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Existence of Multiple Positive Solutions to Three-Point Boundary Value Problems on Time Scales

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Abstract

We consider the three-point even order boundary value problem on time scales,

$$(-1)^n y^{\Delta^{(2n)}}(t) = f(t, y(t)), \;\; t \in [a, c],$$

$$\alpha_{i+1} y^{\Delta^{(2i)}}(b) + \beta_{i+1} y^{\Delta^{(2i+1)}}(a) = y^{\Delta^{(2i)}}(a),$$

$$\gamma_{i+1} y^{\Delta^{(2i)}}(b) = y^{\Delta^{(2i)}}(\sigma(c)), \quad 0 \le i \le n-1,$$

where $n \geq 1$, $a < b < \sigma(c)$, σ(c) is right-dense and $f : [a, \sigma(c)] \times \mathbb{R} \to \mathbb{R}$ is continuous. First, we establish the existence of at least three positive solutions by using the well-known Leggett–Williams fixed point theorem. We also establish the existence of at least $2m - 1$ positive solutions for arbitrary positive integer m.

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1 Introduction

A time scale $\mathbb{T}$ is any nonempty closed subset of $\mathbb{R}$. Hilger [15] initially introduced time scales with the twin goals of unifying the continuous and discrete calculus and extending the results to a dynamic calculus for general time scales. Some other earlier papers in this area include Agarwal and Bohner [1], Anderson [3], Avery and Anderson [5], Erbe and Peterson [12]. For an excellent introduction to the overall area of dynamic equations on time scales, we refer to the recent text books by Bohner and Peterson [7, 8]. In this paper, we establish the existence of multiple positive solutions to even order three-point boundary value problem on time scales,

$$(-1)^n y^{\Delta^{(2n)}}(t) = f(t, y(t)), \ \ t \in [a, c], \tag{1.1}$$

$$\alpha_{i+1} y^{\Delta^{(2i)}}(b) + \beta_{i+1} y^{\Delta^{(2i+1)}}(a) = y^{\Delta^{(2i)}}(a), \tag{1.2}$$

$$\gamma_{i+1} y^{\Delta^{(2i)}}(b) = y^{\Delta^{(2i)}}(\sigma(c)), \quad 0 \le i \le n-1, \tag{1.3}$$

where $n \ge 1$, $a < b < \sigma(c)$, σ(c) is right-dense and $f : [a, \sigma(c)] \times \mathbb{R} \to \mathbb{R}$ is continuous and we assume that the coefficients $\alpha_i, \beta_i, \gamma_i$ are real and satisfy the following condition, called condition (A):

$$0 \le \alpha_i < \frac{\sigma(c) - \gamma_i b + (\gamma_i - 1)(a - \beta_i)}{\sigma(c) - b}, \quad \beta_i \ge 0,$$

$$0 < \gamma_i < \frac{\sigma(c) - a + \beta_i}{b - a + \beta_i} \quad \text{for each} \quad 1 \le i \le n.$$

The study of the existence of positive solutions of the even order boundary value problems (BVPs) arises in a variety of different areas of applied mathematics and physics. In the modeling of nonlinear diffusion via nonlinear sources, thermal ignition of gases, and in chemical concentrates in biological problem [13]. In these applied settings, only positive solutions are meaningful. The existence of positive solutions are studied by many authors. To mention a few, we list some papers, Eloe and Henderson [9–11], Erbe and Wang [13] for at least one positive solution and then Anderson [2], Anderson and Avery [4], Avery and Peterson [6], Henderson and Kaufmann [14] for multiple positive solutions.

This paper is organized as follows. In Section 2, we state some preliminaries on time scales. In Section 3, we state and prove some lemmas which are needed in our main results. In Section 4, we establish the existence of at least three positive solutions of the BVP (1.1)–(1.3) by using the Leggett–Williams fixed point theorem. In Section 5, we establish the existence of at least $2m-1$ positive solutions of the BVP (1.1)–(1.3) for arbitrary positive integer m.

2 Preliminaries

By an interval we mean the intersection of the real interval with a given time scale. The time scale $\mathbb{T}$ may be connected or disconnected. To overcome this topological difficulty, the concept of jump operators is introduced in the following way. The operators σ and ρ from $\mathbb{T}$ to $\mathbb{T}$, defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ are called jump operators. If σ is bounded above and ρ is bounded below, then we define $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ and $\rho(\min \mathbb{T}) = \min \mathbb{T}$. These operators allow us to classify the points of time scale $\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, left-dense if $\rho(t) = t$, right-scattered if $\sigma(t) > t$, left-scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$ and dense if $\rho(t) = t = \sigma(t)$. The set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty. \end{cases}$$

Finally, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Definition 2.1. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there exists a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \ \leq \ \epsilon|\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the delta (or Hilger) *derivative* of f at t.

