International Journal of Difference Equations
ISSN 0973-6069, Volume 4, Number 2, pp. 233–251 (2009)
http://campus.mst.edu/ijde

Existence and Uniqueness for Nonlinear Discrete Sturm–Liouville Problems

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Abstract

In this paper we study the existence and uniqueness of Sturm–Liouville discrete problem. We concentrate on purely discrete approaches based on matrix formulations of nonlinear problems. Our main tools include degree theory and variational methods. Our results involve higher-order problems, noninvertible left-hand side difference operators and discontinuous right-hand sides.

AMS Subject Classifications: 39A12, 34B15.
Keywords: Boundary value problem, difference equations, variational methods, degree theory, coincidence degree.

1 Introduction

The development in numerical analysis has propelled interest in difference equations and their relationship to their differential counterparts. The theory of discrete nonlinear boundary value problems has often been connected (e.g., Gaines [5]) to the study of corresponding topics in differential equations and investigation of differences between the two approaches. This spirit remains in the recent publications (see e.g., Mawhin, Thompson, Tonkes [9] or Bereanu, Mawhin [2]). Furthermore, there is a new emerging tool which supports research in this direction and unifies discrete and continuous approach in a simple way, the time scales calculus (the seminal works dates back to Hilger [6]). Consequently, there have emerged interesting contributions which encompass both continuous and discrete versions of nonlinear problems (see e.g., their survey Bohner, Peterson [3]).

Received September 8, 2008; Accepted June 24, 2009
Communicated by Christopher Tisdell

This paper can be seen as a part of this research stream. Our main contribution is in the fact that we use a pure discrete technique with no continuous counterpart. We investigate a class of nonlinear Sturm–Liouville discrete problems, transform it into matrix equations and use degree theory and variational methods to obtain existence and uniqueness theorems. Using the properties of discrete problems (matrix structure, finite dimension, etc.) we obtain existence and uniqueness results which improve existing results (see e.g., their collection Agarwal [1]) by considering discontinuous right-hand side functions.

Our problem of interest is the nonlinear discrete Sturm–Liouville problem

$$\begin{cases} -\Delta\left(p(t-1)\Delta x(t-1)\right) + q(t)x(t) = f(t,x(t)), \quad t = a, a+1, \dots, b, \\ \alpha x(a-1) - \beta \Delta x(a-1) = C, \quad \gamma x(b+1) + \delta \Delta x(b) = D, \end{cases} \tag{1.1}$$

where $a, b \in \mathbb{Z}$, $\alpha, \beta, \gamma, \delta, C, D \in \mathbb{R}$ and p, q satisfy

$$p(t) > 0 \text{ and } q(t) \geq 0, \tag{pq}$$

for $t = a, a+1, \dots, b$, and its two extensions. First, we consider the problem (1.1) with nonconstant steps. Second, we extend (1.1) into the $2n$-th order problem. A similar approach has been used to study periodic problems in Stehlík [12].

The paper is organized as follows. First, in Section 2 we extend (1.1) to the heterogeneous case. We use the time scale notation to show that the time scale calculus provides an important by-product, the language for problems with nonconstant steps. Second, in Section 3 we formulate the matrix version of the boundary value problem and we investigate its properties. Next, we apply fixed point theory (Section 4), degree theory (Section 5), and variational methods (Section 6) to obtain the existence and uniqueness results for (2.2). Finally, in Section 7 we extend these results for $2n$-th order boundary value problems.

2 Nonconstant Steps

In order to include the problem with nonconstant steps and to simplify complicated expressions, we use the notation of the time scales calculus (see Bohner, Peterson [3] for more details). Namely, throughout this paper $\mathbb{T}$ denotes a set of discrete points, $[a, b]_{\mathbb{T}}$ its subset lying between a and b, $\sigma(t)$ maps the point t to its successor and $\mu(t) = \sigma(t) - t$ measures the gap between the point and its successor. The predecessor of t is denoted by $\rho(t)$. The generalized difference has the form

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\mu(t)}. \tag{2.1}$$

Using these notions, we study the generalization of the problem (1.1) with nonconstant steps, i.e.,

$$\begin{cases} -\left(p(\rho(t))x^{\Delta}(\rho(t))\right)^{\Delta} + q(t)x(t) = f(t,x(t)), \quad \text{on } [a,b]_{\mathbb{T}} \\ \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = C, \quad \gamma x(\sigma(b)) + \delta x^{\Delta}(b) = D. \end{cases} \tag{2.2}$$

Remark 2.1. For the sake of lucidity, we introduce the following notations. We suppose that there are N points in interval $[a,b]_{\mathbb{T}}$. We denote the value of the function x in the following way.

$$[x_0, x_1, \dots, x_{N-1}, x_N, x_{N+1}] := [x(\rho(a)), x(a), x(\sigma(a)), \dots, x(\rho(b)), x(b), x(\sigma(b))]\,.$$

In the same way, we can write e.g., $p_1 = p(a)$, $q_2 = q(\sigma(a))$ or $f_N(x_N) = f(b, x(b))$. This notation complies with our intention to transform the problem to the nonlinear problem in $\mathbb{R}^N$, i.e., the solution is going to be a vector in $\mathbb{R}^N$.

