanwhile we define an operator $A : K \to \mathbb{B}$ by

$$\begin{aligned} (Ax)(t) = &\frac{\beta t + \gamma}{d}\left(\int_0^T (T-s)f(s,x(s))\nabla s - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)f(s,x(s))\nabla s\right) \\ &- \int_0^t (t-s)f(s,x(s))\nabla s. \end{aligned}$$

Note that from the definition of the operator A, if $x \in K$, then $Ax \in K$. This implies $A : K \to K$. In addition, A is a completely continuous operator. Then, applying the Schauder fixed point theorem, we can obtain the following result.

Theorem 2.6. *Assume that v and w are lower and upper solutions of the BVP* (1.1), (1.2)*, respectively, and $v(t) \leq w(t)$ on $[0, T]$. Then there exists a solution x of the BVP* (1.1), (1.2) *such that $v(t) \leq x(t) \leq w(t)$ on $[0, T]$.*

Furthermore, if $f(t, x)$ satisfies a monotonicity condition in x for each $t \in [0, T]$, then we have uniqueness of the solution x of the BVP (1.1), (1.2).

Theorem 2.7. *Assume that $f(t, x)$ is strictly decreasing in x for each $t \in [0, T]$. Then the solutions of the BVP* (1.1), (1.2) *are unique.*

3 Main Result

First we define the sector $[v, w]$ for every $v, w \in \mathbb{D}$ by

$$[v, w] := \{x(t) \in \mathbb{D} : \ v(t) \leq x(t) \leq w(t), \ t \in [0, T]\}.$$

In the following result, note that $f^{(i)}(t, x)$, $i = 0, 1, \cdots, k + 1$, are the usual partial derivatives of f of x on $\mathbb{R}$.

Theorem 3.1. *Assume that*

(i) *$v_0, w_0 \in \mathbb{D}$ are lower and upper solutions of the BVP* (1.1), (1.2) *respectively, such that $v_0 \leq w_0$ on $[0, T]$;*

(ii) *$f^{(i)}(t, x)$, $i = 0, 1, \cdots, k$, are continuous on $[0, T] \times [v_0, w_0]$ with $f^{(1)}(t, x) < 0$ such that $f^{(k+1)}(t, x)$ exists and is ld-continuous with respect to t on $[0, T]$;*

(iii) *There exist $M, N > 0$ such that $-M \leq \dfrac{f^{(k+1)}(t, x)}{(k + 1)!} \leq N$ on $[0, T] \times [v_0, w_0]$.*

Then there exist monotone sequences $\{v_n\}$ and $\{w_n\}$ which converge uniformly to the unique solution x of the BVP (1.1), (1.2). *Moreover, the order of convergence of sequences $\{v_n\}$ and $\{w_n\}$ is $k + 1$.*

Proof. First we define for $k \geq 1$,

$$F(t, x; u_0) = \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t, u_0)(x - u_0)^i - M(x - u_0)^{k+1} \tag{3.1}$$

and

$$G(t, x; u_0) = \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t, u_0)(x - u_0)^i + \begin{cases} -M(x - u_0)^{k+1}, & k + 1 \quad \text{is odd}, \\ N(x - u_0)^{k+1}, & k + 1 \quad \text{is even}, \end{cases} \tag{3.2}$$

where x and u_0 are functions of t on $[0,T]$. Consider now the BVPs

$$x^{\Delta\nabla}(t) + F(t,x;v_0) = 0, \quad t \in [0,T] \tag{3.3}$$

with the boundary condition (1.2) and

$$x^{\Delta\nabla}(t) + G(t,x;w_0) = 0, \quad t \in [0,T] \tag{3.4}$$

with the boundary condition (1.2). Now we shall show that v_0 and w_0 are the lower and upper solutions of the BVP (3.3), (1.2) and the BVP (3.4), (1.2), respectively. In view of the hypothesis (i) and (iii), we have

