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Dynamics of a Rational Recursive Sequence

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Abstract

In this paper, the dynamics of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}}, \quad n \in \mathbb{N}_0,$$

where the initial conditions are arbitrary nonzero real numbers, is studied. Moreover, the solutions are obtained.

AMS Subject Classifications: 39A10.
Keywords: Difference equations, recursive sequences, periodic solution.

1 Introduction

This paper studies the dynamics of the solutions of recursive sequences satisfying

$$x_{n+1} = \frac{x_{n-5}}{\pm 1 \pm x_{n-1}x_{n-3}x_{n-5}}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where the initial conditions are arbitrary nonzero real numbers. Also, we get explicit forms of the solutions.

Recently there has been a great interest in studying the qualitative properties of rational difference equations. The study of rational difference equations of order greater than one is quite challenging and rewarding because some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations. However, there have not been any effective general methods to deal with the global behavior of

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rational difference equations of order greater than one so far. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Aloqeili [1] obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_n x_{n-1}}.$$

Cinar [2–4] considered the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{x_{n-1}}{-1 + a x_n x_{n-1}}, \quad x_{n+1} = \frac{a x_{n-1}}{1 + b x_n x_{n-1}}.$$

Elabbasy et al [5] investigated the global stability, periodicity character and gave the solution of a special case of the recursive sequence

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}.$$

Elabbasy et al [7] studied the global stability, periodicity character, boundedness and obtained the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Elabbasy et al [8] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{d x_{n-l} x_{n-k}}{c x_{n-s} - b} + a.$$

Karatas et al [10] obtained the form of the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Simsek et al [17] found explicit forms of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}.$$

For related work see also [13–22]. For the systematical studies of rational and nonrational difference equations, one can refer to the papers [1–12] and references therein.

Here, we recall some notations and results which will be useful in our investigation. Let I be some interval of real numbers and let

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions

$$x_{-k}, x_{-k+1}, \ldots, x_0 \in I,$$

the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n \in \mathbb{N}_0 \tag{1.2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$ [12]. A point $\overline{x} \in I$ is called an equilibrium point of (1.2) if

$$\overline{x} = f(\overline{x}, \overline{x}, \ldots, \overline{x}).$$

That is, $x_n = \overline{x}$ for $n \geq 0,$ is a solution of (1.2), or equivalently, $\overline{x}$ is a fixed point of f.

Definition 1.1 (Stability)**.** (i) The equilibrium point $\overline{x}$ of (1.2) is called *locally stable* if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all

$$x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$$

with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of (1.2) is called *locally asymptotically stable* if $\overline{x}$ is a locally stable solution of (1.2) and there exists $\gamma > 0$, such that for all

$$x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I$$

with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + \ldots + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of (1.2) is called a *global attractor* if for all

$$x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0 \in I,$$

we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of (1.2) is called *globally asymptotically stable* if $\overline{x}$ is locally stable and $\overline{x}$ is also a global attractor of (1.2).

(v) The equilibrium point $\overline{x}$ of (1.2) is called *unstable* if $\overline{x}$ is not locally stable.

The linearized equation of (1.2) about the equilibrium $\overline{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\overline{x}, \overline{x}, \ldots, \overline{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem 1.2 (see [11]). *Assume that* $p, q \in \mathbb{R}$ *and* $k \in \mathbb{N}_0$. *Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n \in \mathbb{N}_0.$$

Remark 1.3. Theorem 1.2 can be easily extended to general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + \ldots + p_k x_n = 0, \quad n \in \mathbb{N}_0, \tag{1.3}$$

where $p_1, p_2, \ldots, p_k \in \mathbb{R}$ and $k \in \mathbb{N}$. Then (1.3) is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Definition 1.4 (Periodicity). A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be *periodic* with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

2 The Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{1 + x_{n-1}x_{n-3}x_{n-5}}$

In this section we give a specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}x_{n-5}}, \quad n \in \mathbb{N}_0, \tag{2.1}$$

where the initial conditions are arbitrary nonzero positive real numbers.

