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Oscillation of a Class of Fourth-Order Nonlinear Mixed Neutral Difference Equations

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Abstract

In this paper sufficient conditions are obtained for oscillation of a class of fourth order nonlinear mixed neutral difference equations of the form

$$\Delta^2\left(\frac{1}{a(n)}\left(\Delta^2\left(y(n)+p(n)y(n-m)\right)\right)^{\alpha}\right) = q(n)f(y(n-\sigma_1)) + r(n)g(y(n+\sigma_2)),$$

where $f, g \in C(\mathbb{R}, \mathbb{R})$ such that $xf(x) > 0$, $xg(x) > 0$ for $x \neq 0$, and α is the ratio of odd positive integers. This problem is studied under various ranges of $p(n)$.

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1 Introduction

In [9, 10], the author has considered a class of nonlinear fourth order difference equations of the form

$$\Delta^2\left(r(n)\Delta^2\left(y(n)+p(n)y(n-m)\right)\right) + q(n)G(y(n-k)) = 0 \tag{1.1}$$

and studied the oscillatory and asymptotic behaviour of solutions under the assumptions

$$\sum_{n=0}^{\infty}\frac{n}{r(n)} = \infty, \qquad \sum_{n=0}^{\infty}\frac{n}{r(n)} < \infty.$$

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It is observed that the associated forced equation

$$\Delta^2\left(r(n)\Delta^2\left(y(n)+p(n)y(n-m)\right)\right)+q(n)G(y(n-k))=f(n)$$

is oscillatory but not the unforced equation (1.1) for various ranges of $p(n)$. This is due to the analysis incorporated there. However, the study of (1.1) is still under progress, as well as the investigation of the oscillatory results. For recent contributions in this area, we refer the reader to [2–4, 7, 8] and the references cited therein.

Agarwal et al [5] has studied the fourth order nonlinear difference equation of the form

$$\Delta^2\left(\frac{1}{a(n)}\left(\Delta^2 x(n)\right)^{\alpha}\right)=q(n)f(x(\sigma_1(n)))+r(n)g(x(\sigma_2(n))), \tag{1.2}$$

where $f,g\in C(\mathbb{R},\mathbb{R})$ such that $xf(x)>0$, $xg(x)>0$, $f'(x)\geq 0$, $g'(x)\geq 0$, $\sigma_1(n)<n$, $\sigma_2(n)>n$ for $n\in N_0=\{n_0,n_0+1,\cdots\}$, and α is the quotient of odd positive integers. They have obtained some oscillation criteria for (1.2).

The prime interest of the present work is to study the oscillatory behaviour of solutions of a class of nonlinear neutral delay difference equations of fourth order of the form

$$\Delta^2\left(\frac{1}{a(n)}\left(\Delta^2\left(y(n)+p(n)y(n-\tau)\right)\right)^{\alpha}\right)$$
$$=q(n)f(y(n-\sigma_1))+r(n)g(y(n+\sigma_2)), \tag{1.3}$$

where Δ is the forward difference operator defined by $\Delta y(n)=y(n+1)-y(n)$, a,p,q and r are real-valued functions defined on N_0 such that $r(n)>0$, $q(n)\geq 0$, and $r(n)\geq 0$ for all $n\in N_0$, $f,g\in C(\mathbb{R},\mathbb{R})$ such that $xf(x)>0$, $xg(x)>0$ for $x\neq 0$, α is the ratio of odd positive integers, and τ, σ_1 and σ_2 are positive constants.

Equation (1.2) is a particular case of (1.3) subject to the range of $p(n)$. Hence the study of (1.3) is more interesting. Indeed, (1.3) can similarly be viewed as

$$\Delta^2\left(\frac{1}{a(n)}\left(\Delta^2\left(y(n)+p(n)y(n-\tau)\right)\right)^{\alpha}\right)$$
$$=q(n)f(y(n-\sigma_1))-r(n)g(y(n+\sigma_2)). \tag{1.4}$$

It is predicted that the study of oscillatory and asymptotic behaviour of solutions of (1.3) and (1.4) is more or less similar. Interestingly, (1.1) is a particular case of (1.3) or (1.4). But unlike the analysis in [9, 10], an attempt is made here to study the oscillatory behaviour of solutions of (1.3).

By a solution of (1.3) we mean a real-valued function $y(n)$ satisfying (1.3) for all large $n\geq n_0\in N_0$. A nontrivial solution $y(n)$ of (1.3) is said to be nonoscillatory if it is either eventually positive or eventually negative; otherwise it is oscillatory. Equation (1.3) is said to be oscillatory if all its solutions are oscillatory.

2 Oscillation Results

We define the operators

$$L_0 z(n) = z(n), \quad L_1 z(n) = \Delta L_0 z(n), \quad L_2 z(n) = \frac{1}{a(n)}(\Delta L_1 z(n))^{\alpha},$$
$$L_3 z(n) = \Delta L_2 z(n) \quad \text{and} \quad L_4 z(n) = \Delta L_3 z(n),$$

where $z(n) = y(n) + p(n)y(n-\tau)$. Thus (1.3) becomes

$$L_4 z(n) = q(n) f(y(n-\sigma_1)) + r(n) g(y(n+\sigma_2)). \tag{2.1}$$

For our use in the sequel, we assume the following conditions:

(H$_1$) $f(uv) \geq f(u)f(v)$, $g(uv) \geq g(u)g(v)$ for $u, v > 0$ and $u, v \in \mathbb{R}$,

(H$_2$) $f(-u) = -f(u)$, $g(-u) = -g(u)$ for all $u \in \mathbb{R}$,

(H$_3$) there exist $\lambda > 0$, $\mu > 0$ such that

$$f(u) + f(v) \geq \lambda f(u+v) \quad \text{and} \quad g(u) + g(v) \geq \mu\, g(u+v)$$

for all $u > 0$, $v > 0$ and $u, v \in \mathbb{R}$,

(H$_4$) $Q(n) = \min\{q(n),\ q(n-\tau)\}$, $R(n) = \min\{r(n),\ r(n-\tau)\}$, $n \geq \tau$,

(H$_5$) $\displaystyle\sum_{n=0}^{\infty} (a(n))^{\frac{1}{\alpha}} = \infty$,

(H$_6$) $\dfrac{f(u^{\frac{1}{\alpha}})}{u} \geq \beta_1 > 0$ $\dfrac{g(u^{\frac{1}{\alpha}})}{u} \geq \beta_2 > 0$, $u \neq 0$,

(H$_7$) $\displaystyle\limsup_{k\to\infty} \sum_{j=k-\sigma_1}^{k-1} Q(j) f[B(j-\sigma_1, n_1)] > \frac{1+f(b)}{\lambda\beta_1}$,