$$v_0^{\Delta\nabla}(t) + F(t,v_0;v_0) = v_0^{\Delta\nabla}(t) + f(t,v_0) \geq 0,$$

$$\begin{aligned} w_0^{\Delta\nabla}(t) + F(t,w_0;v_0) &= w_0^{\Delta\nabla}(t) + \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t,v_0)(w_0 - v_0)^i - M(w_0 - v_0)^{k+1} \\ &= w_0^{\Delta\nabla}(t) + f(t,w_0) - \frac{f^{(k+1)}(t,\eta_1)}{(k+1)!}(w_0 - v_0)^{k+1} - M(w_0 - v_0)^{k+1} \\ &\leq w_0^{\Delta\nabla}(t) + f(t,w_0) \\ &\leq 0, \end{aligned}$$

where $v_0 \leq \eta_1 \leq w_0$. This implies that v_0 and w_0 are lower and upper solutions of the BVP (3.3), (1.2), respectively. Hence, by Theorem 2.6, there exists a solution v_1 of the BVP (3.3), (1.2) such that $v_0(t) \leq v_1(t) \leq w_0(t)$ on $[0,T]$.

Similarly, v_0 and w_0 are lower and upper solutions of the BVP (3.4), (1.2), respectively. If $k+1$ is odd, then we have

$$\begin{aligned} v_0^{\Delta\nabla}(t) + G(t,v_0;w_0) &= v_0^{\Delta\nabla}(t) + \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t,w_0)(v_0 - w_0)^i - M(v_0 - w_0)^{k+1} \\ &= v_0^{\Delta\nabla}(t) + f(t,v_0) - \frac{f^{(k+1)}(t,\eta_2)}{(k+1)!}(v_0 - w_0)^{k+1} - M(v_0 - w_0)^{k+1} \\ &\geq v_0^{\Delta\nabla}(t) + f(t,v_0) \\ &\geq 0; \end{aligned}$$

If $k+1$ is even, then we get

$$\begin{aligned} v_0^{\Delta\nabla}(t) + G(t,v_0;w_0) &= v_0^{\Delta\nabla}(t) + \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t,w_0)(v_0 - w_0)^i + N(v_0 - w_0)^{k+1} \\ &= v_0^{\Delta\nabla}(t) + f(t,v_0) - \frac{f^{(k+1)}(t,\eta_2)}{(k+1)!}(v_0 - w_0)^{k+1} + N(v_0 - w_0)^{k+1} \\ &\geq v_0^{\Delta\nabla}(t) + f(t,v_0) \\ &\geq 0, \end{aligned}$$

where $v_0 \le \eta_2 \le w_0$. Thus, we have

$$v_0^{\Delta\nabla}(t) + G(t, v_0; w_0) \ge 0.$$

In addition, we have

$$w_0^{\Delta\nabla}(t) + G(t, w_0; w_0) = w_0^{\Delta\nabla}(t) + f(t, w_0) \le 0.$$

Then, applying Theorem 2.6, there exists a solution w_1 of the BVP (3.4), (1.2) such that $v_0(t) \le w_1(t) \le w_0(t)$ on $[0, T]$.

Next we prove $v_1 \le w_1$ on $[0, T]$. To prove this, using Taylor's theorem, we show that v_1 is a lower solution of the BVP (1.1), (1.2) and w_1 is an upper solution:

$$\begin{aligned}
0 &= v_1^{\Delta\nabla}(t) + F(t, v_1; v_0) \\
&= v_1^{\Delta\nabla}(t) + \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t, v_0)(v_1 - v_0)^i - M(v_1 - v_0)^{k+1} \\
&= v_1^{\Delta\nabla}(t) + f(t, v_1) - \frac{f^{(k+1)}(t, \eta_3)}{(k+1)!}(v_1 - v_0)^{k+1} - M(v_1 - v_0)^{k+1} \\
&\le v_1^{\Delta\nabla}(t) + f(t, v_1).
\end{aligned}$$

