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Time Scales Inequalities

George A. Anastassiou
University of Memphis
Department of Mathematical Sciences
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here first we collect and develop necessary background on time scales required for this article. Then we present time scales integral inequalities of types: Poincaré, Sobolev, Opial, Ostrowski and Hilbert–Pachpatte. We give also the generalized analogs of all these inequalities involving high order delta derivatives of functions on time scales. We finish with lots of applications: all these inequalities on the specific time scales $\mathbb{R}$, $\mathbb{Z}$ and $q^{\overline{\mathbb{Z}}}$, $q > 1$.

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1 Preliminaries

Here mainly we follow [6]. We are also inspired by [4, 5].

Definition 1.1. A time scale is an arbitrary nonempty closed subset of the real numbers, e.g., $\mathbb{R}$, $\mathbb{Z}$, $q^{\mathbb{N}_0} = \{q^k | k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, q > 1\}$.

Definition 1.2. If $\mathbb{T}$ is a time scale, then we define the forward jump operator $\sigma : \mathbb{T} \longmapsto \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T} | s > t\}$, $\forall t \in \mathbb{T}$; the backward jump operator $\rho : \mathbb{T} \longmapsto \mathbb{T}$ by $\rho(t) = \sup\{s \in \mathbb{T} | s < t\}$, $\forall t \in \mathbb{T}$; and the graininess function $\mu : \mathbb{T} \to \mathbb{R}_+ = [0, \infty)$, by $\mu(t) = \sigma(t) - t$, $\forall t \in \mathbb{T}$. Furthermore for a function $f : \mathbb{T} \to \mathbb{R}$, we define $f^\sigma(t) = f(\sigma(t))$, $\forall t \in \mathbb{T}$; and $f^\rho(t) = f(\rho(t))$, $\forall t \in \mathbb{T}$. In this definition we use $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if t is the maximum of $\mathbb{T}$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if t is the minimum of $\mathbb{T}$).

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We call $t \in \mathbb{T}$ right-scattered if $t < \sigma(t)$, $t \in \mathbb{T}$ right-dense if $t = \sigma(t)$, $t \in \mathbb{T}$ left-scattered if $\rho(r) < t$, $t \in \mathbb{T}$ left-dense if $\rho(t) = t$, $t \in \mathbb{T}$ isolated if $\rho(t) < t < \sigma(t)$, $t \in \mathbb{T}$ dense if $\rho(t) = t = \sigma(t)$. We notice that ρ is an increasing function, so is $\rho^2(t) = \rho(\rho(t)), \ldots$, so that $\rho^n(t) = \rho\left(\rho^{n-1}(t)\right)$ is increasing in t for $n \in \mathbb{N}$. Since $\mathbb{T}$ is closed subset of $\mathbb{R}$ we have that $\sigma(t)$, $\rho(t) \in \mathbb{T}$, for $t \in \mathbb{T}$.

Definition 1.3 (see [6]). A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous (denoted by C_{rd}) if it is continuous at right-dense points of $\mathbb{T}$ and its left-sided limits are finite at left-dense points of $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is rd-continuous iff f is continuous. Also, if $\mathbb{T} = \mathbb{Z}$, then any function defined on $\mathbb{Z}$ is rd-continuous (see [7]).

Definition 1.4 (see [6]). If $\sup \mathbb{T} < \infty$ and $\sup \mathbb{T}$ is left-scattered, we let $\mathbb{T}^k := \mathbb{T} - \{\sup \mathbb{T}\}$, otherwise we let $\mathbb{T}^k := \mathbb{T}$ the time scale. In summary,

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - (\rho(\sup \mathbb{T}), \sup \mathbb{T}], \text{ if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, \text{ if } \sup \mathbb{T} = \infty. \end{cases}$$

Definition 1.5 (see [6]). Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|, \quad \forall\, s \in U.$$

We call $f^{\Delta}(t)$ the delta (or Hilger [8]) derivative of f at t. If $\mathbb{T} = \mathbb{R}$, then $f^{\Delta} = f'$, whereas if $\mathbb{T} = \mathbb{Z}$, then $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$, the usual forward difference operator.