(H$_8$) $\displaystyle\limsup_{k\to\infty} \sum_{j=k}^{k+\sigma_2-1} R(j) g[A(j+\sigma_2, k+\sigma_2)] > \frac{1+g(b)}{\mu\beta_2}$,

(H$_9$) $\displaystyle\limsup_{k\to\infty} \sum_{j=k-\sigma_1}^{k-1} Q(j) f[C(k-\sigma_1, j-\sigma_1)] > \frac{1+f(b)}{\lambda\beta_1}$,

(H$_{10}$) $\displaystyle\limsup_{k\to\infty} \sum_{j=k}^{k+\sigma_2-1} r(j) g[A(j+\sigma_2, k+\sigma_2)] > \frac{1}{\beta_2}$, (H$_{11}$) $\limsup_{k\to\infty} \sum_{j=k-\sigma_1}^{k-1} q(j) f[B(j-\sigma_1, n_1)] > \frac{1}{\beta_1}$,

(H$_{12}$) $\limsup_{k\to\infty} \sum_{j=k-\sigma_1}^{k-1} q(j) f[C(k-\sigma_1, j-\sigma_1)] > \frac{1}{\beta_1}$,

(H$_{13}$) $\limsup_{k\to\infty} \sum_{j=k+\tau-\sigma_1-1}^{k-1} q(j) f[D(k+\tau-\sigma_1-1, j+\tau-\sigma_1-1)] > \frac{1}{\beta_1 f(\frac{1}{b})}, \tau \le \sigma_1$,

(H$_{14}$) $\frac{f^{\frac{1}{\alpha}}(u)}{4} \ge \beta_3 > 0, \frac{g^{\frac{1}{\alpha}}(u)}{u} \ge \beta_4 > 0, u \ne 0$,

(H$_{15}$) $\limsup_{k\to\infty} \sum_{t_3=k}^{k+\sigma_2-1} \sum_{t_2=k}^{t_3-1} \left[a(t_2) \sum_{t_1=k}^{t_2-1} \sum_{t=k}^{t_1-1} R(t) \right]^{\frac{1}{\alpha}} > \frac{(1+g(b))^{\frac{1}{\alpha}}}{\mu^{\frac{1}{\alpha}} \beta_4}$,

(H$_{16}$) $\limsup_{k\to\infty} B[k-\tau, n_1] \left(\sum_{s=k}^{\infty} Q(s) \right)^{\frac{1}{\alpha}} > \frac{(1+f(b))^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}} \beta_3}$,

(H$_{17}$) $\limsup_{k\to\infty} \sum_{t_3=k-\sigma_1}^{k-1} \sum_{t_2=t_3}^{k-1} \left[a(t_2) \sum_{t_1=t_2}^{k-1} \sum_{s=t_1}^{k-1} Q(s) \right]^{\frac{1}{\alpha}} > \frac{(1+f(b))^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}} \beta_3}$,

(H$_{18}$) $\limsup_{n\to\infty} \sum_{t_3=n}^{n+\sigma_2-1} \sum_{t_2=n}^{t_3-1} \left(a(t_2) \sum_{t_1=n}^{t_2-1} \sum_{t=n}^{t_1-1} r(t) \right)^{\frac{1}{\alpha}} > \frac{1}{\beta_4}$,

(H$_{19}$) $\limsup_{n\to\infty} B[n-\sigma_1, n_1] \left(\sum_{s=n}^{\infty} q(s) \right)^{\frac{1}{\alpha}} > \frac{1}{\beta_3}$,

(H$_{20}$) $\limsup_{n\to\infty} \sum_{t_3=n-\sigma_1}^{n-1} \sum_{t_2=t_3}^{n-1} \left[a(t_2) \sum_{t_1=t_2}^{n-1} \sum_{s=t_1}^{n-1} q(s) \right]^{\frac{1}{\alpha}} > \frac{1}{\beta_3}$,

(H$_{21}$) $\limsup_{n\to\infty} D[n, n-\sigma_1] \left(\sum_{t=n}^{t_1-1} q(t) \right)^{\frac{1}{\alpha}} > \frac{b}{\beta_3}$, where

$$\begin{aligned} A[k,m] &= \sum_{\ell=m}^{k-2}(k-\ell-1)(\ell-m)^{\frac{1}{\alpha}}a^{\frac{1}{\alpha}}(\ell), \\ B[k,n_1] &= \sum_{\ell=n_1}^{k-2}(k-\ell-1)(k+1-\ell)^{\frac{1}{\alpha}}a^{\frac{1}{\alpha}}(\ell), \\ C[k,m] &= \sum_{\ell=m}^{k-1}(\ell-m+1)(k+1-\ell)^{\frac{1}{\alpha}}a^{\frac{1}{\alpha}}(\ell), \\ D[k,m] &= \sum_{\ell=m}^{k-1}(k+1-\ell)^{\frac{1}{\alpha}}a^{\frac{1}{\alpha}}(\ell). \end{aligned}$$

Theorem 2.1. *Let* $0 \le p(n) \le b < \infty$. *If* ($H_1$)–($H_9$) *hold, then* (1.3) *is oscillatory.*

Proof. Suppose to the contrary that $y(n)$ is a nonoscillatory solution of (1.3) for $n \ge n_0$. Without loss of generality we may assume that $y(n) > 0$ for $n \ge n_0$ due to (H_2). Ultimately, it follows from (2.1) that $L_4 z(n) \ge 0$ eventually and hence $L_i z(n)$, $i = 1, 2, 3$ are eventually of one sign. In what follows, we shall consider the following three cases:

(a) $L_i z(n) > 0$, $i = 1, 2, 3$ for $n \ge n_1 > n_0 + \max\{\tau, \sigma_1\}$,

(b) $L_i z(n) > 0$, $i = 1, 2$, $L_3 z(n) < 0$ for $n \ge n_1$,

(c) $(-1)^i L_i z(n) > 0$, $i = 1, 2, 3$ for $n \ge n_1$.