3 Matrix Formulation

For the sake of simplicity, let us start with the problem (2.2) in the case when $q \equiv 0$, i.e., we consider the problem

$$\begin{cases} -\big(p(\rho(t))x^{\Delta}(\rho(t))\big)^{\Delta} = f(t, x(t)), \quad \text{on } [a,b]_{\mathbb{T}} \\ \alpha x(\rho(a)) - \beta x^{\Delta}(\rho(a)) = C, \quad \gamma x(\sigma(b)) + \delta x^{\Delta}(b) = D \end{cases} \tag{3.1}$$

Using (2.1), we can rewrite the problem (3.1) as a system of nonlinear equations in $\mathbb{R}^{N+2}$ (first, for better understanding, let the symbol $\bullet$ represent a nonzero entry in the three-band left-hand side matrix):

$$\left[\begin{array}{ccccccc|c} \bullet & \bullet & & & & & & C \\ \bullet & \bullet & \bullet & & & & & \mu_0 f_1(x_1) \\ & \bullet & \bullet & \bullet & & & & \mu_1 f_2(x_2) \\ & & \ddots & \ddots & \ddots & & & \vdots \\ & & & \ddots & \ddots & \ddots & & \vdots \\ & & & & \bullet & \bullet & \bullet & \mu_{N-1} f_N(x_N) \\ & & & & & \bullet & \bullet & D \end{array}\right]. \tag{3.2}$$

If we take into account the first two lines of this system

$$\left[\begin{array}{ccccc|c} \dfrac{\alpha\mu_0 + \beta}{\mu_0} & -\dfrac{\beta}{\mu_0} & 0 & 0 & \dots & C \\ -\dfrac{p_0}{\mu_0} & \dfrac{p_0}{\mu_0} + \dfrac{p_1}{\mu_1} & -\dfrac{p_1}{\mu_1} & 0 & \dots & \mu_0 f_1(x_1) \\ & \ddots & \ddots & \ddots & & \vdots \end{array}\right],$$

and suppose that

$$\alpha\mu_0 + \beta \neq 0, \tag{$\alpha\beta$}$$

then we can multiply the first line by $\dfrac{p_0}{\alpha\mu_0+\beta}$ and add it to the second line to obtain

$$\left[\begin{array}{ccccc|c} \dfrac{\alpha\mu_0+\beta}{\mu_0} & -\dfrac{\beta}{\mu_0} & 0 & 0 & \dots & C \\ 0 & -\dfrac{p_0\beta}{\mu_0(\alpha\mu_0+\beta)}+\dfrac{p_0}{\mu_0}+\dfrac{p_1}{\mu_1} & -\dfrac{p_1}{\mu_1} & 0 & \dots & \mu_0 f_1(x_1)+\dfrac{Cp_0}{\alpha\mu_0+\beta} \\ & \ddots & \ddots & \ddots & & \vdots \end{array}\right]$$

and similarly, the last two lines

$$\left[\begin{array}{ccccc|c} & \ddots & \ddots & \ddots & & \vdots \\ \dots & 0 & -\dfrac{p_{N-1}}{\mu_{N-1}} & \dfrac{p_{N-1}}{\mu_{N-1}}+\dfrac{p_N}{\mu_N} & -\dfrac{p_N}{\mu_N} & \mu_{N-1}f_N(x_N) \\ \dots & 0 & 0 & -\dfrac{\delta}{\mu_N} & \dfrac{\gamma\mu_N+\delta}{\mu_N} & D \end{array}\right]$$

can be transformed, assuming that

$$\gamma\mu_N+\delta\neq 0, \tag{$\gamma\delta$}$$

to

$$\left[\begin{array}{ccccc|c} & \ddots & \ddots & \ddots & & \vdots \\ \dots & 0 & -\dfrac{p_{N-1}}{\mu_{N-1}} & \dfrac{p_{N-1}}{\mu_{N-1}}+\dfrac{p_N}{\mu_N}-\dfrac{p_N\delta}{\mu_N(\gamma\mu_N+\delta)} & 0 & \mu_{N-1}f_N(x_N)+D\dfrac{p_N}{\gamma\mu_N+\delta} \\ \dots & 0 & 0 & -\dfrac{\delta}{\mu_N} & \dfrac{\gamma\mu_N+\delta}{\mu_N} & D \end{array}\right].$$

These two steps enable us to reduce the problem (3.2) in $\mathbb{R}^{N+2}$ to the problem in $\mathbb{R}^N$:

$$\widehat{A}x=F(x), \tag{3.3}$$

where $\widehat{A}$ is a three-band matrix having the form

$$\widehat{A}:=\begin{bmatrix} \alpha\mu_0\dfrac{p_0}{\mu_0}+\dfrac{p_1}{\mu_1} & -\dfrac{p_1}{\mu_1} & & & \\ -\dfrac{p_1}{\mu_1} & \dfrac{p_1}{\mu_1}+\dfrac{p_2}{\mu_2} & -\dfrac{p_2}{\mu_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & -\dfrac{p_{N-2}}{\mu_{N-2}} & \dfrac{p_{N-2}}{\mu_{N-2}}+\dfrac{p_{N-1}}{\mu_{N-1}} & -\dfrac{p_{N-1}}{\mu_{N-1}} \\ & & & -\dfrac{p_{N-1}}{\mu_{N-1}} & \dfrac{p_{N-1}}{\mu_{N-1}}+\gamma\mu_N\dfrac{p_N}{\mu_N} \end{bmatrix}, \tag{3.4}$$

and $F : \mathbb{R}^N \to \mathbb{R}^N$ is a function defined by

$$F(x) = \begin{bmatrix} F_1(x_1) \\ F_2(x_2) \\ \vdots \\ F_{N-1}(x_{N-1}) \\ F_N(x_N) \end{bmatrix} := \begin{bmatrix} \mu_0 f_1(x_1) + C\dfrac{p_0}{\alpha\mu_0 + \beta} \\ \mu_1 f_2(x_2) \\ \vdots \\ \mu_{N-2} f_{N-1}(x_{N-1}) \\ \mu_{N-1} f_N(x_N) + D\dfrac{p_N}{\gamma\mu_N + \delta} \end{bmatrix}. \tag{3.5}$$

Obviously, if we find a solution of the operator equation (3.3) we have a solution of (3.2) (the values of x_0 and x_{N+1} can be determined straightforwardly from the boundary conditions). Consequently, we have a solution of the boundary value problem (3.1).

Remark 3.1. Without loss of generality we assume in the sequel that the conditions ($\alpha\beta$) and ($\gamma\delta$) hold. Note that if this is not the case, we can compute the values of x_1 or x_N from the boundary conditions and hence reduce the order of our problem.