On the other hand, if $k + 1$ is odd, then we have

$$\begin{aligned}
0 &= w_1^{\Delta\nabla}(t) + G(t, w_1; w_0) \\
&= w_1^{\Delta\nabla}(t) + \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t, w_1)(w_1 - w_0)^i - M(w_1 - w_0)^{k+1} \\
&= w_1^{\Delta\nabla}(t) + f(t, w_1) - \frac{f^{(k+1)}(t, \eta_4)}{(k+1)!}(w_1 - w_0)^{k+1} - M(w_1 - w_0)^{k+1} \\
&\ge w_1^{\Delta\nabla}(t) + f(t, w_1),
\end{aligned}$$

and if $k + 1$ is even, then we get

$$\begin{aligned}
0 &= w_1^{\Delta\nabla}(t) + G(t, w_1; w_0) \\
&= w_1^{\Delta\nabla}(t) + \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t, w_1)(w_1 - w_0)^i + N(w_1 - w_0)^{k+1} \\
&= w_1^{\Delta\nabla}(t) + f(t, w_1) - \frac{f^{(k+1)}(t, \eta_4)}{(k+1)!}(w_1 - w_0)^{k+1} + N(w_1 - w_0)^{k+1} \\
&\ge w_1^{\Delta\nabla}(t) + f(t, w_1),
\end{aligned}$$

where $v_0 \le \eta_3 \le v_1$, $w_1 \le \eta_4 \le w_0$. Hence, we can conclude from the above estimates that v_1, and w_1 are lower and upper solution of the BVP (1.1), (1.2), respectively, and it follows from Theorem 2.3 that $v_1 \le w_1$ on $[0, T]$. Consequently these results yield

$$v_0 \le v_1 \le w_1 \le w_0 \quad \text{on} \quad [0, T].$$

Now suppose that for some $n > 1$, $v_0 \le v_n \le w_n \le w_0$ on $[0, T]$ and

$$v_n^{\Delta\nabla}(t) + f(t, v_n(t)) \ge 0,\ \beta v_n(0) - \gamma v_n^{\Delta}(0) = 0,\ v_n(T) - \sum_{i=1}^{m-2} a_i v_n(\xi_i) = 0, \quad (3.5)$$

$$w_n^{\Delta\nabla}(t) + f(t, w_n(t)) \ge 0,\ \beta w_n(0) - \gamma w_n^{\Delta}(0) = 0,\ w_n(T) - \sum_{i=1}^{m-2} a_i w_n(\xi_i) = 0. \quad (3.6)$$

We show that

$$v_n \le v_{n+1} \le w_{n+1} \le w_n \quad \text{on} \quad [0, T],$$

where v_{n+1} and w_{n+1} are the solutions of the BVPs

$$x^{\Delta\nabla}(t) + F(t, v_{n+1}; v_n) = 0 \quad (3.7)$$

and

$$x^{\Delta\nabla}(t) + G(t, w_{n+1}; w_n) = 0 \quad (3.8)$$

with the boundary condition (1.2), respectively. It easily follows by the definition of F and (3.5) that

$$v_n^{\Delta\nabla}(t) + F(t, v_n; v_n) = v_n^{\Delta\nabla}(t) + f(t, v_n) \ge 0$$

and

$$\begin{aligned} w_n^{\Delta\nabla}(t) + F(t, w_n; v_n) &= w_n^{\Delta\nabla}(t) + \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t, w_n)(w_n - v_n)^i - M(w_n - v_n)^{k+1} \\ &= w_n^{\Delta\nabla}(t) + f(t, w_n) - \frac{f^{(k+1)}(t, \eta_5)}{(k+1)!}(w_n - v_n)^{k+1} - M(w_n - v_n)^{k+1} \\ &\le w_n^{\Delta\nabla}(t) + f(t, w_n) \\ &\le 0, \end{aligned}$$