If f is delta differentiable for every $t \in \mathbb{T}^\kappa$, then we say that $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$. If f and g are two delta differentiable functions at t, then fg is delta differentiable at t and $(fg)^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t)$.

Definition 2.2. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.3. Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F which is pre-differentiable with region of differentiation D such that $F^\Delta(t) = f(t)$ holds for all $t \in D$ is called a *pre-antiderivative* of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f.

Definition 2.4. Let β be a real Banach space. A nonempty closed convex set κ is called a *cone* of β if it satisfies the following conditions:

(1). $u \in \beta$, $\sigma \geq 0$, implies $\sigma u \in \kappa$,

(2). $u \in \kappa$, $-u \in \kappa$ implies $u = 0$.

Definition 2.5. Let X and Y be Banach spaces and $T : X \to Y$. T is said to be *completely continuous*, if T is continuous, and for each bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a convergent subsequence.

3 Green's Function and Bounds

To state and prove the main results of this paper, we need the following lemmas. Let $G_i(t, s)$ be Green's function for the boundary value problems,

$$-y^{\Delta^2}(t) = 0, \quad t \in [a, c], \tag{3.1}$$

$$\alpha_i y(b) + \beta_i y^{\Delta}(a) = y(a), \tag{3.2}$$

$$\gamma_i y(b) = y(\sigma(c)), \tag{3.3}$$

for $1 \leq i \leq n$. First, we need a few results on the related second order homogeneous boundary value problem (3.1)–(3.3).

Lemma 3.1. *For $1 \leq i \leq n$, let $d_i = (\gamma_i - 1)(a - \beta_i) + (1 - \alpha_i)\sigma(c) + b(\alpha_i - \gamma_i)$. The homogeneous boundary value problem* (3.1)–(3.3) *has only the trivial solution if and only if $d_i \neq 0$.*

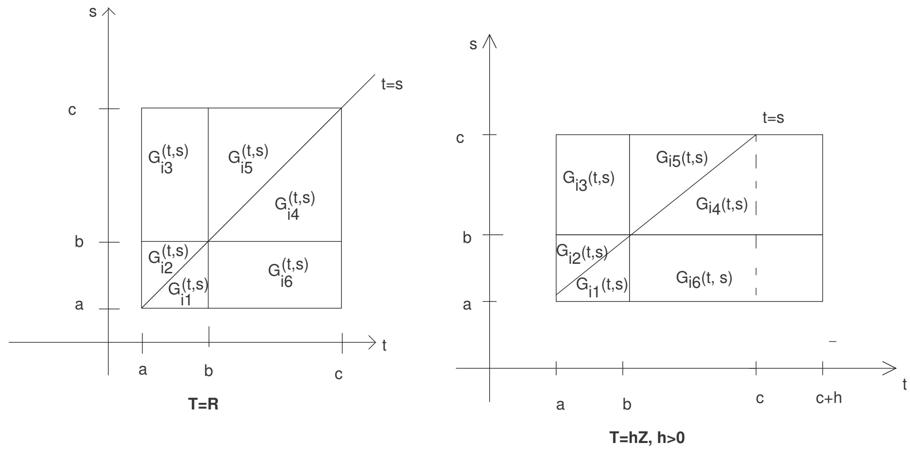
Lemma 3.2. *For $1 \leq i \leq n$, Green's function $G_i(t, s)$ for the homogeneous boundary value problem* (3.1)–(3.3)*, is given by*

$$G_i(t,s) = \begin{cases} G_i(t,s)_{t \in [a,b]} = \dfrac{1}{d_i} \begin{cases} G_{i_1}(t,s), & a < \sigma(s) < t \leq b < \sigma(c) \\ G_{i_2}(t,s), & a \leq t < s < b < \sigma(c) \\ G_{i_3}(t,s), & a \leq t < b < s < \sigma(c) \end{cases} \\ \\ G_i(t,s)_{t \in [b,\sigma(c)]} = \dfrac{1}{d_i} \begin{cases} G_{i_4}(t,s), & a < b < \sigma(s) < t \leq \sigma(c) \\ G_{i_5}(t,s), & a < b \leq t < s < \sigma(c) \\ G_{i_6}(t,s), & a \leq \sigma(s) < b < t < \sigma(c), \end{cases} \end{cases} \tag{3.4}$$