Lemma 3.2. *Let us suppose that $\alpha \geq 0$, $\gamma \geq 0$. Then the matrix $\widehat{A}$ is positive semidefinite. Moreover, if $\alpha + \gamma > 0$, then the matrix $\widehat{A}$ is positive definite.*

Proof. First, let us take into account the case when $\alpha > 0$ or $\gamma > 0$. Then

- $\widehat{A}$ is symmetric and irreducible,
- $\widehat{a}_{ii} \geq \sum\limits_{\substack{k=1 \\ k \neq i}}^{N} |\widehat{a}_{ik}|$ for $i = 1, 2, \ldots, N$, but unequal for at least one value of i ($i = 1$ if $\alpha > 0$ or $i = N$ if $\gamma > 0$).

These two conditions are sufficient for the positive definiteness of $\widehat{A}$ (see e.g., Schwarz [11, Theorem 1-5]).

Let us suppose that $\alpha = \gamma = 0$ and let us suppose that $\lambda < 0$ is an eigenvalue of $\widehat{A}$. This implies, following the above arguments, that the matrix $\widehat{A} - \lambda I$ is positive definite and thus $\det\left(\widehat{A} - \lambda I\right) > 0$. But this is a contradiction with λ being an eigenvalue. Thus $\widehat{A}$ has only nonnegative eigenvalues and since it is symmetric, it is positive semidefinite. This completes the proof. □

Remark 3.3. Note that the case $\alpha = \gamma = 0$ corresponds to the problem with the Neumann boundary conditions. In this case, $\lambda = 0$ is a simple eigenvalue corresponding to the eigenvector $v_0 = [1, 1, \ldots, 1]^T$. In all the remaining cases the matrix $\widehat{A}$ is positive definite.

Let us return to the problem (2.2) (with nonzero q). In this case, we can repeat the above procedure and transform the problem (2.2) into the operator equation in $\mathbb{R}^N$:

$$Ax = F(x), \tag{3.6}$$

where F is defined in (3.5) and A is the $N \times N$ matrix having the form:

$$A := \widehat{A} + Q,$$

where $\widehat{A}$ is defined in (3.4) and Q is the $N \times N$ diagonal matrix:

$$Q := \text{diag}\left[\mu_0 q_1, \mu_1 q_2, \dots, \mu_{N-1} q_N\right].$$

Using Lemma 3.2 we can characterize the definiteness of the matrix A.

Lemma 3.4. *Let us suppose that $\alpha \geq 0, \gamma \geq 0$ and $q_k \geq 0$ for $k = 1, 2, \dots, N$. Then the matrix A is positive semidefinite. If, moreover one of the coefficients q_k, α or γ is positive, i.e.,*

$$\alpha + \gamma + \sum_{i=1}^{N} q_i > 0,$$

then the matrix A is positive definite.

Proof. The fact that $q_k \geq 0$ implies that the matrix Q is at least positive semidefinite. We can repeat the arguments from Lemma 3.2 to claim that the positivity of either α, γ or any q_k is sufficient for the positive definiteness of A. □

4 Application of Fixed Point Theory

If A is a positive definite matrix, then there exists its inverse A^{-1} and we can rewrite the operator equation (3.6) as

$$x = A^{-1} F(x). \tag{4.1}$$

Moreover, we denote the minimal eigenvalue of A by $\lambda_{\min}$. The maximal graininess $\mu_{\max}$ is defined by

$$\mu_{\max} := \max_{k=1,2,\dots,N} \{\mu_k\}.$$

We first use the Banach contraction principle.

Theorem 4.1. *Let us suppose that $\alpha \geq 0, \gamma \geq 0$ and $q_k \geq 0$ for $k = 1, 2, \dots, N$ and*

$$\alpha + \gamma + \sum_{i=1}^{N} q_i > 0.$$

If for each $t \in [a, b]_{\mathbb{T}}$, the functions $f(t, \cdot)$ satisfy a Lipschitz condition with a constant $K \in \mathbb{R}$ such that

$$|K| < \frac{\lambda_{\min}}{\mu_{\max}}, \tag{4.2}$$

then BVP (2.2) *has a unique solution.*

Proof. To use the contraction principle we must show that the operator $A^{-1}F$ is a contraction with a constant $\alpha < 1$. We have that

$$\|F(u) - F(v)\| = \sum_{k=1}^{N} |F_k(u_k) - F_k(v_k)| = \sum_{k=1}^{N} \mu_{k-1}|f_k(u_k) - f_k(v_k)|$$
$$\leq K\mu_{\max} \sum_{k=1}^{N} |u_k - v_k| = K\mu_{\max}\|u - v\|,$$

which implies that

$$\|A^{-1}F(u) - A^{-1}F(v)\| \leq \|A^{-1}\|\|F(u) - F(v)\| \leq \frac{|K|\mu_{\max}}{\lambda_{\min}}\|u - v\|.$$

Thus $A^{-1}F$ is a contraction mapping with $\alpha = \dfrac{|K|\mu_{\max}}{\lambda_{\min}} < 1$ and the equation (4.1) has a unique fixed point. □

The following example illustrates the necessity of the presence of $\mu_{\max}$ in (4.2).

Example 4.2. Let us take into account the following two time scales:

$$\mathbb{T}_1 = \mathbb{N} = \{1, 2, 3, 4, \ldots, 1023, 1024, 1025, \ldots\},$$
$$\mathbb{T}_2 = 2^{\mathbb{N}_0} = \{1, 2, 4, 8, \ldots, 512, 1024, 2048, \ldots\},$$

and the boundary value problem (with $\varepsilon > 0$)

$$\begin{cases} -\left(p(\rho(t))x^{\Delta}(\rho(t))\right)^{\Delta} + q(t)x(t) = \varepsilon\cos(x(t)), \quad \text{on } [2, 1024]_{\mathbb{T}_i}, \\ x(1) = C, \quad x^{\Delta}(1024) = D. \end{cases}$$

Since the function $\varepsilon\cos(x(t))$ is Lipschitz continuous with the constant ε, we obtain the uniqueness of solution if p and q are such that

$$\lambda_{\min} > \varepsilon, \quad \text{on } \mathbb{T}_1,$$
$$\lambda_{\min} > 512\varepsilon, \quad \text{on } \mathbb{T}_2.$$

However, note that $\lambda_{\min}$ depends on the time scale too (because of the presence of graininess function in $\widehat{A}$ and Q, see (3.4) and (5.5)).