where $v_n \le \eta_5 \le w_n$. This implies that v_n is a lower solution of the BVP (3.7), (1.2) and w_n is an upper solution. Hence, by Theorem 2.6, there exists a solution v_{n+1} of the BVP (3.7), (1.2) such that

$$v_n(t) \le v_{n+1}(t) \le w_n(t) \quad \text{on} \quad [0, T].$$

Similar arguments yield that there exists a solution $w_{n+1}(t)$ of the BVP (3.8), (1.2) such that

$$v_n(t) \le w_{n+1}(t) \le w_n(t) \quad \text{on} \quad [0, T].$$

In order to prove $v_{n+1} \le w_{n+1}$ on $[0, T]$, we show that v_{n+1} and w_{n+1} are lower and the upper solutions of the BVP (1.1) and (1.2), respectively. Since v_{n+1} is the solution of the BVP (3.7), (1.2), we get

$$
\begin{aligned}
0 &= v_{n+1}^{\Delta\nabla}(t) + F(t, v_{n+1}; v_n) \\
&= v_{n+1}^{\Delta\nabla}(t) + \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(t, v_n)(v_{n+1} - v_n)^i - M(v_{n+1} - v_n)^{k+1} \\
&= v_{n+1}^{\Delta\nabla}(t) + f(t, v_{n+1}) - \frac{f^{(k+1)}(t, \eta_6)}{(k+1)!}(v_{n+1} - v_n)^{k+1} - M(v_{n+1} - v_n)^{k+1} \\
&\le v_{n+1}^{\Delta\nabla}(t) + f(t, v_{n+1}).
\end{aligned}
$$

This implies that v_{n+1} is a lower solution of the BVP (1.1), (1.2). In a similar manner, we can prove that w_{n+1} is an upper solution of the BVP (1.1), (1.2). Hence, it follows from Theorem2.3 that $v_{n+1} \le w_{n+1}$ on $[0, T]$. This proves that

$$v_n \le v_{n+1} \le w_{n+1} \le w_n \text{ on } [0, T].$$

Now by induction we can conclude that for all $n \in \mathbb{N}$

$$v_0 \le v_1 \le \cdots \le v_n \le w_n \le \cdots \le w_1 \le w_0 \quad \text{on} \quad [0, T],$$

where for each $n \in \mathbb{N}$, v_{n+1} and w_{n+1} satisfy the BVPs (3.7), (1.2) and (3.8), (1.2).

Since $[0, T]$ is compact and the sequence is monotone and bounded, the sequence $\{v_n\}$ converges uniformly to some function x, where

$$
\begin{aligned}
v_{n+1}(t) = \frac{\beta t + \gamma}{d} &\int_0^T (T - s)F(s, v_{n+1}; v_n)\nabla s \\
&- \frac{\beta t + \gamma}{d} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)F(s, v_{n+1}; v_n)\nabla s \\
&- \int_0^t (t - s)F(s, v_{n+1}; v_n)\nabla s.
\end{aligned}
$$

Note that $\lim_{n\to\infty} F(t, v_{n+1}; v_n) = f(t, x(t))$ and $\int_0^T (T-s)\nabla s \le T^2$, $\int_0^{\xi_i} (\xi_i - s)\nabla s \le \xi_i^2$ and $\int_0^t (t-s)\nabla s \le T^2$. Using the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} \int_0^T (T - s)F(s, v_{n+1}; v_n)\nabla s = \int_0^T (T - s)f(s, x(s))\nabla s,$$

$$\lim_{n\to\infty} \int_0^{\xi_i} (\xi_i - s)F(s, v_{n+1}; v_n)\nabla s = \int_0^{\xi_i} (\xi_i - s)f(s, x(s))\nabla s$$

and

$$\lim_{n\to\infty} \int_0^t (t - s)F(s, v_{n+1}; v_n)\nabla s = \int_0^t (t - s)f(s, x(s))\nabla s.$$

Hence we get

$$x(t) = \frac{\beta t + \gamma}{d} \left(\int_0^T (T-s) f(s, x(s)) \nabla s - \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) f(s, x(s)) \nabla s \right) - \int_0^t (t-s) f(s, x(s)) \nabla s.$$

A similar argument can be used for the sequence $\{w_n\}$. Hence x is the unique solution of the BVP (1.1), (1.2) as desired.