Case (a). Assume that $L_i z(n) > 0$, $i = 1, 2, 3$ for $n \ge n_1$. Then for $\ell \ge m + 1 \ge n_1 + 1$,

$$L_2 z(\ell) - L_2 z(m) = \sum_{j=m}^{\ell-1} L_3 z(j) \ge (\ell - m) L_3 z(m)$$

implies that

$$L_2 z(\ell) \ge (\ell - m) L_3 z(m),$$

that is,

$$\Delta^2 z(\ell) \ge (\ell - m)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(\ell) L_3^{\frac{1}{\alpha}} z(m). \tag{2.2}$$

Using the discrete Taylor formula [1], $z(n)$ can be written for $k \geq n_1 + 2$ as

$$\begin{aligned} z(k) &= z(n_1) + (k-n_1)\Delta z(n_1) + \sum_{\ell=n_1}^{k-2}(k-\ell-1)\Delta^2 z(\ell) \\ &\geq z(n_1) + (k-n_1)\Delta z(n_1) + \sum_{\ell=n_1}^{k-2}(k-\ell-1)(\ell-m)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(\ell) L_3^{\frac{1}{\alpha}} z(m) \\ &> \sum_{\ell=n_1}^{k-2}(k-\ell-1)(\ell-m)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(\ell) L_3^{\frac{1}{\alpha}} z(m) \\ &\geq \sum_{\ell=m}^{k-2}(k-\ell-1)(\ell-m)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(\ell) L_3^{\frac{1}{\alpha}} z(m) \\ &= A[k,m] L_3^{\frac{1}{\alpha}} z(m), \end{aligned}$$

where $k \geq m + 2 \geq n_1 + 2$. Consequently,

$$z(j+\sigma_2) \geq A[j+\sigma_2, k+\sigma_2] L_3^{\frac{1}{\alpha}} z(k+\sigma_2),\ j+\sigma_2 \geq k+\sigma_2+2 \geq n_1+2. \quad (2.3)$$

From (2.1), it follows that

$$L_4 z(n) \geq r(n) g(y(n+\sigma_2))$$

and thus

$$\begin{aligned} L_4 z(n) + g(b) L_4 z(n-\tau) &\geq r(n) g(y(n+\sigma_2)) + r(n-\tau) g(y(n-\tau+\sigma_2)) g(b) \\ &\geq R(n)[g(y(n+\sigma_2)) + g(b) g(y(n-\tau+\sigma_2))] \\ &\geq R(n)[g(y(n+\sigma_2)) + g(by(n-\tau+\sigma_2))] \\ &\geq \mu R(n) g(y(n+\sigma_2) + by(n+\sigma_2-\tau)) \\ &\geq \mu R(n) g(z(n+\sigma_2)), \end{aligned}$$

where $z(n+\sigma_2) \leq y(n+\sigma_2) + by(n+\sigma_2-\tau)$. Hence using (2.3),

$$\begin{aligned} L_4 z(j) + g(b) L_4 z(j-\tau) &\geq \mu R(j) g(z(j+\sigma_2)) \\ &\geq \mu R(j) g\left[A(j+\sigma_2, k+\sigma_2) L_3^{\frac{1}{\alpha}} z(k+\sigma_2)\right] \\ &\geq \mu R(j) g\left[A(j+\sigma_2, k+\sigma_2)\right] g\left[L_3^{\frac{1}{\alpha}} z(k+\sigma_2)\right] \end{aligned}$$

due to (H_1), (H_3) and (H_4). Summing the last inequality from k to $k+\sigma_2-1$, we get

$$\begin{aligned} &L_3 z(k+\sigma_2) - L_3 z(k) + g(b) L_3 z(k+\sigma_2-\tau) - g(b) L_3 z(k-\tau) \\ &\qquad \geq \mu g\left[L_3^{\frac{1}{\alpha}} z(k+\sigma_2)\right] \sum_{j=k}^{k+\sigma_2-1} R(j) g\left[A(j+\sigma_2, k+\sigma_2)\right], \end{aligned}$$

that is,

$$\frac{L_3 z(k+\sigma_2) + g(b) L_3 z(k+\sigma_2-\tau)}{\mu g\left[L_3^{\frac{1}{\alpha}} z(k+\sigma_2)\right]} \geq \sum_{j=k}^{k+\sigma_2-1} R(j) g\left[A(j+\sigma_2, k+\sigma_2)\right].$$

Since $L_3 z(n)$ is nondecreasing, the last inequality becomes

$$\frac{1+g(b)}{\mu} \frac{L_3 z(k+\sigma_2)}{g\left[L_3^{\frac{1}{\alpha}} z(k+\sigma_2)\right]} \geq \sum_{j=k}^{k+\sigma_2-1} R(j) g\left[A(j+\sigma_2, k+\sigma_2)\right]. \tag{2.4}$$

Using (H_6), the inequality (2.4) can be written as

$$\sum_{j=k}^{k+\sigma_2-1} R(j) g\left[A(j+\sigma_2, k+\sigma_2)\right] \leq \frac{1+g(b)}{\mu \beta_2},$$

which is a contradiction to our hypothesis (H_8).

Case (b). Let $L_i z(n) > 0$, $i = 1, 2$ and $L_3 z(n) < 0$ for $n \geq n_1$. Then for $k+1 \geq \ell > n_1$, we have

$$L_2 z(k+1) - L_2 z(\ell) = \sum_{j=\ell}^{k} L_3 z(j),$$

that is,

$$L_2 z(\ell) \geq (k+1-\ell)(-L_3 z(k)).$$

Hence

$$\Delta^2 z(\ell) \geq (k+1-\ell)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(\ell)\ (-L_3^{\frac{1}{\alpha}} z(k)).$$

Applying Taylor's formula to $z(\ell)$, we obtain for $k \geq n_1 + 2$

$$\begin{aligned} z(k) &\geq \sum_{\ell=n_1}^{k-2} (k-\ell-1) \Delta^2 z(\ell) \\ &\geq \left(-L_3^{\frac{1}{\alpha}} z(k)\right) \sum_{\ell=n_1}^{k-2} (k-\ell-1)(k+1-\ell)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(\ell) \\ &= B[k, n_1] \left(-L_3^{\frac{1}{\alpha}} z(k)\right), \end{aligned}$$

and hence

$$z(j-\sigma_1) \geq B[j-\sigma_1, n_1] \left[-L_3^{\frac{1}{\alpha}} z(j-\sigma_1)\right], \quad j-\sigma_1 \geq n_1 + 2. \tag{2.5}$$

Since (2.1) can be reduced to

$$L_4 z(n) \geq q(n) f(y(n-\sigma_1)),$$

by Case (a), we obtain

$$L_4z(n) + f(b)L_4z(n-\tau) \geq \lambda Q(n)f(z(n-\sigma_1)) \tag{2.6}$$

due to (H_1), (H_3) and (H_4). Consequently,

$$L_4z(j) + f(b)L_4z(j-\tau) \geq \lambda Q(j)f\left[B(j-\sigma_1, n_1)\right] f\left[-L_3^{\frac{1}{\alpha}}z(j-\sigma_1)\right]$$

due to (2.5). Proceeding as in Case (a), we obtain a contradiction to our hypothesis (H_7).