The following result is a simple application of the Brouwer fixed point theorem.

Theorem 4.3. *Let us suppose that $\alpha \geq 0, \gamma \geq 0$ and $q_k \geq 0$ for $k = 1, 2, \ldots, N$ and*

$$\alpha + \gamma + \sum_{i=1}^{N} q_i > 0.$$

If $f : \{1, 2, \ldots, N\} \times \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function, then BVP (2.2) *has a solution.*

Proof. Note first, that the continuity and boundedness of $F_k(\cdot)$ is equivalent with the same properties of $f_k(\cdot)$ for all $k = 1, 2, \dots, N$. The operator $A^{-1}F$ is continuous, since F_k are continuous and A^{-1} is a regular matrix. Moreover, there exists a constant $M > 0$ such that $|F_k(u)| < M$, for each $k = 1, 2, \dots, N$ and $u \in \mathbb{R}$. If we define $R \in \mathbb{R}$ by $R = \dfrac{\sqrt{NM}}{\lambda_{\min}}$, then for each $u \in B(o, R)$ ($B(o, R)$ is a ball in $\mathbb{R}^N$) we have that

$$\|A^{-1}G(u)\| \leq \|A^{-1}\|\|F(u)\| \leq \frac{1}{\lambda_{\min}}\sqrt{\sum_{k=1}^{N} f_k(u_k)} < \frac{\sqrt{NM}}{\lambda_{\min}} = R.$$

Thus $A^{-1}G(B(o, R)) \subset B(o, R)$ and since $B(o, R)$ is a closed, nonempty, convex and bounded subset of $\mathbb{R}^N$ we have from the Brouwer fixed point theorem that there exists a fixed point $x \in \mathbb{R}^N$ of (4.1). Thus the problem (2.2) has a solution. □

5 Application of Degree Theory

The conditions from the previous section are easily verified but they can be restricting. Therefore, we use here the degree theory to enlarge the set of right-hand side functions for which our problem has a solution. In other words, the results in this section provide existence for right-hand side functions which are neither bounded nor Lipschitz continuous.

Theorem 5.1. *Let us suppose that $\alpha \geq 0, \gamma \geq 0$ and $q_k \geq 0$ for $k = 1, 2, \dots, N$ and*

$$\alpha + \gamma + \sum_{i=1}^{N} q_i > 0. \tag{5.1}$$

Moreover, let us assume that for all $k = 1, 2, \dots, N$, the functions f_k are continuous and that there exists $R > 0$ such that for each $u \in \mathbb{R}$ with $|u| \leq R$

$$\langle u, F_k(u) \rangle \leq 0. \tag{5.2}$$

Then the problem (2.2) *has a solution.*

Proof. First, the assumption implies (see Lemma 3.4) that we can, as in the previous section, look for the fixed point of the operator $T(x) = A^{-1}F(x)$. The regularity of A^{-1} and the continuity of F yield the continuity of T. We define a convex open set $U \subset \mathbb{R}^N$ by

$$U := \left\{x \in \mathbb{R}^N : |x_i| < R \text{ for all } i = 1, 2, \dots, N\right\}.$$

Let us define the homotopy $H : [0, 1] \times U \to \mathbb{R}^N$ by

$$H(t, x) := (1 - t)x + t(x - T(x)).$$

Now let us suppose that $H(t, \overline{x}) = o$ for some $t \in [0, 1]$ and $\overline{x} \in \partial U$. But multiplying the equivalent version of this equation $(1 - t)\overline{x} + t(\overline{x} - A^{-1}F(\overline{x})) = o$ by $\overline{x}^T A$ from left implies that

$$\overline{x}^T A\overline{x} - t\overline{x}^T F(\overline{x}) = \overline{x}^T A\overline{x} - t\sum_{i=1}^{N} \langle(\overline{x})_i, F_k((\overline{x})_i)\rangle \geq \overline{x}^T A\overline{x} > 0,$$

where we used Lemma 3.4 and (5.2). Consequently, by the homotopy invariance property of the Brouwer degree, we obtain that

$$\deg(T, U, o) = \deg(I, U, o) = 1.$$

Hence, T has a fixed point in U and therefore BVP (2.2) has a solution. $\square$

Remark 5.2. Note that it is essential to consider the assumption (5.2) with F_k's and not only with f_k's (see (3.5), for the difference between f_k and F_k). Whereas in the proofs of Theorem 4.1 and Theorem 4.3 the distinction did not play the role since the Lipschitz continuity and boundedness of f_k's are equivalent to these properties of F_k's (with different constants). But in this case the difference is significant since the set of conditions (5.2), which is required in the proof, is equivalent to $\langle u, f_k(u)\rangle \leq 0$ if and only if $C = 0$ and $D = 0$.

At this stage, the natural question arises. Can we extend this result also for the Neumann boundary conditions? We use here the coincidence degree by Jean Mawhin to show that this can be simply done. We rely on the following statement.

Theorem 5.3 (see [4, Theorem 5.2.16])**.** *Let A : Dom $A \subset X \to X$ be a Fredholm operator of index zero, Ω a bounded open subset of a Banach space X and let $B(I - Q)F$ be a compact operator from $\overline{\Omega}$ to X. Assume further that*

- *$Ax - \lambda F(x) \neq o$ for $x \in \partial\Omega \cap DomA, \lambda \in (0, 1)$,*
- *deg$(\Lambda QF|_{Ker\ A\cap\Omega}$, Ker $A \cap \Omega, o) \neq 0$.*

Then the equation $Ax = F(x)$ has a solution.