Theorem 2.1. *Let* $\{x_n\}_{n=-5}^{\infty}$ *be a solution of* (2.1). *Then for* $n \in \mathbb{N}_0$

$$\begin{aligned}
x_{6n-5} &= \frac{f \prod_{i=0}^{n-1} (1 + 3ibdf)}{\prod_{i=0}^{n-1} (1 + (3i+1)bdf)}, \quad x_{6n-2} = \frac{c \prod_{i=0}^{n-1} (1 + (3i+1)ace)}{\prod_{i=0}^{n-1} (1 + (3i+2)ace)}, \\
x_{6n-4} &= \frac{e \prod_{i=0}^{n-1} (1 + 3iace)}{\prod_{i=0}^{n-1} (1 + (3i+1)ace)}, \quad x_{6n-1} = \frac{b \prod_{i=0}^{n-1} (1 + (3i+2)bdf)}{\prod_{i=0}^{n-1} (1 + (3i+3)bdf)}, \\
x_{6n-3} &= \frac{d \prod_{i=0}^{n-1} (1 + (3i+1)bdf)}{\prod_{i=0}^{n-1} (1 + (3i+2)bdf)}, \quad x_{6n} = \frac{a \prod_{i=0}^{n-1} (1 + (3i+2)ace)}{\prod_{i=0}^{n-1} (1 + (3i+3)ace)},
\end{aligned}$$

where $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$x_{6n-11} = \frac{f\prod_{i=0}^{n-2}(1+3ibdf)}{\prod_{i=0}^{n-2}(1+(3i+1)bdf)}, \quad x_{6n-8} = \frac{c\prod_{i=0}^{n-2}(1+(3i+1)ace)}{\prod_{i=0}^{n-2}(1+(3i+2)ace)},$$

$$x_{6n-10} = \frac{e\prod_{i=0}^{n-2}(1+3iace)}{\prod_{i=0}^{n-2}(1+(3i+1)ace)}, \quad x_{6n-7} = \frac{b\prod_{i=0}^{n-2}(1+(3i+2)bdf)}{\prod_{i=0}^{n-2}(1+(3i+3)bdf)},$$

$$x_{6n-9} = \frac{d\prod_{i=0}^{n-2}(1+(3i+1)bdf)}{\prod_{i=0}^{n-2}(1+(3i+2)bdf)}, \quad x_{6n-6} = \frac{a\prod_{i=0}^{n-2}(1+(3i+2)ace)}{\prod_{i=0}^{n-2}(1+(3i+3)ace)}.$$

Now, it follows from (2.1) that

$$\begin{aligned}
x_{6n-5} &= \frac{x_{6n-11}}{1+x_{6n-7}x_{6n-9}x_{6n-11}} \\
&= \frac{\dfrac{f\prod_{i=0}^{n-2}(1+3ibdf)}{\prod_{i=0}^{n-2}(1+(3i+1)bdf)}}{1+\dfrac{b\prod_{i=0}^{n-2}(1+(3i+2)bdf)}{\prod_{i=0}^{n-2}(1+(3i+3)bdf)}\dfrac{d\prod_{i=0}^{n-2}(1+(3i+1)bdf)}{\prod_{i=0}^{n-2}(1+(3i+2)bdf)}\dfrac{f\prod_{i=0}^{n-2}(1+3ibdf)}{\prod_{i=0}^{n-2}(1+(3i+1)bdf)}} \\
&= \frac{f\prod_{i=0}^{n-2}(1+3ibdf)}{\prod_{i=0}^{n-2}(1+(3i+1)bdf)\left(1+\dfrac{b}{\prod_{i=0}^{n-2}(1+(3i+3)bdf)}df\prod_{i=0}^{n-2}(1+3ibdf)\right)} \\
&= \frac{f\prod_{i=0}^{n-2}(1+3ibdf)}{\prod_{i=0}^{n-2}(1+(3i+1)bdf)\left(1+\dfrac{bdf}{(1+(3n-3)bdf)}\right)}\frac{(1+(3n-3)bdf)}{(1+(3n-3)bdf)}
\end{aligned}$$

$$= \frac{f\prod_{i=0}^{n-1}(1+3ibdf)}{\prod_{i=0}^{n-2}(1+(3i+1)bdf)\{(1+(3n-3)bdf)+bdf\}}$$

$$= \frac{f\prod_{i=0}^{n-1}(1+3ibdf)}{\prod_{i=0}^{n-2}(1+(3i+1)bdf)\{1+(3n-2)bdf\}}.$$