It remains to show the order of convergence of the sequences $\{v_n\}$ and $\{w_n\}$ is $k+1$. To this end, we set $p_{n+1} = x(t) - v_{n+1}$ and $q_{n+1} = w_{n+1} - x(t)$ on $[0, T]$. Notice that $p_{n+1} \geq 0$ and $q_{n+1} \geq 0$ on $[0, T]$.

$$\begin{aligned}
p_{n+1} &= \frac{\beta t + \gamma}{d} \int_0^T (T-s)(f(s, x(s)) - F(s, v_{n+1}; v_n)) \nabla s \\
&= \frac{\beta t + \gamma}{d} \int_0^T (T-s) \left(f(s, x(s)) - \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(s, v_n)(v_{n+1} - v_n)^i \right) \nabla s \\
&\quad + \frac{\beta t + \gamma}{d} \int_0^T (T-s) M (v_{n+1} - v_n)^{k+1} \nabla s \\
&\quad - \frac{\beta t + \gamma}{d} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) \left(f(s, x(s)) - \sum_{i=0}^{k} \frac{1}{i!} f^{(i)}(s, v_n)(v_{n+1} - v_n)^i \right) \nabla s \\
&\quad - \frac{\beta t + \gamma}{d} \int_0^{\xi_i} (\xi_i - s) M (v_{n+1} - v_n)^{k+1} \nabla s - \int_0^t (t-s) M (v_{n+1} - v_n)^{k+1} \nabla s \\
&\quad - \int_0^t (t-s) \left(f(s, x(s)) - \sum_{i=0}^{k} \frac{1}{i!} f^{(k+1)}(s, v_n)(v_{n+1} - v_n)^i \right) \nabla s \\
&= \frac{\beta t + \gamma}{d} \int_0^T (T-s) \left(f(s, x(s)) - f(s, v_{n+1}) + \frac{f^{(k+1)}(s, \zeta)}{(k+1)!} (v_{n+1} - v_n)^{k+1} \right) \nabla s \\
&\quad + \frac{\beta t + \gamma}{d} \int_0^T (T-s) M (v_{n+1} - v_n)^{k+1} \nabla s \\
&\quad - \frac{\beta t + \gamma}{d} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)(f(s, x(s)) - f(s, v_{n+1})) \nabla s \\
&\quad - \frac{\beta t + \gamma}{d} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) \left(\frac{f^{(k+1)}(s, \zeta)}{(k+1)!} (v_{n+1} - v_n)^{k+1} + M (v_{n+1} - v_n)^{k+1} \right) \nabla s \\
&\quad - \int_0^t (t-s) \left(f(s, x(s)) - f(s, v_{n+1}) + \frac{f^{(k+1)}(s, \zeta)}{(k+1)!} (v_{n+1} - v_n)^{k+1} \right) \nabla s \\
&\quad - \int_0^t (t-s) M (v_{n+1} - v_n)^{k+1} \nabla s.
\end{aligned}$$