Case (c). Using the fact that $L_3z(n) < 0$, we obtained from Case (b) that

$$\Delta^2 z(\ell) \geq (k+1-\ell)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(\ell)\left(-L_3^{\frac{1}{\alpha}}z(k)\right),\ k+1 \geq \ell \geq n_1.$$

Using the discrete Taylor formula for $z(m)$ for $k-1 \geq m > n_1$, it is easy to verify that

$$\begin{aligned} z(m) &\geq \sum_{\ell=m}^{k-1}(\ell+1-m)\Delta^2 z(\ell) \\ &\geq -L_3^{\frac{1}{\alpha}}z(k)\sum_{\ell=m}^{k-1}(\ell+1-m)(k+1-\ell)^{\frac{1}{\alpha}}a^{\frac{1}{\alpha}}(\ell) \\ &= C[k,m]\left(-L_3^{\frac{1}{\alpha}}z(k)\right), \quad k-1 \geq m > n_1. \end{aligned}$$

Hence

$$z(j-\sigma_1) \geq C[k-\sigma_1, j-\sigma_1]\left[-L_3^{\frac{1}{\alpha}}z(k-\sigma_1)\right], \quad k-\sigma_1-1 \geq j-\sigma_1 \geq n_1.$$

Using the same type of reasoning as in Case (b), we find

$$L_4z(j) + f(b)L_4z(j-\tau) \geq \lambda Q(j)f\left[C(k-\sigma_1, j-\sigma_1)\right] f\left[-L_3^{\frac{1}{\alpha}}z(k-\sigma_1)\right]$$

for $k-\sigma_1-1 \geq j-\sigma_1 \geq n_1$. Therefore

$$\begin{aligned} &\sum_{j=k-\sigma_1}^{k-1}\left[L_4z(j) + f(b)L_4z(j-\tau)\right] \\ &\qquad \geq \lambda f\left[-L_3^{\frac{1}{\alpha}}z(k-\sigma_1)\right]\sum_{j=k-\sigma_1}^{k-1} Q(j)f\left[C(k-\sigma_1, j-\sigma_1)\right]. \end{aligned}$$

The rest of the proof follows from Case (a). □

Theorem 2.2. *Let $-\infty < -b \leq p(n) \leq 0$ and $b > 0$. If (H_1), (H_2), (H_5), (H_6) and (H_{10})–(H_{13}) hold, then every solution of* (1.3) *oscillates.*

Proof. Let $y(n)$ be a nonoscillatory solution of (1.3) such that $y(n) > 0$ for $n \geq n_0$. Proceeding as in the proof of the Theorem 2.1, we have three Cases (a), (b) and (c). Further, from the said cases, it follows that $z(n)$ is monotonic. Hence $z(n) > 0$ or $z(n) < 0$ for $n \geq n_1 > n_0 + \max\{\tau, \sigma_1\}$. Suppose the former holds. In what follows we shall consider Cases (a), (b) and (c).

Using the same type of reasoning as in Case (a) of Theorem 2.1, we get the inequality (2.3). From (2.1), it follows that $L_4 z(n) \geq r(n) g(y(n+\sigma_2))$ and hence

$$L_4 z(n) \geq r(n) g(z(n+\sigma_2)),$$

where $z(n) = y(n) + p(n)y(n-\tau) \leq y(n)$. Consequently,

$$\begin{aligned} L_4 z(j) &\geq r(j) g\left[A(j+\sigma_2, k+\sigma_2) L_3^{\frac{1}{\alpha}} z(k+\sigma_2)\right] \\ &\geq r(j) g\left[A(j+\sigma_2, k+\sigma_2)\right] g\left[L_3^{\frac{1}{\alpha}} z(k+\sigma_2)\right] \end{aligned}$$

due to the inequality (2.3). Summing the last inequality from k to $k+\sigma_2-1$, we obtain

$$L_3 z(k+\sigma_2) - L_3 z(k) \geq g\left[L_3^{\frac{1}{\alpha}} z(k+\sigma_2)\right] \sum_{j=k}^{k+\sigma_2-1} r(j) g\left[A(j+\sigma_2, k+\sigma_2)\right],$$

that is,

$$\sum_{j=k}^{k+\sigma_2-1} r(j) g\left[A(j+\sigma_2, k+\sigma_2)\right] \leq \frac{L_3 z(k+\sigma_2)}{g\left[L_3^{\frac{1}{\alpha}} z(k+\sigma_2)\right]} \leq \frac{1}{\beta_2},$$

a contradiction to (H_{10}). Case (b) and Case (c) can similarly be dealt with.

Assume that the later holds. Then $z(n) \geq -by(n-\tau)$ implies that $y(n-\sigma_1) \geq \left(-\frac{1}{b}\right) z(n+\tau-\sigma_1)$, for $n \geq n_2 > n_1$. Here also we consider the above three possible cases. It is easy to verify that Cases (a) and (b) imply that $z(n) > 0$, for $n \geq n_2$, a contradiction. Hence Case (c) is the desired case. Since $L_3 z(n) < 0$, we have

$$\Delta^2 z(\ell) \geq (k+1-\ell)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(\ell) \left(-L_3^{\frac{1}{\alpha}} z(k)\right), \quad k+1 \geq \ell \geq n_2,$$

that is,

$$\sum_{\ell=m}^{k-1} \Delta^2 z(\ell) \geq \left(-L_3^{\frac{1}{\alpha}} z(k)\right) \sum_{\ell=m}^{k-1} (k+1-\ell)^{\frac{1}{\alpha}} a^{\frac{1}{\alpha}}(\ell).$$

Consequently,

$$-\Delta z(m) > -\Delta z(m) + \Delta z(k) \geq D[k,m] \left(-L_3^{\frac{1}{\alpha}} z(k)\right)$$

implies that

$$-z(m+1) > -z(m+1) + z(m) \geq D[k,m]\left(-L_3^{\frac{1}{\alpha}} z(k)\right). \tag{2.7}$$

Substituting $k+\tau-\sigma_1-1$ and $j+\tau-\sigma_1-1$ for k and m respectively in (2.7), we get

$$-z(j+\tau-\sigma_1) \geq \left(-L_3^{\frac{1}{\alpha}} z(k+\tau-\sigma_1-1)\right) D[k+\tau-\sigma_1-1, j+\tau-\sigma_1-1].$$

From (2.1), it follows that

$$\begin{aligned}
&L_4 z(j) > q(j) f(y(j-\sigma_1)) \\
&\geq\ q(j) f\left(-\frac{1}{b}\, z(j+\tau-\sigma_1)\right) \\
&\geq\ q(j) f\left[\frac{1}{b}\left(-L_3^{\frac{1}{\alpha}} z(k+\tau-\sigma_1-1)\right) D(k+\tau-\sigma_1-1, j+\tau-\sigma_1-1)\right] \\
&\geq\ q(j) f\left(\frac{1}{b}\right) f\left(-L_3^{\frac{1}{\alpha}} z(k+\tau-\sigma_1-1)\right) f[D(k+\tau-\sigma_1-1, j+\tau-\sigma_1-1)].
\end{aligned}$$