Hence, we have

$$x_{6n-5} = \frac{f\prod_{i=0}^{n-1}(1+3ibdf)}{\prod_{i=0}^{n-1}(1+(3i+1)bdf)}.$$

Similarly, one can prove the other relations. The proof is complete. □

Theorem 2.2. *Equation* (2.1) *has only the trivial equilibrium point which is always not locally asymptotically stable.*

Proof. For the equilibrium points of (2.1), we can write

$$\overline{x} = \frac{\overline{x}}{1+\overline{x}^3}.$$

Then

$$\overline{x} + \overline{x}^4 = \overline{x},$$

i.e.,

$$\overline{x}^4 = 0.$$

Thus the equilibrium point of (2.1) is $\overline{x} = 0$. Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be the function defined by

$$f(u,v,w) = \frac{u}{1+uvw}.$$

Therefore it follows that

$$f_u(u,v,w) = \frac{1}{(1+uvw)^2}, \quad f_v(u,v,w) = \frac{-u^2w}{(1+uvw)^2}, \quad f_w(u,v,w) = \frac{-u^2v}{(1+uvw)^2}.$$

We see that

$$f_u(\overline{x},\overline{x},\overline{x}) = 1, \quad f_v(\overline{x},\overline{x},\overline{x}x) = 0, \quad f_w(\overline{x},\overline{x},\overline{x}) = 0.$$

The proof now follows by using Theorem 1.2. □

Theorem 2.3. *Every positive solution* x *of* (2.1) *is bounded and* $\lim_{n\to\infty} x_n = 0.$

Proof. It follows from (2.1) that

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-1}x_{n-3}x_{n-5}} \leq x_{n-5}.$$

Then the subsequences $\{x_{6n-5}\}_{n=0}^{\infty}$, $\{x_{6n-4}\}_{n=0}^{\infty}$, $\{x_{6n-3}\}_{n=0}^{\infty}$, $\{x_{6n-2}\}_{n=0}^{\infty}$, $\{x_{6n-1}\}_{n=0}^{\infty}$, $\{x_{6n}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$M = \max\{x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}.$$

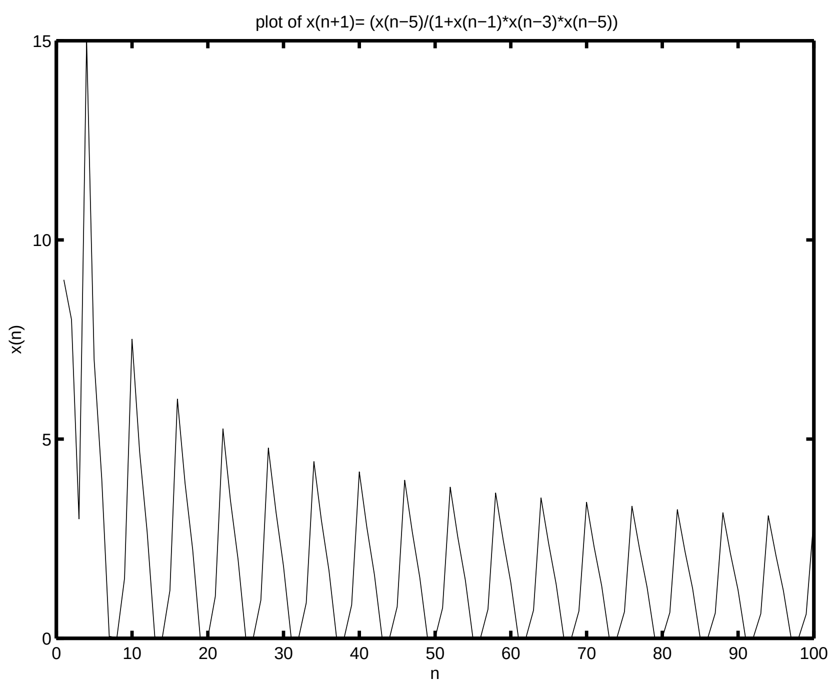
This completes the proof. □

Remark 2.4. Equation (2.1) has no prime period two solution.

To illustrate the results of this section, we now consider numerical examples which represent different types of solutions to (2.1).