The operator Λ in this theorem is the homeomorphism between Ker A and ImA. Using this result we can prove the solvability for the Neumann case only for slightly stricter assumptions, i.e., strict inequality in condition (5.2).

Theorem 5.4. *Let us suppose that $\alpha = \gamma = 0$ and $q \equiv 0$. Moreover, let us assume that for all $k = 1, 2, \ldots, N$, the functions f_k are continuous and that there exists $R > 0$ such that for each $u \in \mathbb{R}$ with $|u| \leq R$*

$$\langle u, F_k(u)\rangle < 0. \tag{5.3}$$

Then the problem (2.2) *has a solution.*

Proof. Obviously, $\lambda_0 = 0$ is a simple eigenvalue of A with a corresponding eigenvector $v_0 = [1, 1, \ldots, 1]^T$, which implies that

$$\text{Ker } A = \mathbb{R}^N \setminus \text{Im } A = \left\{ v \in \mathbb{R}^N : v = tv_0, t \in \mathbb{R} \right\}. \tag{5.4}$$

The projection matrix Q which maps $\mathbb{R}^N$ to this subspace is

$$Q = \frac{1}{N} \begin{bmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix}. \tag{5.5}$$

The restriction of A to $\mathbb{R}^N \setminus \text{Ker } A$ has a bounded inverse B and thus the operator $B(I - Q)$ is compact. As in the previous theorem we define Ω as the N-dimensional cube

$$\Omega := \left\{ x \in \mathbb{R}^N : |x_i| < R \text{ for all } i = 1, 2, \ldots, N \right\}.$$

The strict inequality in (5.3) is used now to prove that $A\overline{x} \neq \lambda F(\overline{x})$ for all $\lambda \in (0, 1)$ and $\overline{x} \in \partial\Omega$. If this hadn't been the case (i.e., if the equality had hold for some $\overline{x}$ and λ), we would have arrived to the following contradiction:

$$\begin{aligned} \overline{x}^T A\overline{x} - \lambda\overline{x}^T F(\overline{x}) &= \overline{x}^T A\overline{x} - \lambda \sum_{i=1}^{N} \langle (\overline{x})_i, f_k((\overline{x})_i) \rangle \\ &\geq -\lambda \sum_{i=1}^{N} \langle (\overline{x})_i, f_k((\overline{x})_i) \rangle > 0, \end{aligned}$$

which verifies the first assumption of Theorem 5.3.

To finish the proof it suffices to show that the second condition there is satisfied as well. From (5.4) we have that $\Lambda = I$ and that

$$\text{Ker } A \cap \Omega = \left\{ v \in \mathbb{R}^N : v = tv_0, t \in (-R, R) \right\}.$$

Let us define a homotopy $H : [0, 1] \times \mathbb{R}^N \to \mathbb{R}^N$ by

$$H(t, x) = [(1 - t)(-x)] + tQF(x).$$

From (5.3) we have that both terms in this expression must have the same signs and thus $H(t, x) \neq 0$ for all $t \in (0, 1)$ and $x \in \partial\left(\text{Ker } A \cap \Omega\right)$. The homotopy invariance property implies that

$$\begin{aligned} 1 &= \deg\left(-I|_{\text{Ker } A\cap\Omega}, \text{Ker } A \cap \Omega, o\right) \\ &= \deg\left(H(0, \cdot)|_{\text{Ker } A\cap\Omega}, \text{Ker } A \cap \Omega, o\right) \\ &= \deg\left(H(1, \cdot)|_{\text{Ker } A\cap\Omega}, \text{Ker } A \cap \Omega, o\right) \\ &= \deg\left(QF|_{\text{Ker } A\cap\Omega}, \text{Ker } A \cap \Omega, o\right), \end{aligned}$$

which completes the proof. □

Remark 5.5. The idea of the proof shows that the conditions (5.2) and (5.3) can be weakened. Indeed, the estimates holds also for the case when we suppose that F satisfies

$$x^T F(x) \le 0 \quad (< 0).$$

for all $x \in \partial U$, where U is some convex subset containing the origin.

6 Application of Variational Methods

In Section 3 we reformulated the boundary value problem as an equation in $\mathbb{R}^N$ (3.6) with a symmetric left-hand side matrix. One of the great advantages of symmetric problems is that we can seek critical points of their potentials. In this section we show that even simple variational techniques yield existence and uniqueness results which do not require continuity of the right-hand side function in (3.6).

Our main tool in this section is the following result.

Theorem 6.1 (see [4, Theorems 6.2.8 and 6.2.11]). *Let H be a Hilbert space. Let $\mathcal{G} : H \to \mathbb{R}$ be a weakly sequentially lower semi-continuous and weakly coercive functional on H. Then $\mathcal{G}$ is bounded from below on H and there exists $u_0 \in H$ such that*

$$\mathcal{G}(u_0) = \inf_{u \in H} \mathcal{G}(u).$$

If the Fréchet derivative $\mathcal{G}'(u_0)$ exists, then

$$\mathcal{G}'(u_0) = o.$$

If, moreover, $\mathcal{G}$ is strictly convex and continuous, then u_0 is uniquely determined.

If f_k (or equivalently F_k) are integrable for each $k = 1, 2, \dots, N$, then there exists a functional $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}$ such that $\nabla \mathcal{F} = F$ and the functional $\mathcal{G} : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\mathcal{G}(x) := \frac{1}{2} x^T A x - \mathcal{F}(x), \tag{6.1}$$

whose extrema are solutions of the equation $Ax = F(x)$. Of course, the functional $\mathcal{G}$ would not exist if A was not a symmetric matrix.

Having all necessary background, we present the existence theorem.