where

$$\begin{aligned}
G_{i_1}(t,s) &= [\gamma_i(t-b) + \sigma(c) - t](\sigma(s) + \beta_i - a), \\
G_{i_2}(t,s) &= [\gamma_i(\sigma(s)-b) + \sigma(c) - \sigma(s)](t + \beta_i - a) + \alpha_i(b - \sigma(c))(t - \sigma(s)), \\
G_{i_3}(t,s) &= [t(1-\alpha_i) + \alpha_i b + \beta_i - a](\sigma(c) - \sigma(s)), \\
G_{i_4}(t,s) &= [\sigma(s)(1-\alpha_i) + \alpha_i b + \beta_i - a](\sigma(c) - t) + \gamma_i(b - a + \beta_i)(t - \sigma(s)), \\
G_{i_5}(t,s) &= [t(1-\alpha_i) + \alpha_i b + \beta_i - a](\sigma(c) - \sigma(s)), \\
G_{i_6}(t,s) &= [\gamma_i(t-b) + \sigma(c) - t](\sigma(s) + \beta_i - a).
\end{aligned}$$

Figure 3.1: Green's function



The graph in Figure 3.1 demonstrates that Green's function for (3.1)–(3.3) should be taken in the form of (3.4). Here $s \in [a, c]$.

Lemma 3.3. *Assume that the condition* (A) *is satisfied. Then, for* $1 \le i \le n$*, Green's function* $G_i(t, s)$ *of the BVP* (3.1)–(3.3) *satisfies the inequality*

$$G_i(t, s) \ge \frac{(t-a)}{(\sigma(c)-a)} G_i(\sigma(c), s), \quad (t, s) \in (a, \sigma(c)) \times (a, c).$$

Proof. Green's function $G_i(t, s)$ is given by (3.4) in six different cases, and in each case we prove the inequality.

(i) Let $\sigma(s) < t$ and fix $s \in [a, b]$. Then

$$G_i(t, s) = \frac{1}{d_i}\{[\gamma_i(t-b) + \sigma(c) - t](\sigma(s) + \beta_i - a)\}$$

and $\dfrac{G_i(t, s)}{G_i(\sigma(c), s)} = \dfrac{[\gamma_i(t-b) + \sigma(c) - t]}{\gamma_i(\sigma(c) - b)} > \dfrac{t-a}{\sigma(c)-a}$ for $0 < \gamma_i < \dfrac{\sigma(c)-a}{b-a}$. Since the inequality $\dfrac{\sigma(c)-a+\beta_i}{b-a+\beta_i} < \dfrac{\sigma(c)-a}{b-a}$ holds, we have

$$G_i(t, s) > \frac{t-a}{\sigma(c)-a} G_i(\sigma(c), s) \quad \text{for} \quad 0 < \gamma_i < \frac{\sigma(c)-a+\beta_i}{b-a+\beta_i}.$$

Now let $\sigma(s) < t$ and $s \in [b, c]$. Then

$$\begin{aligned} G_i(t, s) &= [\sigma(s)(1-\alpha_i) + \alpha_i b + \beta_i - a](\sigma(c) - t) + \gamma_i(b - a + \beta_i)(t - \sigma(s)) \\ &= G_i(\sigma(c), s) + \frac{1}{d_i}\{[(\gamma_i - 1)(a - \beta_i) + (1-\alpha_i)\sigma(s) + b(\alpha_i - \gamma_i)](\sigma(c) - t)\}. \end{aligned}$$

Since $(\gamma_i - 1)(a - \beta_i) + (1 - \alpha_i)\sigma(s) + b(\alpha_i - \gamma_i) > 0$, we get

$$\frac{t-a}{\sigma(c)-a} G_i(\sigma(c), s) < G_i(t, s).$$

(ii) Let $t \leq s$ and $s \in [a, b]$. Then

$$G_i(t,s) = \frac{1}{d_i}\{[\gamma_i(\sigma(s)-b) + \sigma(c) - \sigma(s)](t + \beta_i - a) + \alpha_i(b - \sigma(c))(t - \sigma(s))\}.$$

Using the inequalities $0 < \gamma_i < \dfrac{\sigma(c) - a + \beta_i}{b - a + \beta_i}$ and

$$\alpha_i(\sigma(s) - t)(b - a + \beta_i)(\sigma(c) - a) + \beta_i(\sigma(c) - t)(\sigma(s) - a + \beta_i) > 0,$$

we obtain

$$\begin{aligned}
\frac{G_i(t,s)}{G_i(\sigma(c),s)} &= \frac{[\gamma_i(\sigma(s)-b) + \sigma(c) - \sigma(s)](t + \beta_i - a) + \alpha_i(b - \sigma(c))(t - \sigma(s))}{\gamma_i(\sigma(c) - b)(\sigma(s) + \beta_i - a)} \\
&> \frac{(\sigma(s) + \beta_i - a)(t + \beta_i - a) + \alpha_i(\sigma(s) - t)(b - a + \beta_i)}{(\sigma(s) + \beta_i - a)(\sigma(c) + \beta_i - a)} \\
&> \frac{t-a}{\sigma(c)-a}.
\end{aligned}$$