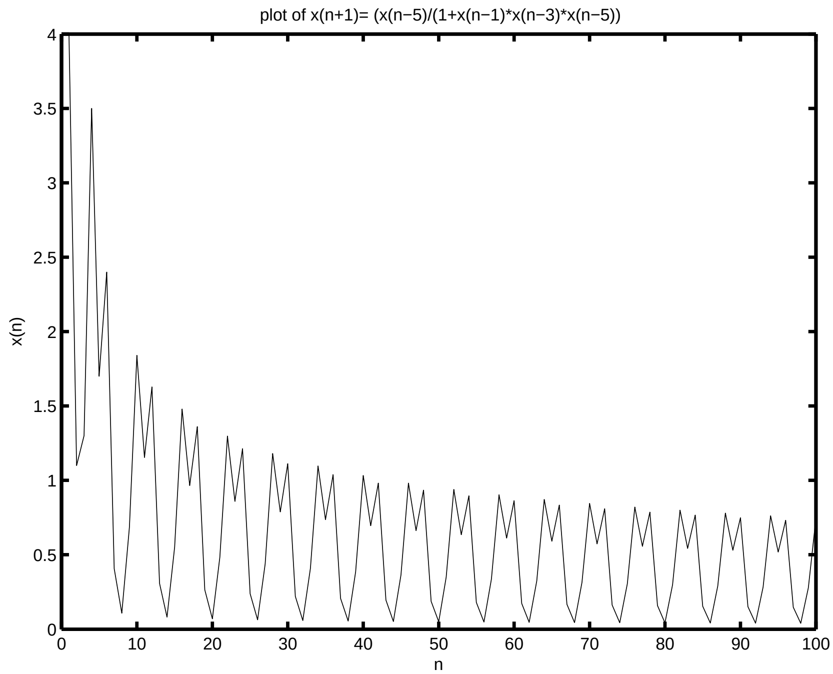
Example 2.5. Assume that $x_{-5} = 9$, $x_{-4} = 8$, $x_{-3} = 3$, $x_{-2} = 15$, $x_{-1} = 7$, $x_0 = 4$. See Figure 2.1.

Figure 2.1: Solution of the Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{1 + x_{n-1}x_{n-3}x_{n-5}}$



Example 2.6. Assume that $x_{-5} = 4$, $x_{-4} = 1.1$, $x_{-3} = 1.3$, $x_{-2} = 3.5$, $x_{-1} = 1.7$, $x_0 = 2.4$. See Figure 2.2,

Figure 2.2: Solution of the Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{1 + x_{n-1}x_{n-3}x_{n-5}}$



3 The Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{1 - x_{n-1}x_{n-3}x_{n-5}}$

Here the specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 - x_{n-1}x_{n-3}x_{n-5}}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

where the initial conditions are arbitrary nonzero real numbers, will be derived.

Theorem 3.1. *Let* $\{x_n\}_{n=-5}^{\infty}$ *be a solution of* (3.1). *Then for* $n \in \mathbb{N}_0$

$$x_{6n-5} = \frac{f\prod_{i=0}^{n-1}(1-3ibdf)}{\prod_{i=0}^{n-1}(1-(3i+1)bdf)}, \quad x_{6n-2} = \frac{c\prod_{i=0}^{n-1}(1-(3i+1)ace)}{\prod_{i=0}^{n-1}(1-(3i+2)ace)},$$

$$x_{6n-4} = \frac{e\prod_{i=0}^{n-1}(1-3iace)}{\prod_{i=0}^{n-1}(1-(3i+1)ace)}, \quad x_{6n-1} = \frac{b\prod_{i=0}^{n-1}(1-(3i+2)bdf)}{\prod_{i=0}^{n-1}(1-(3i+3)bdf)},$$

$$x_{6n-3} = \frac{d\prod_{i=0}^{n-1}(1-(3i+1)bdf)}{\prod_{i=0}^{n-1}(1-(3i+2)bdf)}, \quad x_{6n} = \frac{a\prod_{i=0}^{n-1}(1-(3i+2)ace)}{\prod_{i=0}^{n-1}(1-(3i+3)ace)},$$

where $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$, *and* $jbdf \neq 1, jace \neq 1$ *for* $j \in \mathbb{N}$.