Hence

$$\begin{aligned} p_{n+1} &\le \frac{\beta T+\gamma}{d}\int_0^T (T-s)\left(\frac{1}{(k+1)!}f^{(k+1)}(s,\zeta)+M\right)p_n^{k+1}\nabla s \\ &\quad + \frac{\beta T+\gamma}{d}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i-s)\left(\frac{1}{(k+1)!}f^{(k+1)}(s,\zeta)+M\right)p_n^{k+1}\nabla s \\ &\quad + \int_0^t (t-s)\left(\frac{1}{(k+1)!}f^{(k+1)}(s,\zeta)+M\right)p_n^{k+1}\nabla s \\ &= \frac{\beta T+\gamma}{d}p_n^{k+1}(\tau_1)\int_0^T (T-s)\left(\frac{1}{(k+1)!}f^{(k+1)}(s,\zeta)+M\right)\nabla s \\ &\quad + \frac{\beta T+\gamma}{d}\sum_{i=1}^{m-2} a_i p_n^{k+1}(\tau_2)\int_0^{\xi_i}(\xi_i-s)\left(\frac{1}{(k+1)!}f^{(k+1)}(s,\zeta)+M\right)\nabla s \\ &\quad + p_n^{k+1}(\tau_3)\int_0^t (t-s)\left(\frac{1}{(k+1)!}f^{(k+1)}(s,\zeta)+M\right)\nabla s, \end{aligned}$$

where $v_n \le \zeta \le v_{n+1}$, $0 \le \tau_1,\ \tau_2,\ \tau_3 \le T$. Thus

$$\begin{aligned} \|p_{n+1}\| &\le \frac{\beta T+\gamma}{d}\left(\|p_n\|^{k+1}(M+N)T^2 + \|p_n\|^{k+1}\sum_{i=1}^{m-2} a_i\xi_i^2\right) + \|p_n\|^{k+1}T^2 \\ &= \left(\frac{\beta T+\gamma}{d}\left((M+N)T^2+\sum_{i=1}^{m-2} a_i\xi_i^2\right)+T^2\right)\|p_n\|^{k+1}, \end{aligned}$$

where $\frac{1}{(k+1)!}f^{(k+1)}(t,x)+M \le M+N$, $\int_0^T (T-s)\nabla s \le T^2$, $\int_0^{\xi_i}(\xi_i-s)\nabla s \le \xi_i^2$ and $\int_0^t (t-s)\nabla s \le T^2$. Similarly we obtain for q_{n+1} the estimate

$$\|q_{n+1}\| \le \left(\frac{\beta T+\gamma}{d}\left(T^2+\sum_{i=1}^{m-2} a_i\xi_i^2\right)+K_0T^2\right)\|q_n\|^{k+1}$$

where $\max\left\{\frac{1}{(k+1)!}f^{(k+1)}(t,x)+M,\ \frac{1}{(k+1)!}f^{(k+1)}(t,x)-N\right\} \le K_0$ on $[0,T]$. The proof is complete. □

Example 3.2. Let us illustrate Theorem 3.1 with a specific time scale

$$\mathbb{T} = \left\{\frac{1}{n} : n \in N_0\right\} \cup \{0\} \cup \left[\frac{1}{2}, 2\right].$$

Consider the BVP

$$\begin{cases} x^{\Delta\nabla}(t) - x^3 - 9x = 0, & t \in [0,1] \subset \mathbb{T}, \\ x(0) - x^{\Delta}(0) = 0, & x(1) - \frac{1}{2}x(\xi_1) = 0, \end{cases} \tag{3.9}$$

where $T = 1$, $m = 3$, $\beta = 1$, $\gamma = 1$, $a_1 = \dfrac{1}{2}$, $\xi_1 = \dfrac{2}{3}$ and $f(t,x) = -x^3 - 9x$. Then $v_0 = -1$ and $w_0 = 1$ are lower and upper solutions of the BVP (3.9). By Theorem 2.6, we conclude that there is a solution in $[-1, 1]$ for $t \in [0, 1]$. Moreover, since f satisfies conditions (ii)–(iii) in Theorem 3.1, we also conclude that there are monotone sequences $\{v_n\}$ and $\{w_n\}$ converging uniformly in $[-1, 1]$ on $[0, 1]$ to the unique solution of the BVP (3.9), which is zero.

Acknowledgments

The authors thank the reviewers for their valuable suggestions and comments.

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