Summing both sides of the above inequality from $k+\tau-\sigma_1-1$ to $k-1$, we have

$$\begin{aligned}
L_3 z(k) - L_3 z(k+\tau-\sigma_1-1) \geq f\left(\frac{1}{b}\right) f\left(-L_3^{\frac{1}{\alpha}} z(k+\tau-\sigma_1-1)\right) \times \\
\times \sum_{j=k+\tau-\sigma_1-1}^{k-1} q(j) f[D(k+\tau-\sigma_1-1, j+\tau-\sigma_1-1)],
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{-L_3 z(k+\tau-\sigma_1-1)}{f(-L_3^{\frac{1}{\alpha}} z(k+\tau-\sigma_1-1))} \\
&\qquad \geq f\left(\frac{1}{b}\right) \sum_{j=k+\tau-\sigma_1-1}^{k-1} q(j) f[D(k+\tau-\sigma_1-1, j+\tau-\sigma_1-1)].
\end{aligned}$$

Hence

$$\limsup_{k\to\infty} \sum_{j=k+\tau-\sigma_1-1}^{k-1} q(j) f[D(k+\tau-\sigma_1-1, j+\tau-\sigma_1-1)] \leq \frac{1}{\beta_1 f(\frac{1}{b})},$$

a contradiction to our hypothesis (H_{13}). This completes the proof. □

Theorem 2.3. *Let* $0 \leq p(n) \leq b < \infty$ *and* $\sigma_1 \leq \tau$. *If* (H_1)–(H_5) *and* (H_{14})–(H_{17}) *hold, then* (1.3) *is oscillatory.*

Proof. Let $y(n)$ be a nonoscillatory solution of (1.3) such that $y(n) > 0$ for $n \geq n_0$. It follows from (2.1) that $L_4 z(n) \geq 0$ for $n \geq n_1 > n_0 + \max\{\tau_1, \sigma_1\}$. Consequently, we have three cases as in Theorem 2.1. Assume that Case (a) holds. Summing the inequality

$$L_4 z(n) + g(b) L_4 z(n-\tau) \geq \mu R(n) g(z(n+\sigma_2))$$

from k to $t_1 - 1$ for $t_1 - 1 \geq k \geq n_1$, we get

$$\begin{aligned} L_3 z(t_1) + g(b) L_3 z(t_1 - \tau) &\geq \mu \sum_{t=k}^{t_1-1} R(t) g(z(t+\sigma_2)) \\ &\geq \mu g(z(k+\sigma_2)) \sum_{t=k}^{t_1-1} R(t) \end{aligned}$$

due to $L_3 z(n) > 0$ and since $z(n)$ is nondecreasing. Hence for $t_2 - 1 \geq t_1 - 1 \geq k \geq n_1$, the last inequality yields

$$\sum_{t_1=k}^{t_2-1} [L_3 z(t_1) + g(b) L_3 z(t_1 - \tau)] \geq \mu \sum_{t_1=k}^{t_2-1} g(z(k+\sigma_2)) \sum_{t=k}^{t_1-1} R(t),$$

that is,

$$L_2 z(t_2) + g(b) L_2 z(t_2 - \tau) \geq \mu g(z(k+\sigma_2)) \sum_{t_1=k}^{t_2-1} \sum_{t=k}^{t_1-1} R(t).$$

Therefore

$$(1 + g(b)) L_2 z(t_2) \geq \mu g(z(k+\sigma_2)) \sum_{t_1=k}^{t_2-1} \sum_{t=k}^{t_1-1} R(t)$$

implies that

$$\Delta^2 z(t_2) \geq \left(\frac{\mu}{1+g(b)}\right)^{\frac{1}{\alpha}} g^{\frac{1}{\alpha}}(z(k+\sigma_2)) \left[a(t_2) \sum_{t_1=k}^{t_2-1} \sum_{t=k}^{t_1-1} R(t)\right]^{\frac{1}{\alpha}}.$$

Summing the last inequality from k to $t_3 - 1$, we get

$$\Delta z(t_3) \geq \left(\frac{\mu}{1+g(b)}\right)^{\frac{1}{\alpha}} g^{\frac{1}{\alpha}}(z(k+\sigma_2)) \sum_{t_2=k}^{t_3-1} \left[a(t_2) \sum_{t_1=k}^{t_2-1} \sum_{t=k}^{t_1-1} R(t)\right]^{\frac{1}{\alpha}},$$

where $t_3 - 1 \geq t_2 - 1 \geq t_1 - 1 \geq k \geq n_1$. Further summing the above inequality from k to $k + \sigma_2 - 1$, we get

$$\sum_{t_3=k}^{k+\sigma_2-1} \Delta z(t_3) \geq \left(\frac{\mu}{1+g(b)}\right)^{\frac{1}{\alpha}} g^{\frac{1}{\alpha}}(z(k+\sigma_2)) \sum_{t_3=k}^{k+\sigma_2-1} \sum_{t_2=k}^{t_3-1} \left[a(t_2) \sum_{t_1=k}^{t_2-1} \sum_{t=k}^{t_1-1} R(t)\right]^{\frac{1}{\alpha}},$$

that is,

$$\frac{z(k+\sigma_2)}{g^{\frac{1}{\alpha}}(z(k+\sigma_2))} \geq \left(\frac{\mu}{1+g(b)}\right)^{\frac{1}{\alpha}} \sum_{t_3=k}^{k+\sigma_2-1} \sum_{t_2=k}^{t_3-1} \left[a(t_2) \sum_{t_1=k}^{t_2-1} \sum_{t=k}^{t_1-1} R(t)\right]^{\frac{1}{\alpha}}.$$

Hence, using (H_{14}) we obtain

$$\limsup_{k\to\infty} \sum_{t_3=k}^{k+\sigma_2-1} \sum_{t_2=k}^{t_3-1} \left[a(t_2) \sum_{t_1=k}^{t_2-1} \sum_{t=k}^{t_1-1} R(t)\right]^{\frac{1}{\alpha}} \leq \frac{(1+g(b))^{\frac{1}{\alpha}}}{\mu^{\frac{1}{\alpha}} \beta_4},$$

a contradiction to our assumption (H_{15}).