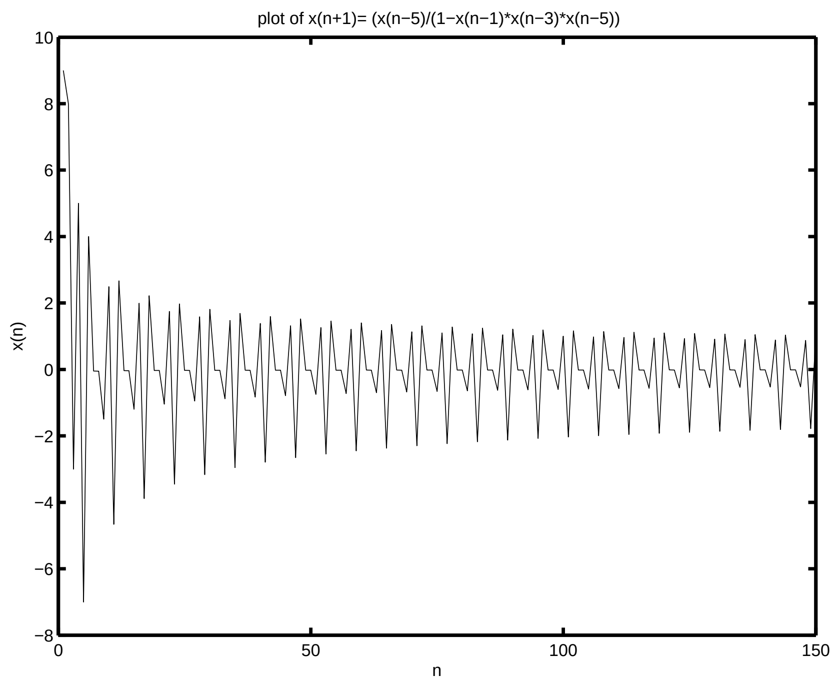
Proof. The proof is similar to the proof of Theorem 2.1 and therefore it will be omitted. □

Theorem 3.2. *Equation* (3.1) *has a unique equilibrium point* $\overline{x} = 0$, *which is not locally asymptotically stable.*

Remark 3.3. Equation (3.1) has no prime period two solution.

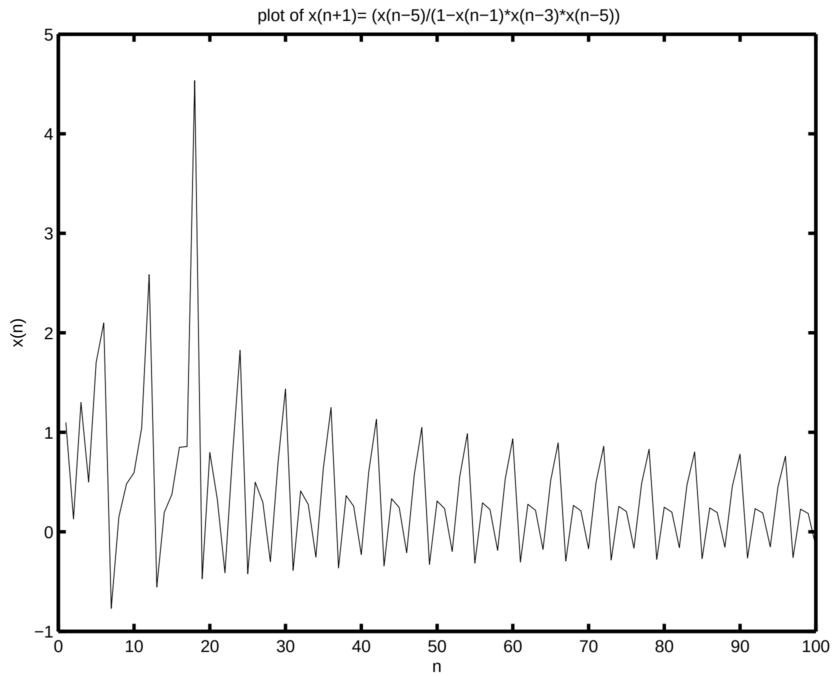
Example 3.4. Figure 3.1 shows the solution when $x_{-5} = 9$, $x_{-4} = 8$, $x_{-3} = -3$, $x_{-2} = 5$, $x_{-1} = -7$, $x_0 = 4$.