Theorem 6.2 (Existence). *Let us suppose that $\alpha \ge 0$, $\gamma \ge 0$ and $q_k \ge 0$ for $k = 1, 2, \dots, N$ and $f : [a, b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is a function such that for each $k = 1, 2, \dots, N$:*

(I) $F_k \in L^1_{loc}(\mathbb{R})$,

(L) *there exists $M > 0$ such that*

$$\lim_{u \to \infty} F_k(u) \le -M \quad \textit{and} \quad \lim_{u \to -\infty} F_k(u) \ge M.$$

Then BVP (2.2) *has a solution.*

Proof. The symmetry of A and the integrability (I) of F_k enable us to look for extrema of the functional $\mathcal{G}$ defined in (6.1). Let us investigate its properties. In order to satisfy the assumptions of Theorem 6.1 we should ensure that the functional $\mathcal{G}$ is weakly coercive and weakly sequentially lower semi-continuous.

Let us study the *weak coercivity* first. The potential $\mathcal{F}$ is the sum of the functions $\mathcal{F}_k(u) := \int\limits_0^u F_k(\tau)\mathrm{d}\tau$, i.e.,

$$\mathcal{F}(x) := \sum_{k=1}^{N} \mathcal{F}_k(x_k) = \sum_{k=1}^{N} \int\limits_0^{x_k} F_k(\tau)\mathrm{d}\tau.$$

The limit condition (L) implies that for each $k = 1, 2, \dots, N$

$$\lim_{u\to\pm\infty} \mathcal{F}_k(u) = \lim_{u\to\pm\infty} \int\limits_0^u F_k(\tau)\mathrm{d}\tau = -\infty.$$

Therefore,

$$\lim_{\|x\|\to\infty} \mathcal{F}(x) = \lim_{\|x\|\to\infty} \sum_{k=1}^{N} \mathcal{F}_k(x_k) = -\infty,$$

which, together with the positive (semi)definiteness of A (see Lemma 3.4), yields

$$\lim_{\|x\|\to\infty} \mathcal{G}(x) \geq \lim_{\|x\|\to\infty} -\mathcal{F}(x) = \infty.$$

Thus $\mathcal{G}$ is a weakly coercive functional.

Second, the *weakly sequential continuity* of $\mathcal{G}$ is a direct consequence of the fundamental theorem of calculus for Lebesgue integration (see e.g., [8, Theorem 23.4]). Indeed, the condition (I) and the fundamental theorem ensure the continuity of $\mathcal{G}$ which, coupled with the finite dimension of $\mathbb{R}^N$, yields the weakly sequential continuity. Thus, assumptions (M) and (L) guarantee the existence of at least one solution of the problem (2.2). □

Having proved the existence of solutions, we exploit the second part of Theorem 6.1 in order to obtain the uniqueness of solutions in the case of discontinuous right-hand side functions.

Theorem 6.3 (Uniqueness)**.** *If, in addition to* (I) *and* (L)*,* $f : [a, b]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *satisfies for* $k = 1, 2, \dots, N$*:*

(M) F_k *is nonincreasing.*

and either

(i) $\alpha + \gamma + \sum_{i=1}^{N} q_i > 0$*, or*

(ii) F_k *is a strictly decreasing function for all* $k = 1, 2, \ldots, N$*,*

then the solution of (2.2) *is unique.*

Proof. In order to obtain the uniqueness from Theorem 6.1 it suffices to ensure the strict convexity of $\mathcal{G}$.

If F_k satisfy (M), then its potential $\mathcal{F}_k(u) := \int_0^u F_k(\tau)\mathrm{d}\tau$ is concave. Consequently $\mathcal{F}$ is concave as well and $-\mathcal{F}$ is convex.

Since the matrix A is positive semidefinite, the functional $\mathcal{G}$ is convex as well. Finally, either part of functional $\mathcal{G}$ can provide *strict* convexity. First, if one of the coefficients α, γ, q_i is strictly positive, then Lemma 3.4 yields positive definiteness of A which correspond to strict convexity of the term $x^T Ax$. Second, if F_k are strictly decreasing, then $\mathcal{F}$ is strictly concave. □

The main advantage of these two results is that they do not require the continuity of right-hand sides. This phenomenon is illustrated by a simple example which also underlines the fact that, in the case of nonhomogeneous boundary conditions, the condition (L) must be verified for F_k and not only for f_k.

Example 6.4. Let us deal with the time scale $\mathbb{T} = 2^{\mathbb{N}_0} = \{1, 2, 4, 8, \ldots, 512, \ldots\}$ and let us consider the Sturm–Liouville problem:

$$\begin{cases} -x^{\Delta\Delta}(\rho(t)) = -\mathrm{sign}(x(t)), & \text{on } [2, 1024]_{\mathbb{T}} \\ x^{\Delta}(1) = C, \quad x(1024) = D. \end{cases} \tag{6.2}$$

First, we realize that $p \equiv 1$ and $q \equiv 0$, $\alpha = \delta = 0$, $\beta = -1$ and $\gamma = 1$. But $\gamma > 0$ and thus the corresponding matrix A is (see Lemma 3.4) positive definite. Since $\mathrm{sign}(u)$ is integrable and nondecreasing, it suffices to verify the assumption (L). For $k = 2, \ldots, 9$ this condition is trivially satisfied since $F_k(u) = -2^{k-1}\mathrm{sign}(u)$. However, in the boundary cases we have to take into account the values of C and D (see (3.5)). Namely, we deal with the following functions:

$$\begin{aligned} F_1(u) &= -\mathrm{sign}(u) + C \\ F_{10}(u) &= -512\mathrm{sign}(u) + \frac{D}{1024}. \end{aligned}$$

From some simple analysis we obtain that the condition (L) is satisfied only if $C \in (-1, 1)$ and $D \in (-2^{19}, 2^{19})$.