Theorem 1.6 (Existence of Antiderivatives [6]). *Let f be rd-continuous. Then f has an antiderivative F satisfying $F^{\Delta} = f$.*

Definition 1.7 (see [6]). If f is rd-continuous and $t_0 \in \mathbb{T}$, then we define the integral

$$F(t) = \int_{t_0}^{t} f(\tau)\,\Delta\tau \text{ for } t \in \mathbb{T}.$$

Therefore for $f \in C_{rd}(\mathbb{T})$ we have by definition

$$\int_a^b f(\tau)\,\Delta\tau = F(b) - F(a),$$

where $F^{\Delta} = f$.

If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,$$

where the integral on the right-hand side is the Riemann integral [7].

If every point in $\mathbb{T}$ is isolated and $a < b$ are in $\mathbb{T}$, then [7]

$$\int_a^b f(t)\,\Delta t = \sum_{t=a}^{\rho(b)} f(t)\,\mu(t)\,.$$

Theorem 1.8 (see [6]). *Let f, g be rd-continuous on $\mathbb{T}$, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then*

(1) $\displaystyle\int_a^b (\alpha f(t) + \beta g(t))\,\Delta t = \alpha \int_a^b f(t)\,\Delta t + \beta \int_a^b g(t)\,\Delta t,$

(2) $\displaystyle\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t,$

(3) $\displaystyle\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t,$

(4) $\displaystyle\int_a^b f(t)\,g^{\Delta}(t)\,\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)\,g(\sigma(t))\,\Delta t,$

(5) $\displaystyle\int_a^a f(t)\,\Delta t = 0,$

(6) $\displaystyle\int_a^b 1\Delta t = b - a.$

Theorem 1.9 (Hölder's Inequality [1]). *Let $a, b \in \mathbb{T}$, $a \leq b$, and $f, g : \mathbb{T} \to \mathbb{R}$ be rd-continuous. Then*

$$\int_a^b |f(t)|\,|g(t)|\,\Delta t \leq \left(\int_a^b |f(t)|^p\,\Delta t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,\Delta t\right)^{\frac{1}{q}},$$

where $p, q > 1$ satisfy $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 1.10 (see [6]). *Let $f, g \in C_{rd}(\mathbb{T})$, $a, b \in \mathbb{T}$, $a \leq b$. Then*

1) *if $|f(t)| \leq g(t)$ on $[a, b) \cap \mathbb{T}$, then* $\displaystyle\left|\int_a^b f(t)\,\Delta t\right| \leq \int_a^b g(t)\,\Delta t,$

2) *if $f(t) \geq 0$, for all $a \leq t < b$ and $t \in \mathbb{T}$, then* $\displaystyle\int_a^b f(t)\,\Delta t \geq 0.$

Corollary 1.11. *Let $f \in C_{rd}(\mathbb{T})\,;\, a, b, c \in \mathbb{T}$, with $c \in [a, b]$; $f(t) \geq 0$, $\forall\, t \in [a, b]$. Then*

$$\int_a^c f(t)\,\Delta t \leq \int_a^b f(t)\,\Delta t.$$

Definition 1.12 (see [6]). For a function $f : \mathbb{T} \to \mathbb{R}$ we consider the second derivative $f^{\Delta\Delta}$ provided f^{Δ} is differentiable on $\mathbb{T}^{k^2} = \left(\mathbb{T}^k\right)^k$ with derivative $f^{\Delta\Delta} = \left(f^{\Delta}\right)^{\Delta} : \mathbb{T}^{k^2} \to \mathbb{R}$. Similarly we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{k^n} \to \mathbb{R}$. Similarly we define $\sigma^2(t) = \sigma(\sigma(t)), \ldots, \sigma^n(t) = \sigma\left(\sigma^{n-1}(t)\right)$, $n \in \mathbb{N}$. For convenience we put $\rho^0(t) = \sigma^0(t) = t$, $f^{\Delta^0} = f$, $\mathbb{T}^{k^0} = \mathbb{T}$.

Notice $\mathbb{T}^{k^n} \subset \mathbb{T}^{k^l}$, $l \in \{0, 1, \ldots, n\}$.