Now let $t \leq s$ and $s \in [b, c]$. Then

$$G_i(t,s) = [t(1-\alpha_i) + \alpha_i b + \beta_i - a](\sigma(c) - \sigma(s)).$$

Since $(t-a)d_i + (\sigma(c) - t)(\alpha_i(b-a) + \beta_i) > 0$ holds, we have

$$\frac{G_i(t,s)}{G_i(\sigma(c),s)} = \frac{t(1-\alpha_i) + \alpha_i b + \beta_i - a}{\gamma_i(b - a + \beta_i)} > \frac{t-a}{\sigma(c)-a}.$$

This completes the proof. □

Lemma 3.4. *Assume that the condition* (A) *is satisfied. Then, for* $1 \leq i \leq n$, *Green's function* $G_i(t,s)$ *given by* (3.4) *possesses the property*

$$G_i(t,s) > 0, \quad (t,s) \in (a, \sigma(c)) \times (a, c).$$

Proof. By Lemma 3.3, it suffices to show that $G_i(\sigma(c), s) > 0$ for $s \in (a, c)$. For $s \in (a, b]$, $G_i(\sigma(c), s) = \dfrac{1}{d_i}\gamma_i(\sigma(c) - b)(\sigma(s) + \beta_i - a) > 0$, and for $s \in [b, c)$, $G_i(\sigma(c), s) = \dfrac{1}{d_i}\gamma_i(b - a + \beta_i)(\sigma(c) - \sigma(s)) > 0$. □

Lemma 3.5. *Assume that the condition* (A) *is satisfied. Then, for* $1 \leq i \leq n$*, Green's function* $G_i(t,s)$ *given by* (3.4) *satisfies*

$$G_i(t,s) \leq \max\left\{G_i(a,s), G_i(\sigma(s),s), \frac{1}{d_i}(b-a+\beta_i)(\sigma(c)-\sigma(s))\right\},$$

$0 < \gamma_i \leq 1,\ (t,s) \in [a,\sigma(c)] \times [a,c]$ *and*

$$G_i(t,s) \leq \max\left\{G_i(\sigma(c),s), G_i(\sigma(s),s)\right\},\ (t,s) \in [a,\sigma(c)] \times [a,c],$$

$1 < \gamma_i < \dfrac{\sigma(c)-a+\beta_i}{b-a+\beta_i}$.

Proof. We prove the inequality in each case of Green's function.

(i) Let $\sigma(s) < t \leq b$ and $s \in [a,b]$. Here $G_i(t,s)$ is nonincreasing in t if $0 < \gamma_i \leq 1$, so that $G_i(t,s) \leq G_i(\sigma(s),s)$. If $1 < \gamma_i < \dfrac{\sigma(c)-a+\beta_i}{b-a+\beta_i}$, however, the function is nondecreasing in t and $G_i(t,s) \leq G_i(\sigma(c),s)$. Now let $\sigma(s) < t \leq \sigma(c)$ and fix $s \in [b,c]$. Here $G_i(t,s)$ is nonincreasing in t if $0 < \gamma_i \leq 1$, so that $G_i(t,s) \leq G_i(\sigma(s),s)$. Let $\gamma_i \in \left(1, \dfrac{\sigma(c)-a+\beta_i}{b-a+\beta_i}\right)$. So $\alpha_i < 1$. If $s \in \left[b, \dfrac{\gamma_i(b-a+\beta_i)-\alpha_i b-\beta_i+a}{1-\alpha_i}\right)$, then $G_i(t,s)$ is nondecreasing in t and $G_i(t,s) \leq G_i(\sigma(c),s)$. In the other case, if $s \in \left(\dfrac{\gamma_i(b-a+\beta_i)-\alpha_i b-\beta_i+a}{1-\alpha_i}, c\right]$, then $G_i(t,s)$ is nonincreasing in t and we have $G_i(t,s) \leq G_i(\sigma(s),s)$.