Figure 3.1: Solution of the Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{1 - x_{n-1}x_{n-3}x_{n-5}}$



Example 3.5. Figure 3.2 shows the solution when $x_{-5} = 1.1$, $x_{-4} = 0.13$, $x_{-3} = 1.3$, $x_{-2} = 0.5$, $x_{-1} = 1.7$, $x_0 = 2.1$.

Figure 3.2: Solution of the Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{1 - x_{n-1}x_{n-3}x_{n-5}}$



4 The Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{-1 + x_{n-1}x_{n-3}x_{n-5}}$

In this section, we investigate the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{-1 + x_{n-1}x_{n-3}x_{n-5}}, \quad n \in \mathbb{N}_0, \tag{4.1}$$

where the initial conditions are arbitrary nonzero real numbers with $x_{-5}x_{-3}x_{-1} \neq 1$ and $x_{-4}x_{-2}x_0 \neq 1$.

Theorem 4.1. *Every solution* $\{x_n\}_{n=-5}^{\infty}$ *of* (4.1) *is periodic with period twelve and is of the form*

$$\left\{ f, e, d, c, b, a, \frac{f}{-1 + bdf}, \frac{e}{-1 + ace}, d(-1 + bdf), \right.$$
$$\left. c(-1 + ace), \frac{b}{-1 + bdf}, \frac{a}{-1 + ace}, f, e, \ldots \right\}.$$

Proof. From (4.1), we see that

$$\begin{aligned} x_1 &= \frac{f}{-1+bdf},\ x_2 = \frac{e}{-1+ace},\ \ x_3 = d(-1+bdf),\ \ x_4 = c(-1+ace), \\ x_5 &= \frac{b}{-1+bdf},\ x_6 = \frac{a}{-1+ace},\ \ x_7 = f = x_{-5},\ \ x_8 = e = x_{-4}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 4.2. *Equation* (4.1) *has two equilibrium points which are* $0, \sqrt[3]{2}$, *and these equilibrium points are not locally asymptotically stable.*

Proof. The proof is the same as the proof of Theorem 2.2 and hence is omitted. □

Theorem 4.3. *Equation* (4.1) *has a periodic solution of period six iff* $ace = bdf = 2$, *and then takes the form* $\{f, e, d, c, b, a, f, e, d, c, b, a, \ldots\}$.

Proof. The proof is obtained from Theorem 4.1. □

Remark 4.4. Equation (4.1) has no prime period two solution.

Example 4.5. Figure 4.1 shows the solution when $x_{-5} = 1.1$, $x_{-4} = 0.8$, $x_{-3} = 1.3$, $x_{-2} = 0.5$, $x_{-1} = -1.7$, $x_0 = 2.1$.

Example 4.6. Figure 4.2 shows the solution when $x_{-5} = 0.1$, $x_{-4} = 0.3$, $x_{-3} = 2$, $x_{-2} = 5$, $x_{-1} = 10$, $x_0 = 4/3$.

5 The Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{-1 - x_{n-1}x_{n-3}x_{n-5}}$

In this section, we study the solutions of the difference equation

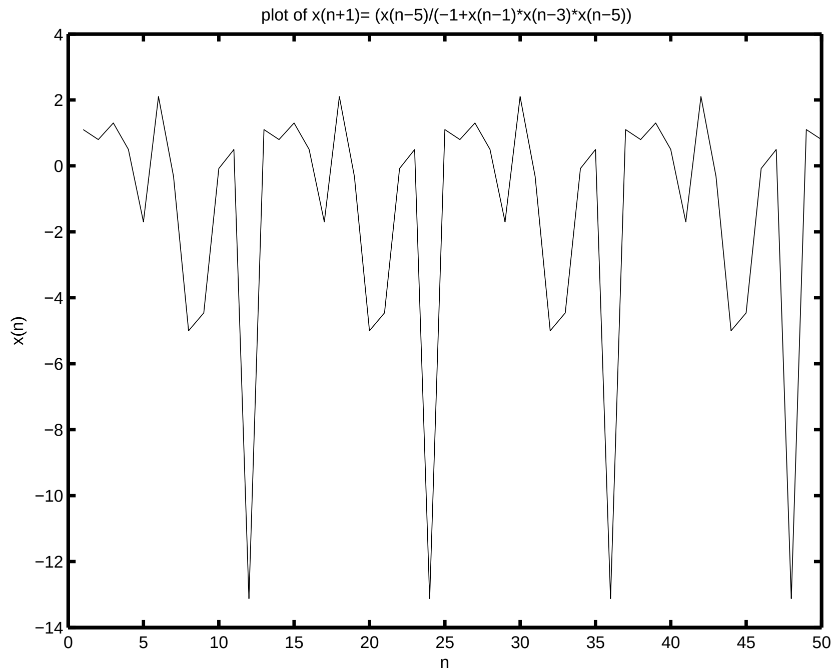
$$x_{n+1} = \frac{x_{n-5}}{-1 - x_{n-1}x_{n-3}x_{n-5}}, \quad n \in \mathbb{N}_0, \tag{5.1}$$