Theorem 1.13 (Taylor's Formula [2]). *Let f be n-times differentiable on $\mathbb{T}^{k^n}$, $t \in \mathbb{T}$, and $\alpha \in \mathbb{T}^{k^{n-1}}$; $h_0(r,s) = 1$, $h_{k+1}(r,s) = \int_s^r h_k(\tau, s)\,\Delta\tau$, $k \in \mathbb{N}_0$. Then*

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha)\, f^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau.$$

Corollary 1.14 (see [2]). *Let f be n-times differentiable on $\mathbb{T}^{k^n}$ and $m \in \mathbb{N}$ with $m < n$. Then, $\forall\, \alpha \in \mathbb{T}^{k^{n-1-m}}$ and $t \in \mathbb{T}^{k^m}$, we have*

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} h_k(t,\alpha)\, f^{\Delta^{k+m}}(\alpha) + \int_{\alpha}^{\rho^{n-m-1}(t)} h_{n-m-1}(t, \sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau.$$

Denote by $C_{rd}^n(\mathbb{T})$ the space of all functions $f \in C_{rd}(\mathbb{T})$ such that $f^{\Delta^i} \in C_{rd}(\mathbb{T})$ for $i = 1, \ldots, n \in \mathbb{N}$. In this last case $\mathbb{T}^k = \mathbb{T}$.

We need the following result.

Theorem 1.15 (Taylor's Formula [3, 7]). *Assume $\mathbb{T}^k = \mathbb{T}$ and $f \in C_{rd}^n(\mathbb{T})$, $n \in \mathbb{N}$ and $s, t \in \mathbb{T}$. Here $h_0(t,s) = 1$, $\forall\, s, t \in \mathbb{T}$; $k \in \mathbb{N}_0$, and*

$$h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\,\Delta\tau, \quad \forall\, s, t \in \mathbb{T}.$$

(then $h_k^{\Delta}(t,s) = h_{k-1}(t,s)$, for $k \in \mathbb{N}$, $\forall\, t \in \mathbb{T}$, for each $s \in \mathbb{T}$ fixed). Then

$$f(t) = \sum_{k=0}^{n-1} f^{\Delta^k}(s)\, h_k(t,s) + \int_s^t h_{n-1}(t, \sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau.$$

Remark 1.16 *(to Theorem 1.15).* By [7], we have $h_1(t,s) = t - s$, $\forall\, s, t \in \mathbb{T}$. So if $t \geq s$, then $h_1(t,s) \geq 0$, $h_2(t,s) \geq 0, \ldots, h_{n-1}(t,s) \geq 0$. However for n odd, $h_{n-1}(t, \sigma(\tau)) \geq 0$ for all $s \leq \tau \leq t$ (see proof of Theorem 2.5). Also it holds [2]

$$h_k(t,s) \leq \frac{(t-s)^k}{k!}, \quad \forall\, t \geq s,\ k \in \mathbb{N}_0.$$

Corollary 1.17 (to Theorem 1.15). *Assume $f \in C_{rd}^n(\mathbb{T})$ and $s, t \in \mathbb{T}$. Let $m \in \mathbb{N}$ with $m < n$. Then*

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} f^{\Delta^{k+m}}(s)\, h_k(t,s) + \int_s^t h_{n-m-1}(t, \sigma(\tau))\, f^{\Delta^n}(\tau)\, \Delta\tau.$$

Proof. Use Theorem 1.15 with n and f substituted by $n-m$ and f^{Δ^m}, respectively. □

Corollary 1.18. *Let $f \in C_{rd}(\mathbb{T})$; $a, b \in \mathbb{T}$, such that $f(t) > 0$, $\forall\, t \in [a,b] \cap \mathbb{T}$, then $\int_a^b f(t)\, \Delta t > 0$.*

Proof. Since $f(t) > 0$, $\forall\, t \in [a,b] \cap \mathbb{T}$ by Theorem 1.10 (2) we get $\int_a^b f(t)\, \Delta t \geq 0$. Assume that $\int_a^b f(t)\, \Delta t = 0$. Then $F(t) = \int_a^t f(t)\, \Delta t = 0$, $\forall\, t \in [a,b] \cap \mathbb{T}$. Thus by [6] we get $F^{\Delta}(t) = f(t) = 0$, $\forall\, t \in [a,b] \cap \mathbb{T}$, a contradiction. □

We need the following result.