7 Higher Order Problems

In this section we generalize the problem (1.1) into a higher order case. We restrict our attention to problems with uniformly distributed points. In general, for higher-order discrete problems we cannot bypass the nonsymmetric character of the problem with heterogeneously distributed points as for the second order case. First, we deal with the $2n$-th order BVP:

$$\begin{cases} (-1)^n \Delta^n (p(k-n)\Delta^n x(k-n)) = f(k, x(k)), \quad k = 1, 2, \ldots, N, \\ \Delta^i x(1-n) = C_i, \quad \Delta^i x(N+n-i) = D_i, \quad i = 0, \ldots, n-1. \end{cases} \tag{7.1}$$

Note that, as in Section 3, we omit the term $q(k)x(k)$ first. Obviously, there exist the corresponding constants $\overline{C}_i, \overline{D}_i \in \mathbb{R}$ such that the problem

$$\begin{cases} (-1)^n \Delta^n (p(k-n)\Delta^n x(k-n)) = f(k, x(k)), \quad k = 1, 2, \ldots, N, \\ x(1-n+i) = \overline{C}_i, \quad x(N+n-i) = \overline{D}_i, \quad i = 0, \ldots, n-1 \end{cases}$$

has the same solution. To rewrite this problem into a vector equation we use first the expression for the n-th difference

$$\Delta^n x(k-n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x(k-n+i),$$

which we apply to rewrite the operator on the left-hand side of (7.1):

$$\Delta^n (p(k-n)\Delta^n x(k-n)) \\ = \sum_{j=0}^{n} (-1)^j \binom{n}{j} p(k-n+j) \left(\sum_{i=0}^{n} (-1)^i \binom{n}{i} x(k-n+i+j) \right).$$

This formula implies that the coefficient by $x(k-n+l)$ in equation corresponding to k is

$$\sum_{\substack{i,j \in \mathbb{N}_0 \\ i+j=l}} (-1)^l \binom{n}{j} \binom{n}{i} p(k-n+j)$$

We can define the $(N+n) \times (N+n)$ diagonal matrix P whose diagonal entries are the values of the function p, i.e.,

$$P := \text{diag}\left\{ p(1-n), p(2-n), \ldots, p(N-1), p(N) \right\},$$

and the subdiagonal rectangular $(N+n)\times N$ matrix $\widetilde{A}_n$ by

$$\widetilde{A}_n = \begin{bmatrix} -\binom{n}{0} & & & \\ \binom{n}{1} & -\binom{n}{0} & & \\ \vdots & \binom{n}{1} & \ddots & \\ (-1)^{n+1}\binom{n}{n} & \ddots & \binom{n}{1} & -\binom{n}{0} \\ & (-1)^{n+1}\binom{n}{n} & \ddots & \binom{n}{1} \\ & & (-1)^{n+1}\binom{n}{n} & \vdots \\ & & & (-1)^{n+1}\binom{n}{n} \end{bmatrix}.$$

Now, the problem (7.1) can be rewritten as an equation in $\mathbb{R}^N$:

$$\widehat{A}_{2n}x = F(x),$$

where the $N\times N$ matrix $\widehat{A}_{2n}$ can be obtained by

$$\widehat{A}_{2n} = \widetilde{A}_n^T P \widetilde{A}_n,$$

which is a direct consequence of the appearance of $(-1)^n$ in (7.1) and the multiplication properties of binomial coefficients (see e.g., [7, Problem 11]).

Lemma 7.1. *If $p, q, k \in \mathbb{Z}$, then the following equality holds:*

$$\sum_{m=0}^{k} \binom{p}{m}\binom{q}{k-m} = \binom{p+q}{k}. \tag{7.2}$$

Finally, the function $F : \mathbb{R}^N \to \mathbb{R}^N$ is defined by $F(x) :=$

$$\begin{bmatrix} f(1,x_1) - \sum_{l=0}^{n-1} \overline{C}_l \left(\sum_{\substack{i,j\in\mathbb{N}_0 \\ i+j=l}} (-1)^l \binom{n}{j} \binom{n}{i} p(1-n+j) \right) \\ f(2,x_2) - \sum_{l=1}^{n-1} \overline{C}_l \left(\sum_{\substack{i,j\in\mathbb{N}_0 \\ i+j=l}} (-1)^l \binom{n}{j} \binom{n}{i} p(2-n+j) \right) \\ f(3,x_3) - \sum_{l=2}^{n-1} \overline{C}_l \left(\sum_{\substack{i,j\in\mathbb{N}_0 \\ i+j=l}} (-1)^l \binom{n}{j} \binom{n}{i} p(3-n+j) \right) \\ \vdots \\ f(N-1,x_{N-1}) - \sum_{l=1}^{n-1} \overline{D}_l \left(\sum_{\substack{i,j\in\mathbb{N}_0 \\ i+j=l}} (-1)^l \binom{n}{j} \binom{n}{i} p(N-1+n+j) \right) \\ f(N,x_N) - \sum_{l=0}^{n-1} \overline{D}_l \left(\sum_{\substack{i,j\in\mathbb{N}_0 \\ i+j=l}} (-1)^l \binom{n}{j} \binom{n}{i} p(N+n+j) \right) \end{bmatrix}.$$

Now we are ready to characterize the definiteness of matrices $\widehat{A}_{2n}$.

Lemma 7.2. *If $p(k) > 0$ for all $k = a-n, \ldots, b$, then the matrix $\widehat{A}_{2n}$ is positive definite.*

Proof. Let us define, for the sake of this proof, the $(N+n)\times(N+n)$ diagonal matrix $\widetilde{P}$ by

$$\widetilde{P} := \operatorname{diag}\left[\sqrt{p(1-n)}, \sqrt{p(2-n)}, \ldots, \sqrt{p(N-1)}, \sqrt{p(N)}\right].$$

Obviously $P = \widetilde{P}^T\widetilde{P}$, which implies that

$$\widehat{A}_{2n} = \widetilde{A}_n^T \widetilde{P}^T \widetilde{P} \widetilde{A}_n = \left(\widetilde{P}\widetilde{A}_n\right)^T \left(\widetilde{P}\widetilde{A}_n\right).$$

The rank of matrices $\widetilde{A}_n$ is n. Thus the rank of $\widetilde{P}\widetilde{A}_n$ is n as well. But this implies that the matrix $\widehat{A}_{2n}$ is positive definite, since for each $x \in \mathbb{R}^N$ such that $\|x\| > 0$ we have that

$$x^T \widehat{A}_{2n} x = x^T \left(\widetilde{P}\widetilde{A}_n\right)^T \left(\widetilde{P}\widetilde{A}_n\right) x = \left\|\widetilde{P}\widetilde{A}_n x\right\| > 0.$$