Let Case (b) hold. From (2.1) and (2.6) it follows that

$$\sum_{s=k}^{\infty} [L_4 z(s) + f(b) L_4 z(s-\tau)] \geq \lambda \sum_{s-k}^{\infty} Q(s) f(z(s-\sigma_1)),$$

that is,

$$-L_3 z(k) - f(b) L_3 z(k-\tau) \geq \lambda f(z(k-\sigma_1)) \sum_{s=k}^{\infty} Q(s)$$

for $k \geq n_0$. Thus for $\sigma_1 \leq \tau$,

$$\begin{aligned} (1+f(b))(-L_3 z(k-\tau)) &\geq \lambda f(z(k-\sigma_1)) \sum_{s=k}^{\infty} Q(s) \\ &\geq \lambda f(z(k-\tau)) \sum_{s=k}^{\infty} Q(s). \end{aligned}$$

From (2.5) it follows that

$$\begin{aligned} z(k-\tau) &\geq B[k-\tau, n_1] \left[-L_3^{\frac{1}{\alpha}} z(k-\tau)\right] \\ &\geq B[k-\tau, n_1] \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(k-\tau)) \left(\sum_{s=k}^{\infty} Q(s)\right)^{\frac{1}{\alpha}} \end{aligned}$$

for $k-\tau \geq n_1+2$. Hence

$$\frac{z(k-\tau)}{f^{\frac{1}{\alpha}}(z(k-\tau))} \geq B[k-\tau, n_1] \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} \left(\sum_{s=k}^{\infty} Q(s)\right)^{\frac{1}{\alpha}}$$

implies that

$$\limsup_{k\to\infty} B[k-\tau, n_1] \left(\sum_{s=k}^{\infty} Q(s)\right)^{\frac{1}{\alpha}} \leq \frac{(1+f(b))^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}} \beta_3},$$

a contradiction to our assumption (H_{16}).

Finally, we consider Case (c). Proceeding, as in Case (b), we obtain

$$(1+f(b))(-L_3 z(t_1-\tau)) \geq \lambda f(z(k-\sigma_1)) \left(\sum_{s=t_1}^{k-1} Q(s)\right)$$

for $k-1 \geq t_1 \geq n_1$. Hence for $k-1 \geq t_2 \geq t_1 > n_1$, we get

$$(1+f(b)) \sum_{t_1=t_2}^{k-1} (-L_3 z(t_1-\tau)) \geq \lambda f(z(k-\sigma_1)) \left(\sum_{t_1=t_2}^{k-1} \sum_{s=t_1}^{k-1} Q(s)\right),$$

that is,

$$(1+f(b)) L_2 z(t_2-\tau) \geq \lambda f(z(k-\sigma_1)) \left[\sum_{t_1=t_2}^{k-1} \sum_{S=t_1}^{k-1} Q(s)\right].$$

Consequently,

$$(1+f(b))^{\frac{1}{\alpha}} \Delta^2 z(t_2-\tau) \geq \lambda^{\frac{1}{\alpha}} \left[a(t_2) \sum_{t_1=t_2}^{k-1} \sum_{S=t_1}^{k-1} Q(s)\right]^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(k-\sigma_1))$$

for $k-1 \geq t_2 \geq t_1 \geq n_1$. Summing the last inequality from t_3 to $k-1 \geq t_3 \geq t_2 \geq t_1 \geq n_1$, we get

$$\Delta z(t_3-\tau) \geq \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(k-\sigma_1)) \sum_{t_2=t_3}^{k-1} \left[a(t_2) \sum_{t_1=t_2}^{k-1} \sum_{S=t_1}^{k-1} Q(s)\right]^{\frac{1}{\alpha}}.$$

Since $\Delta z(n)$ is nondecreasing, the above inequality becomes

$$\Delta z(t_3) \geq \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(k-\sigma_1)) \sum_{t_2=t_3}^{k-1} \left[a(t_2) \sum_{t_1=t_2}^{k-1} \sum_{s=t_1}^{k-1} Q(s)\right]^{\frac{1}{\alpha}}.$$

Thus,

$$\sum_{t_3=k-\sigma_1}^{k-1} \Delta z(t_3) \geq \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(k-\sigma_1)) \sum_{t_3=k-\sigma_1}^{k-1} \sum_{t_2=t_3}^{k-1} \left[a(t_2) \sum_{t_1=t_2}^{k-1} \sum_{s=t_1}^{k-1} Q(s)\right]^{\frac{1}{\alpha}},$$

that is,

$$z(k) \geq \left(\frac{\lambda}{1+f(b)}\right)^{\frac{1}{\alpha}} f^{\frac{1}{\alpha}}(z(k)) \sum_{t_3=k-\sigma_1}^{k-1} \sum_{t_2=t_3}^{k-1} \left[a(t_2) \sum_{t_1=t_2}^{k-1} \sum_{s=t_1}^{k-1} Q(s)\right]^{\frac{1}{\alpha}}$$

implies that

$$\limsup_{k\to\infty}\sum_{t_3=k-\sigma_1}^{k-1}\sum_{t_2=t_3}^{k-1}\left[a(t_2)\sum_{t_1=t_2}^{k-1}\sum_{s=t_1}^{k-1}Q(s)\right]^{\frac{1}{\alpha}}\le\frac{(1+f(b))^{\frac{1}{\alpha}}}{\lambda^{\frac{1}{\alpha}}\beta_3},$$

a contradiction to (H_{17}). Hence the proof is complete. □

Theorem 2.4. *Assume that* $-\infty<-b\le p(n)\le 0$ *and* $b>0$*. If* (H_2), (H_5), (H_{14}) *and* (H_{18})–(H_{21}) *hold, then* (1.3) *is oscillatory.*

Proof. Proceeding as in the proof of the Theorem 2.2, we obtain

$$L_4z(n)\le r(n)g(z(n+\sigma)).$$

Hence for $t_1-1\ge n\ge n_1$,

$$\begin{aligned}\sum_{t=n}^{t_1-1}L_4z(t) &\ge \sum_{t=n}^{t_1-1}r(t)g(z(t+\sigma_2))\\ &\ge g(z(n+\sigma_2))\sum_{t=n}^{t_1-1}r(t)\end{aligned}$$

due to $L_3z(n)>0$, $z(n)>0$ and since $z(n)$ is nondecreasing. Thus the above inequality becomes

$$L_3z(t_1)\ge g(z(n+\sigma_2))\sum_{t=n}^{t_1-1}r(t).$$

Summing the last inequality from n to $t_2-1\ge t_1-1\ge n\ge n_1$, we get

$$L_2z(t_2)\ge g(z(n+\sigma_2))\left(\sum_{t_1=n}^{t_2-1}\sum_{t=n}^{t_1-1}r(t)\right),$$

that is,

$$\Delta^2z(t_2)\ge g^{\frac{1}{\alpha}}(z(n+\sigma_2))\left(a(t_2)\sum_{t_1=n}^{t_2-1}\sum_{t=n}^{t_1-1}r(t)\right)^{\frac{1}{\alpha}}.$$