Lemma 1.19. *Let the time scale $\mathbb{T}$ be such that $\mathbb{T}^k = \mathbb{T}$. Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, such that $h_0(t,s) \equiv 1$, $\forall\, s, t \in \mathbb{T}$, and $h_{k+1}(t,s) = \int_s^t h_k(\tau, s)\, \Delta\tau$, $\forall\, s, t \in \mathbb{T}$, for all $k \in \mathbb{N}_0$. Then $h_k(t,s)$ is continuous in $s \in \mathbb{T}$, $k \in \mathbb{N}_0$, for each fixed $t \in \mathbb{T}$; and continuous in $t \in \mathbb{T}$ for each fixed $s \in \mathbb{T}$. Also it holds that $h_k(t, \sigma(s))$ is rd-continuous in $s \in \mathbb{T}$ for each fixed $t \in \mathbb{T}$; for all $k \in \mathbb{N}_0$.*

Proof. Consider also $g_k : \mathbb{T}^2 \to \mathbb{R}$, $k = 0, 1, \ldots, n$, such that $g_0(t,s) \equiv 1$, $\forall\, s, t \in \mathbb{T}$; and $g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau), s)\, \Delta\tau$, $\forall\, s, t \in \mathbb{T}$, for $k \in \mathbb{N}_0$. By [6], we have that

$$h_k^{\Delta}(t,s) = h_{k-1}(t,s), \ \ k \in \mathbb{N}, \forall\, t \in \mathbb{T},$$

for each fixed $s \in \mathbb{T}$. Also we have

$$g_k^{\Delta}(t,s) = g_{k-1}(\sigma(t), s), \quad k \in \mathbb{N}, \ \forall\, t \in \mathbb{T},$$

for each fixed $s \in \mathbb{T}$. Clearly $g_1(t,s) = h_1(t,s) = t - s$, $\forall\, s, t \in \mathbb{T}$. By [6, Theorem 1.112] we obtain that

$$h_k(t,s) = (-1)^k g_k(s,t), \quad \forall\, t, s \in \mathbb{T}, \text{for all } \ k \in \mathbb{N}_0.$$

By [6, Theorem 1.16(i)], we have that since g_k is differentiable for any $t \in \mathbb{T}$ (the first variable), then it is continuous for any $t \in \mathbb{T}$; for all $k \in \mathbb{N}_0$. Thus, by the last equation just above, we obtain that $h_k(t,s)$ is continuous in $s \in \mathbb{T}$; and of course h_k is also continuous in $t \in \mathbb{T}$; for all $k \in \mathbb{N}_0$. By [6, Theorem 1.60(iii)], we have that the jump operator σ is rd-continuous, and by the same [6, Theorem 1.60(v)], we get that $h_k(t, \sigma(s))$ is rd-continuous, for all $k \in \mathbb{N}_0$. The lemma now is proved. □

2 Main Results

In this article we assume $\mathbb{T}^k = \mathbb{T}$. We present first a time scales Poincaré type inequality.

Theorem 2.1. *Let* $f \in C_{rd}^n(\mathbb{T})$*,* n *is an odd number,* $a, b \in \mathbb{T}$*;* $a \le b$*;* $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$*. Assume* $f^{\Delta^k}(a) = 0$*,* $k = 0, 1, \ldots, n-1$*. Here* σ *is continuous and* $h_{n-1}(t, s)$ *jointly continuous. Then*

$$\int_a^b |f(t)|^q \Delta t \le \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{q}{p}} \Delta t \right) \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right).$$

Proof. Since $f^{\Delta^k}(a) = 0$, $k = 0, 1, \ldots, n-1$, by Theorem 1.15 we get

$$f(t) = \int_a^t h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau,$$

$\forall\, t \in [a, b] \cap \mathbb{T}$, where $a, b \in \mathbb{T}$. Hence

$$\begin{aligned} |f(t)| &\le \int_a^t h_{n-1}(t, \sigma(\tau)) \left| f^{\Delta^n}(\tau) \right| \Delta\tau \\ &\le \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_a^t \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right)^{\frac{1}{q}} \\ &\le \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right)^{\frac{1}{q}}. \end{aligned}$$

Therefore

$$|f(t)|^q \le \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{q}{p}} \left(\int_a^b \left| f^{\Delta^n}(\tau) \right|^q \Delta\tau \right), \tag{2.1}$$

for all $a \le t \le b$. Next by integrating (2.1) we are proving the claim. □

Next we give a time scales Sobolev type inequality.