(ii) Let $a \leq t \leq s$ and fix $s \in [a,b]$. Then $G_i(t,s)$ is increasing in t for all $t \in [a,s]$, for any $\gamma_i \in \left(0, \dfrac{\sigma(c)-a+\beta_i}{b-a+\beta_i}\right)$. Therefore $G_i(t,s) \leq G_i(\sigma(s),s)$. Now let $a \leq t < s \leq \sigma(c)$ and fix $s \in [b,c]$. Let $\gamma_i \in (0,1]$. If $\alpha_i \in (0,1)$, then $G_i(t,s)$ is nondecreasing in t and $G_i(t,s) \leq G_i(\sigma(s),s)$. For $\alpha_i > 1$, $G_i(t,s)$ is nonincreasing in t and $G_i(t,s) \leq G_i(a,s)$. If $\alpha_i = 1$, then $G_i(t,s)$ is constant in t and $G_i(t,s) = \dfrac{1}{d_i}(b-a+\beta_i)(\sigma(c)-\sigma(s))$. If $1 < \gamma_i < \dfrac{\sigma(c)-a+\beta_i}{b-a+\beta_i}$, then we get $\alpha_i < 1$. Thus $G_i(t,s)$ is nondecreasing in t, so that $G_i(t,s) \leq G_i(\sigma(s),s)$. $\square$

Lemma 3.6. *Assume that the condition* (A) *holds. For fixed* $s \in [a,c]$*, and* $1 \leq i \leq n$*, Green's function* $G_i(t,s)$ *in* (3.4) *satisfies*

$$\min_{t \in [b,\sigma(c)]} G_i(t,s) \geq m_i \parallel G_i(\cdot,s) \parallel, \tag{3.5}$$

where

$$m_i = \min\left\{\begin{array}{l} \dfrac{\gamma_i(\sigma(c)-b)}{\sigma(c)-a+\gamma_i(a-b)}, \dfrac{\gamma_i(b-a+\beta_i)}{\sigma(c)(1-\alpha_i)+\alpha_i b+\beta_i-a}, \\ \dfrac{\gamma_i(b-a+\beta_i)}{\alpha_i(b-a)+\beta_i}, \dfrac{\gamma_i}{b-a+\beta_i}, \dfrac{b-a+\beta_i}{\sigma(c)-a+\beta_i} \end{array}\right\},$$

and $\parallel . \parallel$ *is defined by* $\parallel x \parallel = \max\{\mid x(t) \mid : t \in [a,\sigma(c)]\}$.

Proof. First consider the case $0 < \gamma_i \leq 1$. From Lemma 3.5,

$$\| G_i(\cdot, s) \| = \max\left\{ G_i(a,s), G_i(\sigma(s),s), \frac{1}{d_i}(b-a+\beta_i)(\sigma(c)-\sigma(s)) \right\}.$$

By using the boundary condition (3.3), we get $G_i(b,s) \geq G_i(\sigma(c),s)$, so that

$$\min_{t\in[b,\sigma(c)]} G_i(t,s) = G_i(\sigma(c),s).$$

For $s \in [a,b]$ we have from cases in (3.4) that

$$G_i(\sigma(c),s) \geq \frac{\gamma_i(\sigma(c)-b)}{\sigma(c)-a+\gamma_i(a-b)} G_i(\sigma(s),s).$$

Let $s \in [b,c]$. If $\alpha_i < 1$, then the inequality

$$G_i(\sigma(c),s) \geq \frac{\gamma_i(b-a+\beta_i)}{\sigma(c)(1-\alpha_i)+\alpha_i b+\beta_i - a} G_i(\sigma(s),s)$$

holds. If $\alpha_i > 1$, then we have

$$G_i(\sigma(c),s) = \frac{\gamma_i(b-a+\beta_i)}{\alpha_i(b-a)+\beta_i} G_i(a,s).$$

If $\alpha_i = 1$, then we get

$$G_i(\sigma(c),s) \geq \frac{\gamma_i}{(b-a+\beta_i)} \frac{1}{d_i}(b-a+\beta_i)(\sigma(c)-\sigma(s)).$$