where the initial conditions are arbitrary nonzero real numbers with $x_{-5}x_{-3}x_{-1} \neq -1$ and $x_{-4}x_{-2}x_0 \neq -1$.

Theorem 5.1. *Every solution* $\{x_n\}_{n=-5}^{\infty}$ *of* (5.1) *is periodic with period twelve and is of the form*

$$\left\{f, e, d, c, b, a, \frac{-f}{1+bdf}, \frac{-e}{1+ace}, -d(1+bdf), -c(1+ace), \frac{-b}{1+bdf}, \frac{-a}{1+ace}, f, e, \ldots\right\}$$

Figure 4.1: Solution of the Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{-1 + x_{n-1}x_{n-3}x_{n-5}}$



Proof. The proof is similar to the proof of Theorem 4.1 and therefore is omitted. □

Theorem 5.2. *Equation* (5.1) *has one equilibrium point which is* 0*, and this equilibrium point is not locally asymptotically stable.*

Theorem 5.3. *Equation* (5.1) *has a periodic solution of period six iff* $ace = bdf = -2$*, and then takes the form* $\{f, e, d, c, b, a, f, e, d, c, b, a, \ldots\}$.

Proof. The proof is obtained from Theorem 5.1. □

Remark 5.4. Equation (5.1) has no prime period two solution.

Example 5.5. We consider $x_{-5} = -3.1$, $x_{-4} = -1.3$, $x_{-3} = 0.2$, $x_{-2} = -9$, $x_{-1} = 7$, $x_0 = 0.4$. See Figure 5.1.

Example 5.6. See Figure 5.2 for the initial conditions $x_{-5} = -0.2$, $x_{-4} = -0.7$, $x_{-3} = 3$, $x_{-2} = 2$, $x_{-1} = 10/3$, $x_0 = 10/7$.

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Figure 4.2: Solution of the Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{-1 + x_{n-1}x_{n-3}x_{n-5}}$

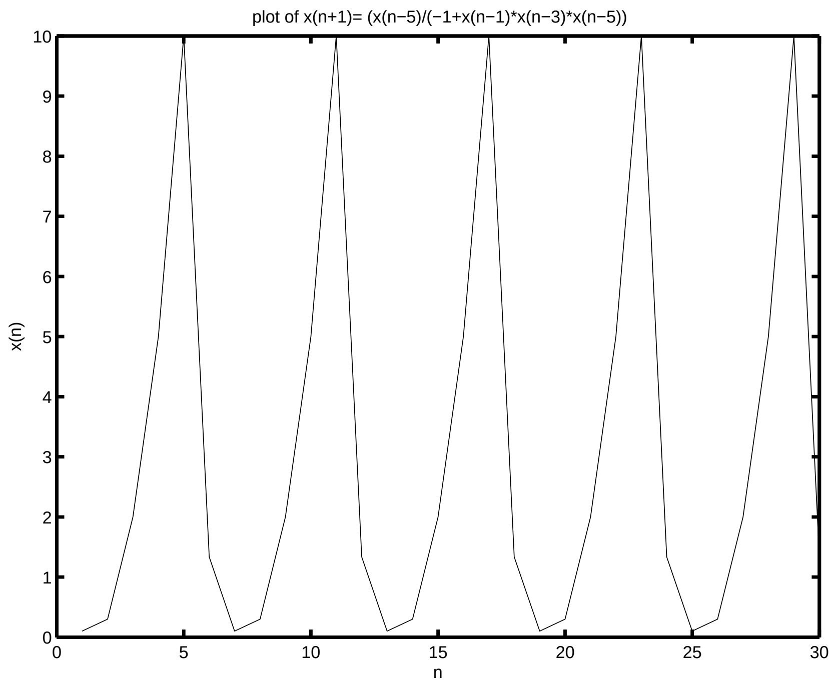

Figure 5.1: Solution of the Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{-1 - x_{n-1}x_{n-3}x_{n-5}}$