Theorem 2.2. *Here all terms and assumptions are as in Theorem 2.1. Let* $r \ge 1$*. Denote* $\|f\|_r = \left(\int_a^b |f(t)|^r \Delta t \right)^{\frac{1}{r}}$*. Then*

$$\|f\|_r \le \left(\int_a^b \left(\int_a^t h_{n-1}(t, \sigma(\tau))^p \Delta\tau \right)^{\frac{r}{p}} \Delta t \right)^{\frac{1}{r}} \left\| f^{\Delta^n} \right\|_q.$$

Proof. As in the proof of Theorem 2.1 we have ($a \le t \le b$)

$$|f(t)| \le \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left(\int_a^b \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}}.$$

Hence

$$|f(t)|^r \le \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{r}{p}} \left(\int_a^b \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{r}{q}},$$

and

$$\int_a^b |f(t)|^r \Delta t \le \left(\int_a^b \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{r}{p}} \Delta t\right) \left(\int_a^b \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{r}{q}}.$$

Next raise both sides of the last inequality to the power $\frac{1}{r}$, thus proving the claim. □

We present a time scales Opial type inequality.

Theorem 2.3. *Let $f \in C_{rd}^n(\mathbb{T})$, n is an odd number, $a, b \in \mathbb{T}$; $a \le b$; $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{\Delta^k}(a) = 0$, $k = 0, 1, \ldots, n-1$, and that $\left|f^{\Delta^n}\right|$ is increasing on $[a,b] \cap \mathbb{T}$. Here σ is continuous and $h_{n-1}(t,s)$ jointly continuous. Then*

$$\int_a^b |f(t)| \left|f^{\Delta^n}(t)\right| \Delta t$$
$$\le (b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right) \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\Delta^n}(t)\right)^{2q} \Delta t\right)^{\frac{1}{q}}.$$

Proof. It holds

$$f(t) = \int_a^t h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau) \Delta\tau,$$

$\forall\, t \in [a,b] \cap \mathbb{T}$, where $a, b \in \mathbb{T}$. Hence

$$\begin{aligned} |f(t)| &\le \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left(\int_a^t \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right)^{\frac{1}{q}} \\ &\le \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left|f^{\Delta^n}(t)\right| (t-a)^{\frac{1}{q}}. \end{aligned}$$

Therefore

$$|f(t)| \left|f^{\Delta^n}(t)\right| \le \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau\right)^{\frac{1}{p}} \left(f^{\Delta^n}(t)\right)^2 (t-a)^{\frac{1}{q}},$$

for all $a \leq t \leq b$. Consequently we derive

$$\int_a^b |f(t)| \left|f^{\Delta^n}(t)\right| \Delta t \leq \left(\int_a^b \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right)^{\frac{1}{p}} \left(f^{\Delta^n}(t)\right)^2 (t-a)^{\frac{1}{q}} \right) \Delta t$$

$$\leq \left(\int_a^b \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right) \Delta t \right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\Delta^n}(t)\right)^{2q} (t-a) \Delta t \right)^{\frac{1}{q}}$$

$$\leq (b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau \right) \Delta t \right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\Delta^n}(t)\right)^{2q} \Delta t \right)^{\frac{1}{q}},$$

proving the claim. □

We make the following remark.

Remark 2.4 *(to Theorems 2.1–2.3 and their proofs).* As we know [6], we have that

$$h_{n-1}^{\Delta}(t,\sigma(\tau)) = h_{n-2}(t,\sigma(\tau)), \quad \forall\, t \in [a,b] \cap \mathbb{T}.$$