Next consider the case when $1 < \gamma_i < \dfrac{\sigma(c)-a+\beta_i}{b-a+\beta_i}$. By the boundary condition (3.3), we have

$$\min_{t\in[b,\sigma(c)]} G_i(t,s) = G_i(b,s).$$

Using Lemma 3.5, we have

$$\| G_i(\cdot, s) \| = \max\{G_i(\sigma(c),s), G_i(\sigma(s),s)\}.$$

By using (3.4) and the cases in the proof of Lemma 3.5, we see that

$$G_i(b,s) \geq \frac{(b-a+\beta_i)}{(\sigma(c)-a+\beta_i)} G_i(\sigma(c),s) \quad \text{for} \quad s \in \left[a, \frac{\gamma_i(b-a+\beta_i)-\alpha_i b-\beta_i+a}{1-\alpha_i}\right)$$

and

$$G_i(b,s) \geq \frac{(b-a+\beta_i)}{\sigma(c)(1-\alpha_i)+\alpha_i b+\beta_i-a} G_i(\sigma(s),s)$$
$$\text{for} \quad s \in \left[\frac{\gamma_i(b-a+\beta_i)-\alpha_i b-\beta_i+a}{1-\alpha_i}, c\right].$$

This completes the proof. □

Lemma 3.7. *Assume that the condition* (A) *is satisfied, and let* $G_i(t,s)$ *be as in* (3.4). *Let us define* $H_1(t,s) = G_1(t,s)$*, and recursively define*

$$H_j(t,s) = \int_a^{\sigma(c)} H_{j-1}(t,r)G_j(r,s)\Delta r \tag{3.6}$$

for $2 \le j \le n$*. Then* $H_n(t,s)$ *is Green's function for the corresponding homogeneous problem* (1.1)–(1.3).

Lemma 3.8. *Assume that the condition* (A) *holds. If we define*

$$K = \prod_{j=1}^{n-1} K_j, \quad L = \prod_{j=1}^{n-1} m_j L_j,$$

then Green's function $H_n(t,s)$ *in Lemma 3.7 satisfies*

$$0 \le H_n(t,s) \le K \parallel G_n(\cdot,s) \parallel, \quad (t,s) \in [a,\sigma(c)] \times [a,c]$$

and

$$H_n(t,s) \ge m_n L \parallel G_n(\cdot,s) \parallel, \quad (t,s) \in [b,\sigma(c)] \times [a,c],$$

where m_n *is given in Lemma 3.6,*

$$K_j = \int_a^{\sigma(c)} \parallel G_j(\cdot,s) \parallel \Delta s > 0, \ 1 \le j \le n,$$

and

$$L_j = \int_b^{\sigma(c)} \parallel G_j(\cdot,s) \parallel \Delta s > 0, \ 1 \le j \le n.$$

Proof. We use induction on n. First, for $n = 1$, from Lemma 3.5, the conclusion holds. Next, we assume that this conclusion holds for $n = k$. In order to prove that this conclusion holds for $n = k + 1$, we use Lemma 3.6 and Lemma 3.7. □

4 Existence of at Least Three Positive Solutions

In this section, we establish the existence of at least three positive solutions for the even order three point boundary value problem (1.1)–(1.3), by using the Leggett–Williams fixed point theorem.

Let E be a real Banach space with cone P. A map $S : P \to [0,\infty)$ is said to be a nonnegative continuous concave functional on P, if S is continuous and

$$S(\lambda x + (1-\lambda)y) \ge \lambda S(x) + (1-\lambda)S(y)$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Let α and β be two numbers such that $0 < \alpha < \beta$, and let S be a nonnegative continuous concave functional on P. We define the convex sets

$$P_\alpha = \{y \in P : \| y \| < \alpha\}$$

and

$$P(S, \alpha, \beta) = \{y \in P : \alpha \leq S(y), \| y \| \leq \beta\}$$

Theorem 4.1 (Leggett–Williams fixed point theorem). *Let $T : \overline{P}_{a_3} \to \overline{P}_{a_3}$ be completely continuous and S be a nonnegative continuous concave functional on P such that $S(y) \leq \| y \|$ for all $y \in \overline{P}_{a_3}$. Suppose that there exist $0 < d < a_1 < a_2 \leq a_3$ such that*

(i) $\{y \in P(S, a_1, a_2) : S(y) > a_1\} \neq \emptyset$ *and* $S(Ty) > a_1$ *for* $y \in P(S, a_1, a_2)$*;*

(ii) $\| Ty \| < d$ *for* $\| y \| \leq d$*;*

(iii) $S(Ty) > a_1$ *for* $y \in P(S, a_1, a_3)$ *with* $\| Ty \| > a_2$*.*

Then T has at least three fixed points y_1, y_2, y_3 in $\overline{P}_{a_3}$ satisfying

$$\| y_1 \| < d, \quad a_1 < S(y_2), \quad \| y_3 \| > d, \quad S(y_3) < a_1.$$

Theorem 4.2. *Assume that there exist numbers a_0, a_1, and a_2 with $0 < a_0 < a_1 < \dfrac{a_1}{M} < a_2$ such that*

$$f(t, y(t)) < \frac{a_0}{\prod_{j=1}^{n} K_j} \text{ for } t \in [a, \sigma(c)] \text{ and } y \in [0, a_0], \tag{4.1}$$

$$f(t, y(t)) > \frac{a_1}{m_n \prod_{j=1}^{n} L_j} \text{ for } t \in [b, \sigma(c)] \text{ and } y \in \left[a_1, \frac{a_1}{M}\right], \tag{4.2}$$

$$f(t, y(t)) < \frac{a_2}{\prod_{j=1}^{n} K_j} \text{ for } t \in [a, \sigma(c)] \text{ and } y \in [0, a_2]. \tag{4.3}$$