This completes the proof. $\square$

Obviously, we can extend the problem (7.1), include the term $q(k)x(k)$ and take into account the following BVP:

$$\begin{cases} (-1)^n\Delta^n(p(k-n)\Delta^n x(k-n)) + q(k)x(k) = f(k,x(k)), & k=1,2,\dots,N \\ \Delta^i x(1-n) = C_i, \quad \Delta^i x(N+n-i) = D_i, & i=0,\dots,n-1. \end{cases} \tag{7.3}$$

In this case we arrive to the operator setting

$$A_{2n}x = F(x),$$

where F is defined as above and $A_{2n} = \widehat{A}_{2n} + Q$ with Q being an $N\times N$ diagonal matrix

$$Q = \operatorname{diag}\{q(1), q(2), \dots, q(N-1), q(N)\}.$$

If $q(k)\geq 0$ for all $k=1,2,\dots,N$, then, using the conclusion of Lemma 7.2, the matrices A_{2n} are positive definite.

We illustrate the above notation and ideas by the following example with $n=2$.

Example 7.3. Let us consider the fourth-order problem:

$$\begin{cases} \Delta^2(p(k-2)\Delta^2 x(k-2)) + q(k)x(k) = f(k,x(k)), & k=\{1,2,\dots,N\} \\ x(-1)=C_0, \quad \Delta x(-1) = C_1, \quad x(N+2) = D_0, \quad \Delta x(N+1) = D_1. \end{cases}$$

If we define $\overline{C}_0 = C_0$, $\overline{C}_1 = C_0 - C_1$, $\overline{D}_0 = D_0$ and $\overline{D}_1 = D_1 + D_0$, this problem is equivalent with

$$\begin{cases} \Delta^2(p(k-2)\Delta^2 x(k-2)) + q(k)x(k) = f(k,x(k)), & k=\{1,2,\dots,N\} \\ x(-1)=\overline{C}_0, \quad x(0) = \overline{C}_1, \quad x(N+2) = \overline{D}_0, \quad x(N+1) = \overline{D}_1. \end{cases}$$

In this case the matrix $\widetilde{A}_2$ is an $(N+2)\times N$ matrix

$$\widetilde{A}_2 = \begin{bmatrix} -1 & & & & & \\ 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \\ & & & & & -1 \end{bmatrix}$$

and the matrix $\widehat{A}_4 = \widetilde{A}_2^T P \widetilde{A}_2$ is an $N\times N$ five-band matrix $\widehat{A}_4 =$

$$\begin{bmatrix} p_{-1}+4p_0+p_1 & -2(p_0+p_1) & p_1 & & \\ -2(p_0+p_1) & p_0+4p_1+p_2 & -2(p_1+p_2) & p_2 & \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ & p_{N-3} & -2(p_{N-3}+p_{N-2}) & p_{N-3}+4p_{N-2}+p_{N-1} & -2(p_{N-2}+p_{N-1}) \\ & & p_{N-2} & -2(p_{N-2}+p_{N-1}) & p_{N-2}+4p_{N-1}+p_N \end{bmatrix}.$$

Finally, the function $F(x)$ is

$$F(x) := \begin{bmatrix} f_1(1,x_1) - p_{-1}\overline{C}_0 + 2(p_{-1}+p_0)\overline{C}_1 \\ f_2(2,x_2) - p_0\overline{C}_1 \\ f_3(3,x_3) \\ \vdots \\ f(N-2,x_{N-2}) \\ f(N-1,x_{N-1}) - p_{N-1}\overline{D}_1 \\ f(N,x_N) - p_N\overline{D}_0 + 2(p_{N-1}+p_N)\overline{D}_1 \end{bmatrix}.$$

Now we are ready to extend the conclusions of the statements for second-order case. The proofs of these statements are almost literal transcriptions of its second-order counterparts and thus omitted.

Theorem 7.4. *Let us suppose that $q_k \geq 0$ for $k = 1,2,\ldots,N$. If for each $k = 1,2,\ldots,N$, the functions $f(k,\cdot)$ satisfy a Lipschitz condition with a constant $K < \lambda_{\min}$, then BVP* (7.3) *has a unique solution.*

Theorem 7.5. *Let us suppose that $q_k \geq 0$ for all $k = 1,2,\ldots,N$. If $f : \{1,2,\ldots,N\} \times \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function, then BVP* (7.3) *has a solution.*

Theorem 7.6. *Let us suppose that $q_k \geq 0$ for $k = 1,2,\ldots,N$ and that for all $k = 1,2,\ldots,N$, the functions f_k are continuous and that there exists $R > 0$ such that for each $u \in \mathbb{R}$ with $|u| \leq R$*

$$\langle u, F_k(u)\rangle \leq 0.$$

Then the problem (7.3) *has a solution.*

We can simply see from Lemma 7.2 that in our setting there is no need for the equivalent of the coincidence degree result corresponding to Theorem 5.4. Similarly, the result obtained via variational methods is significantly shortened.

Theorem 7.7. *Let us suppose $q_k \geq 0$ for $k = 1,2,\ldots,N$ and $f : \{1,2,\ldots,N\} \times \mathbb{R} \to \mathbb{R}$ is a function such that for each $k = 1,2,\ldots,N$:*

(I) $F_k \in L^1_{loc}$,

(M) F_k *is nonincreasing,*

(L) *there exists $M > 0$ such that*

$$\lim_{u\to\infty} F_k(u) \leq -M \quad \textit{and} \quad \lim_{u\to-\infty} F_k(u) \geq M.$$

Then BVP (2.2) *has a unique solution.*

Acknowledgements

The author gratefully acknowledges the support by the Ministry of Education, Youth and Sports of the Czech Republic, Research Plan No. MSM 4977751301.

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