Consequently,

$$\sum_{t_2=n}^{t_3-1}\Delta^2z(t_2)\ge g^{\frac{1}{\alpha}}(z(n+\sigma_2))\sum_{t_2=n}^{t_3-1}\left(a(t_2)\sum_{t_1=n}^{t_2-1}\sum_{t=n}^{t_1-1}r(t)\right)^{\frac{1}{\alpha}}$$

implies that

$$\Delta z(t_3)\ge g^{\frac{1}{\alpha}}(z(n+\sigma_2))\sum_{t_2=n}^{t_3-1}\left(a(t_2)\sum_{t_1=n}^{t_2-1}\sum_{t=n}^{t_1-1}r(t)\right)^{\frac{1}{\alpha}}.$$

Further summing the above inequality from n to $n+\sigma_2-1$, we have

$$z(n+\sigma_2) \geq g^{\frac{1}{\alpha}}(z(n+\sigma_2)) \sum_{t_3=n}^{n+\sigma_2-1} \sum_{t_2=n}^{t_3-1} \left(a(t_2) \sum_{t_1=n}^{t_2-1} \sum_{t=n}^{t_1-1} r(t) \right)^{\frac{1}{\alpha}},$$

that is,

$$\frac{z(n+\sigma_2)}{g^{\frac{1}{\alpha}}(z(n+\sigma_2))} \geq \sum_{t_3=n}^{n+\sigma_2-1} \sum_{t_2=n}^{t_3-1} \left(a(t_2) \sum_{t_1=n}^{t_2-1} \sum_{t=n}^{t_1-1} r(t) \right)^{\frac{1}{\alpha}}.$$

Hence

$$\limsup_{n\to\infty} \sum_{t_3=n}^{n+\sigma_2-1} \sum_{t_2=n}^{t_3-1} \left(a(t_2) \sum_{t_1=n}^{t_2-1} \sum_{t=n}^{t_1-1} r(t) \right)^{\frac{1}{\alpha}} \leq \frac{1}{\beta_4},$$

a contradiction to (H_{18}).

Consider Case (b). From (2.1), it follows that

$$\begin{aligned} -L_3 z(n) &\geq \sum_{s=n}^{\infty} q(s) f(y(s-\sigma_1)) \\ &\geq \sum_{s=n}^{\infty} q(s) f(z(s-\sigma_1)) \\ &\geq f(z(n-\sigma_1)) \sum_{s=n}^{\infty} q(s), \end{aligned}$$

where $z(n) \leq y(n)$ for $n \geq n_1$. Using the inequality (2.5), we have

$$\begin{aligned} z(n-\sigma_1) &\geq B[n-\sigma_1, n_1] \left(-L_3^{\frac{1}{\alpha}} z(n-\sigma_1) \right) \\ &\geq B[n-\sigma_1, n_1] f^{\frac{1}{\alpha}}(z(n-\sigma_1)) \left(\sum_{s=n}^{\infty} q(s) \right)^{\frac{1}{\alpha}}, \end{aligned}$$

where $(-L_3 z(n))$ is nonincreasing and $z(n)$ is nondecreasing for $n-\sigma_1 \geq n_1+2$. Hence

$$\limsup_{n\to\infty} B[n-\sigma_1, n_1] \left(\sum_{s=n}^{\infty} q(s) \right)^{\frac{1}{\alpha}} \leq \frac{1}{\beta_3},$$

a contradiction to (H_{19}).

Suppose Case (c) holds. For $n-1 \geq t_1 \geq n_1$, (2.1) yields

$$\begin{aligned} -L_3 z(t_1) &\geq \sum_{t=t_1}^{n-1} q(t) f(y(t-\sigma_1)) \\ &\geq \sum_{t=t_1}^{n-1} q(t) f(z(t-\sigma_1)) \\ &\geq f(z(n-\sigma_1)) \sum_{t=t_1}^{n-1} q(t). \end{aligned}$$

The rest of the proof follows from Case (c) of Theorem 2.3.

Next, we assume that $z(n) < 0$ for $n \geq n_1$. Then $z(n) \geq -by(n-\tau)$ implies that $y(n-\sigma_1) \geq (-\frac{1}{b})z(n+\tau-\sigma_1)$ for $n \geq n_2 > n_1$. In this case also, we consider the three Cases (a), (b) and (c). It is easy to verify that Cases (a) and (b) are not possible. Let Case (c) hold. From the inequality (2.7), it follows that

$$-z(n-\sigma_1) \geq D[n, n-\sigma_1] \left(-L_3^{\frac{1}{\alpha}} z(n)\right). \tag{2.8}$$

On the other hand, (2.1) yields

$$\begin{aligned} L_4 z(n) &> q(n) f(y(n-\sigma_1)) \\ &\geq q(n) f\left(-\frac{1}{b} z(n+\tau-\sigma_1)\right) \\ &\geq q(n) f\left(-\frac{1}{b} z(n-\sigma_1)\right) \end{aligned}$$

and hence for $t_1 - 1 \geq n \geq n_2$,

$$\begin{aligned} \sum_{t=n}^{t_1-1} L_4 z(t) &\geq \sum_{t=n}^{t_1-1} q(t) f\left(-\frac{1}{b} z(t-\sigma_1)\right) \\ &\geq f\left(-\frac{1}{b} z(n-\sigma_1)\right) \sum_{t=n}^{t_1-1} q(t), \end{aligned}$$

that is,

$$-L_3 z(n) \geq f\left(-\frac{1}{b} z(n-\sigma_1)\right) \sum_{t=n}^{t_1-1} q(t).$$

Consequently, (2.8) becomes

$$-z(n-\sigma_1) \geq D[n, n-\sigma_1] f^{\frac{1}{\alpha}} \left(-\frac{1}{b} z(n-\sigma_1)\right) \left(\sum_{t=n}^{t_1-1} q(t)\right)^{\frac{1}{\alpha}}.$$

Thus

$$\frac{-z(n-\sigma_1)}{f^{\frac{1}{\alpha}}\left(-\frac{1}{b}z(n-\sigma_1)\right)} \geq D[n,n-\sigma_1]\left(\sum_{t=n}^{t_1-1} q(t)\right)^{\frac{1}{\alpha}}$$

implies that

$$\limsup_{n\to\infty} D[n,n-\sigma_1]\left(\sum_{t=n}^{t_1-1} q(t)\right)^{\frac{1}{\alpha}} \leq \frac{b}{\beta_3},$$

a contradiction to our assumption (H_{21}). Hence the proof is complete. □

Example 2.5. Consider

$$\Delta^2\left[\left(\frac{n}{2}\right)\Delta^2\left(y(n)+\frac{1}{3}(1+(-1)^n)y(n-2)\right)\right] = \frac{32}{3}n\,y^3(n-2)+\frac{32}{3}y^5(n+4), \quad n\geq 0, \quad (2.9)$$

where $0 \leq p(n) = \frac{1}{3}(1+(-1)^n) \leq \frac{2}{3}$, $a(n) = \frac{2}{n}$, $\alpha = 1$, $f(u) = u^3$ and $g(u) = u^5$. Clearly, all the conditions of Theorem 2.1 are satisfied. Hence (2.9) is oscillatory. In particular, $y(n) = (-1)^n$ is one of the oscillatory solutions of (2.9).