Then the BVP (1.1)–(1.3) *has at least three positive solutions, where*

$$M = m_n \prod_{j=1}^{n-1} \frac{m_j L_j}{K_j}.$$

Proof. Let the Banach space $E = C[a, \sigma(c)]$ be equipped with the norm

$$\| y \| = \max_{t \in [a, \sigma(c)]} | y(t) | .$$

We denote

$$P = \{y \in E : y(t) \geq 0, \ t \in [a, \sigma(c)]\}.$$

Then, it is obvious that P is a cone in E. For $y \in P$, we define

$$S(y) = \min_{t \in [b, \sigma(c)]} \mid y(t) \mid$$

and

$$(Ty)(t) = \int_a^{\sigma(c)} H_n(t,s) f(s, y(s)) \Delta s, \ \ t \in [a, \sigma(c)].$$

It is easy to check that S is a nonnegative continuous concave functional on P with $S(y) \leq \parallel y \parallel$ for $y \in P$ and that $T : P \to P$ is completely continuous, and fixed points of T are solutions of the BVP (1.1)–(1.3). First, we prove that, if there exists a positive number r such that $f(t, y(t)) < \dfrac{r}{\prod_{j=1}^n K_j}$ for $t \in [a, \sigma(c)]$ and $y \in [0, r]$, then $T : \overline{P}_r \to P_r$. Indeed, if $y \in \overline{P}_r$, then for $t \in [a, \sigma(c)]$,

$$\begin{aligned}
(Ty)(t) &= \int_a^{\sigma(c)} H_n(t,s) f(s, y(s)) \Delta s \\
&< \frac{r}{\prod_{j=1}^n K_j} \int_a^{\sigma(c)} H_n(t,s) \Delta s \\
&\leq \frac{r}{\prod_{j=1}^n K_j} K \int_a^{\sigma(c)} \parallel G_n(\cdot, s) \parallel \Delta s = r.
\end{aligned}$$

Thus, $\parallel Ty \parallel < r$, that is, $Ty \in P_r$. Hence, we have shown that if (4.1) and (4.3) hold, then T maps $\overline{P}_{a_0}$ into P_{a_0} and $\overline{P}_{a_2}$ into P_{a_2}. Next, we show that

$$\left\{ y \in P\left(S, a_1, \frac{a_1}{M}\right) : S(y) > a_1 \right\} \neq \emptyset$$

and $S(Ty) > a_1$ for all $y \in P\left(S, a_1, \dfrac{a_1}{M}\right)$. In fact, the constant function

$$\frac{a_1 + a_1/M}{2} \in \left\{ y \in P\left(S, a_1, \frac{a_1}{M}\right) : S(y) > a_1 \right\}.$$

Moreover, for $y \in P\left(S, a_1, \dfrac{a_1}{M}\right)$, we have

$$\frac{a_1}{M} \geq \parallel y \parallel \geq y(t) \geq \min_{t \in [b, \sigma(c)]} y(t) = S(y) \geq a_1$$

for all $t \in [b, \sigma(c)]$. Thus, in view of (4.2) we see that

$$\begin{aligned}
S(Ty) &= \min_{t \in [b, \sigma(c)]} \int_a^{\sigma(c)} H_n(t,s) f(s, y(s)) \Delta s \\
&\geq \min_{t \in [b, \sigma(c)]} \int_b^{\sigma(c)} H_n(t,s) f(s, y(s)) \Delta s \\
&> \frac{a_1}{m_n \prod_{j=1}^n L_j} m_n L \int_b^{\sigma(c)} \parallel G_n(\cdot, s) \parallel \Delta s = a_1
\end{aligned}$$

as required. Finally, we show that if $y \in P(S, a_1, a_2)$ and $\| Ty \| > \dfrac{a_1}{M}$, then $S(Ty) > a_1$. To see this, we suppose that $y \in P(S, a_1, a_2)$ and $\| Ty \| > \dfrac{a_1}{M}$. Then, by Lemma 3.8, we have