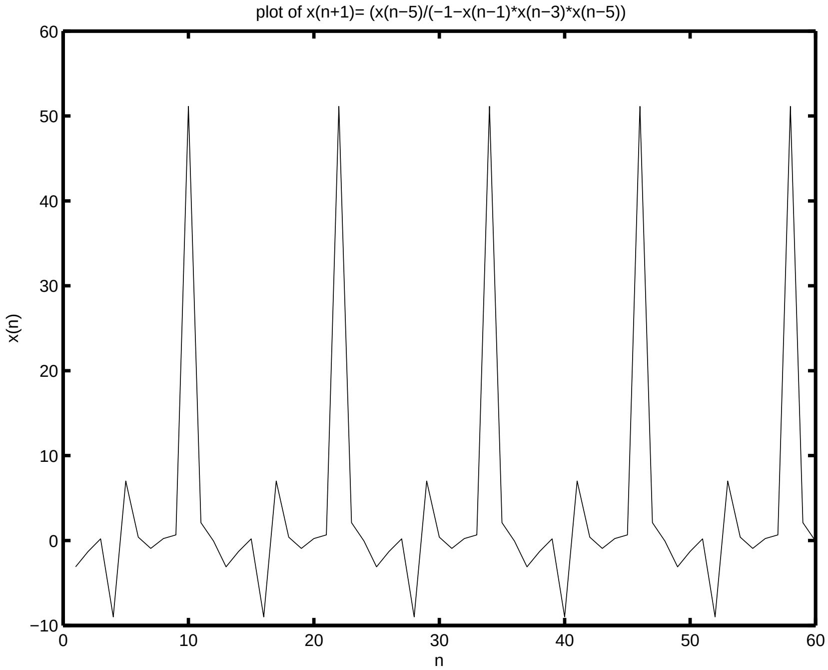

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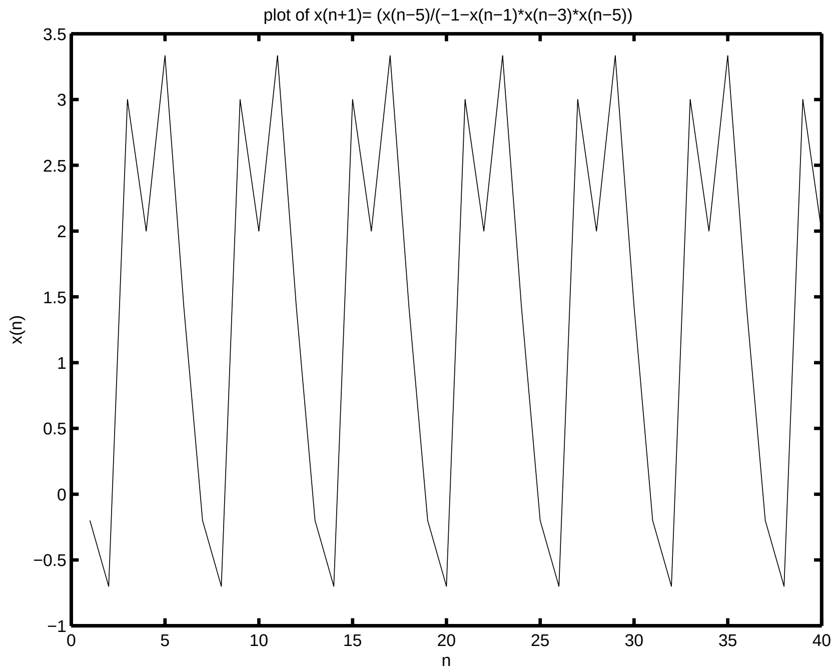
Figure 5.2: Solution of the Difference Equation $x_{n+1} = \dfrac{x_{n-5}}{-1 - x_{n-1}x_{n-3}x_{n-5}}$



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