Theorem 2.6. *Let $-\infty < -b \leq p(n) \leq 0$, $b > 0$ and $\tau \leq \sigma_1$. Assume that (H_1) and (H_2) hold. If*

$$\Delta x(n) - r(n)g\left(A\left[n+\sigma_2, n+\frac{\sigma_2}{2}\right]\right)g\left(x^{\frac{1}{\alpha}}\left[n+\frac{\sigma_2}{2}\right]\right) = 0, \quad (2.10)$$

$$\Delta u(n) + q(n)f(B[n-\sigma_1,\ n_1])f\left(u^{\frac{1}{\alpha}}(n-\sigma_1)\right) = 0, \quad (2.11)$$

$$\Delta v(n) + q(n)f\left(C\left[n-\frac{\sigma_1}{2}, n-\sigma_1\right]\right)f\left(v^{\frac{1}{\alpha}}\left(n-\frac{\sigma_1}{2}\right)\right) = 0, \quad (2.12)$$

and

$$\Delta w(n) + q(n)f\left(\frac{1}{b}\right)f(D[n+\tau-\sigma_1-1, n+2(\tau-\sigma_1-1)])\times \times f\left(w^{\frac{1}{\alpha}}(n+\tau-\sigma_1-1)\right) = 0 \quad (2.13)$$

are oscillatory, then every solution of (1.3) *oscillates.*

Proof. Let $y(n)$ be a nontrivial eventually positive solution of (1.3). Proceeding as in the proof of Theorem 2.2, we have three Cases (a), (b) and (c).

For Case (a), $L_i z(n) > 0$, $i = 1, 2, 3$ for $n \geq n_1$. Then $z(k) \geq A[k,m]L_3^{\frac{1}{\alpha}}z(m)$ for $k \geq m+2 \geq n_1+2$. Hence

$$z(n+\sigma_2) \geq A\left[n+\sigma_2, n+\frac{\sigma_2}{2}\right]L_3^{\frac{1}{\alpha}}z\left(n+\frac{\sigma_2}{2}\right)$$

for $n+\sigma_2 \geq n+\frac{\sigma_2}{2}+2 \geq n_1+2$. Consequently, (1.3) yields

$$\begin{aligned}\Delta L_3 z(n) &\geq r(n)g(y(n+\sigma_2))\\ &\geq r(n)g(z(n+\sigma_2))\\ &\geq r(n)g\left(A\left[n+\sigma_2, n+\frac{\sigma_2}{2}\right]\right) g\left(L_3^{\frac{1}{\alpha}} z\left(n+\frac{\sigma_2}{2}\right)\right)\end{aligned}$$

for $n \geq n_1$ due to (H_1). Thus

$$\Delta L_3 z(n) - r(n)g\left(A\left[n+\sigma_2, n+\frac{\sigma_2}{2}\right]\right) g\left(L_3^{\frac{1}{\alpha}} z\left(n+\frac{\sigma_2}{2}\right)\right) \geq 0$$

has a positive solution $L_3 z(n)$ for $n \geq n_1$. By [6, Corollary 7.4.1], (2.10) has an eventually positive solution, a contradiction to our hypothesis. Case (b) and Case (c) are similar.

Next, we suppose that $z(n) < 0$ for $n \geq n_0$. Hence we have three Cases (a), (b) and (c) and out of which Case (c) is the desired case. Proceeding as in Theorem 2.2, we obtain

$$L_4 z(n) \geq q(n) f\left(\frac{1}{b}\right) f\left(-L_3^{\frac{1}{\alpha}} z(n+\tau-\sigma_1-1)\right) \times \\ \times f\left[D(n+\tau-\sigma_1-1, n+2(\tau-\sigma_1-1))\right]$$

for $n \geq n_1$. Therefore,

$$\Delta L_3 z(n) + f\left(\frac{1}{b}\right) q(n) f\left[D(n+\tau-\sigma_1-1, n+2(\tau-\sigma_1-1))\right] \times \\ \times f\left(-L_3^{\frac{1}{\alpha}} z(n+\tau-\sigma_1-1)\right) \geq 0.$$

By [6, Corollary 7.4.1], (2.13) has a negative solution, a contradiction to our hypothesis. This completes the proof. □

Theorem 2.7. *Let* $0 \leq p(n) \leq b < 1$. *Assume that* (H_1) *and* (H_2) *hold. If*

$$\Delta x(n) - g(1-b) r(n) g\left(A\left[n+\sigma_2, n+\frac{\sigma_2}{2}\right]\right) g\left(x^{\frac{1}{\alpha}}\left[n+\frac{\sigma_2}{2}\right]\right) = 0$$

and

$$\Delta u(n) + g(1-b) q(n) f(B[n-\sigma_1, n_1]) f\left(u^{\frac{1}{\alpha}}(n-\sigma_1)\right) = 0$$

are oscillatory, then every unbounded solution of (1.3) *is oscillatory.*

Proof. Suppose to the contrary that $y(n)$ is an unbounded nonoscillatory solution of (1.3) such that $y(n) > 0$ for $n \geq n_0$. Using the same type of reasoning as in Theorem

2.1, we consider three Cases (a), (b) and (c). For each of Cases (a) and (b), $z(n)$ is nondecreasing for $n \geq n_1 > n_0$. Hence for $n \geq n_1$,

$$\begin{aligned}(1-p(n))z(n) &< z(n)-p(n)z(n-\tau)\\ &= y(n)-p(n)p(n-\tau)y(n-2\tau) < y(n).\end{aligned}$$

It is easy to check Cases (a) and (b) as in the proofs of Theorems 2.1 and 2.6. For Case (c), $z(n)$ happens to be bounded due to $(-1)^i L_i z(n) > 0$, $i = 1, 2, 3$. Hence this case does not exist. The proof is complete. □

Remark 2.8. When $p(n) = 0$, the results of [5] follow from Theorems 2.2, 2.4 and 2.6. Hence for $p(n) \leq 0$, the work of [5] is a particular case of the present work. We note that Theorems 2.1, 2.3 and 2.7 are different in their own right.

Remark 2.9. Due to the method employed here, it is evident to consider the assumption (H_5). However, we cannot apply the present method in the case of

(H_{22}) $$\sum_{n=0}^{\infty} a^{\frac{1}{\alpha}}(n) < \infty.$$

It seems that some more conditions or a different method along with (H_{22}) is necessary to see that (1.3) is oscillatory.

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