$$\begin{aligned} S(Ty) &= \min_{t\in[b,\sigma(c)]} \int_a^{\sigma(c)} H_n(t,s) f(s,y(s))\Delta s \\ &\geq \min_{t\in[b,\sigma(c)]} m_n L \int_a^{\sigma(c)} \| G_n(\cdot,s) \| f(s,y(s))\Delta s \\ &\geq m_n L \int_a^{\sigma(c)} \| G_n(\cdot,s) \| f(s,y(s))\Delta s \end{aligned}$$

for all $t \in [a, \sigma(c)]$. Thus

$$S(Ty) \geq \frac{m_n L}{K} \max_{t\in[a,\sigma(c)]} \int_a^{\sigma(c)} H_n(t,s) f(s,y(s))\Delta s = \frac{m_n L}{K} \| Ty \| > \frac{m_n L}{K}\frac{a_1}{M} = a_1.$$

To sum up, all the hypotheses of Theorem 4.1 are satisfied. Hence T has at least three fixed points, that is, the BVP (1.1)–(1.3) has at least three positive solutions y_1, y_2 and y_3 such that

$$\| y_1 \| < a_0, \quad a_1 < \min_{t\in[b,\sigma(c)]} y_2(t), \quad \| y_3 \| > a_0, \quad \min_{t\in[b,\sigma(c)]} y_3(t) < a_1.$$

This completes the proof. □

5 Existence of Multiple Positive Solutions

In this section, we establish the existence of at least $2m-1$ positive solutions for the BVP (1.1)–(1.3), by using induction on m.

Theorem 5.1. *Let m be an arbitrary positive integer. Assume that there exist numbers a_i, $1 \leq i \leq m$, and b_j, $1 \leq j \leq m-1$, with*

$$0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < \ldots < a_{m-1} < b_{m-1} < \frac{b_{m-1}}{M} < a_m$$

such that

$$f(t,y(t)) < \frac{a_i}{\prod_{j=1}^n K_j} \text{ for } t \in [a,\sigma(c)] \text{ and } y \in [0,a_i],\ 1 \leq i \leq m \tag{5.1}$$

and

$$f(t,y(t)) > \frac{b_j}{m_n \prod_{j=1}^n L_j} \text{ for } t \in [b,\sigma(c)] \text{ and } y \in \left[b_j, \frac{b_j}{M}\right],\ 1 \leq j \leq m-1. \tag{5.2}$$

Then the BVP (1.1)–(1.3) has at least $2m-1$ positive solutions in $\overline{P}_{a_m}$.

Proof. We use induction on m. First, for $m = 1$, we know from (5.1) that $T : \overline{P}_{a_1} \to P_{a_1}$. Then it follows from Schauder's fixed point theorem that the BVP (1.1)–(1.3) has at least one positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $m = k$. In order to prove that this conclusion holds for $m = k + 1$, we suppose that there exist numbers $a_i (1 \leq i \leq k + 1)$ and $b_j (1 \leq j \leq k)$ with

$$0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < \ldots < a_k < b_k < \frac{b_k}{M} < a_{k+1}$$

such that

$$f(t, y(t)) < \frac{a_i}{\prod_{j=1}^{n} K_j} \text{ for } t \in [a, \sigma(c)] \text{ and } y \in [0, a_i],\ 1 \leq i \leq k + 1 \tag{5.3}$$

and

$$f(t, y(t)) > \frac{b_j}{m_n \prod_{j=1}^{n} L_j} \text{ for } t \in [b, \sigma(c)] \text{ and } y \in \left[b_j, \frac{b_j}{M}\right],\ 1 \leq j \leq k. \tag{5.4}$$

By assumption, the BVP (1.1)–(1.3) has at least $2k - 1$ positive solutions u_i, $i = 1, 2, \ldots, 2k - 1$, in $\overline{P}_{a_k}$. At the same time, it follows from Theorem 4.2, (5.3) and (5.4) that the BVP (1.1)–(1.3) has at least three positive solutions u, v and w in $\overline{P}_{a_{k+1}}$ such that $\| u \| < a_k$, $b_k < \min_{t \in [b, \sigma(c)]} v(t)$, $\| w \| > a_k$, $\min_{t \in [b, \sigma(c)]} w(t) < b_k$. Obviously, v and w are different from u_i, $i = 1, 2, \ldots, 2k - 1$. Therefore, the BVP(1.1)–(1.3) has at least $2k + 1$ positive solutions in $\overline{P}_{a_{k+1}}$ which shows that this conclusion also holds for $m = k + 1$. □

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