Also $(h_{n-1}(t,\sigma(t)))^p$ is continuous at (t,t), $t > a$; $p > 1$. By the chain rule, [6, Theorem 1.90], we get that $(h_{n-1}(t,\sigma(\tau))^p)^{\Delta}$ exists in $t \in \mathbb{T}$, where τ is fixed in $\mathbb{T}$; $p > 1$, and

$$\begin{aligned}((h_{n-1}(t,\sigma(\tau)))^p)^{\Delta} &= p\left\{ \int_0^1 (h_{n-1}(t,\sigma(\tau)) + \right. \\ &\quad \left. h\mu(t)h_{n-2}(t,\sigma(\tau)))^{p-1} dh \right\} h_{n-2}(t,\sigma(\tau)).\end{aligned}$$

Here by assumption σ is continuous and $h_{n-1}(t,s)$ is jointly continuous. So that $(h_{n-1}(t,\sigma(\tau)))^p$ is jointly continuous in (t,τ), that is rd-continuous in t and τ; $p \geq 1$. Here $\mathbb{T}^k = \mathbb{T}$, and by Lemma 1.19 we get that $h_{n-2}(t,\sigma(\tau))$ is continuous in t and τ. By bounded convergence theorem, using the last formula above, we get that $((h_{n-1}(t,\sigma(\tau)))^p)^{\Delta}$ is continuous in t and τ; $p > 1$, and thus rd-continuous in t and τ.

Consider now the function

$$u(t) = \int_a^t h_{n-1}(t,\sigma(\tau))^p \Delta\tau, \quad \forall\, t \in [a,b] \cap \mathbb{T}.$$

Clearly $u(a) = 0$. Furthermore, by [6, Theorem 1.117], we get

$$\begin{aligned}u^{\Delta}(t) &= \int_a^t (h_{n-1}(t,\sigma(\tau))^p)^{\Delta} \Delta\tau + (h_{n-1}(\sigma(t),\sigma(t)))^p \\ &= \int_a^t (h_{n-1}(t,\sigma(\tau))^p)^{\Delta} \Delta\tau.\end{aligned}$$

That is, $u(t)$ is differentiable, hence continuous and therefore rd-continuous on $[a,b] \cap \mathbb{T}$.

We proceed with a time scales Ostrowski type inequality.

Theorem 2.5. *Let $f \in C_{rd}^{n}(\mathbb{T})$, n is odd, $a, b, c \in \mathbb{T}$ with $a \leq c \leq b$. Assume that $f^{\Delta^k}(c) = 0$, $k = 1, \dots, n-1$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\Delta t - f(c)\right| \leq \frac{[h_{n+1}(a,c) + h_{n+1}(b,c)]}{b-a}\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

Proof. By assumptions and Theorem 1.15, we obtain

$$f(t) - f(c) = \int_c^t h_{n-1}(t, \sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau, \quad \forall\, t \in [a,b]\cap\mathbb{T}.$$

Hence

$$\begin{aligned} E(x) &:= \frac{1}{b-a}\int_a^b f(t)\,\Delta t - f(c) \\ &= \frac{1}{b-a}\int_a^b f(t)\,\Delta t - \frac{1}{b-a}\int_a^b f(c)\,\Delta t = \frac{1}{b-a}\int_a^b (f(t) - f(c))\,\Delta t. \end{aligned}$$

Thus

$$|E(x)| \leq \frac{1}{b-a}\int_a^b |f(t) - f(c)|\,\Delta t.$$

However we observe that ($c \leq t \leq b$)

$$\begin{aligned} |f(t) - f(c)| &\leq \int_c^t h_{n-1}(t,\sigma(\tau))\left|f^{\Delta^n}(\tau)\right|\Delta\tau \\ &\leq \left(\int_c^t h_{n-1}(t,\sigma(\tau))\,\Delta\tau\right)\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}. \end{aligned}$$

Also when $a \leq t \leq c$, we have

$$|f(t) - f(c)| = \left|\int_t^c h_{n-1}(t,\sigma(\tau))\, f^{\Delta^n}(\tau)\,\Delta\tau\right|$$

$$\leq \int_t^c |h_{n-1}(t,\sigma(\tau))|\left|f^{\Delta^n}(\tau)\right|\Delta\tau \leq \left(\int_t^c |h_{n-1}(t,\sigma(\tau))|\,\Delta\tau\right)\left\|f^{\Delta^n}\right\|_{\infty,[a,b]\cap\mathbb{T}}.$$

Since $h_1(t,s) = t - s$, if $t \leq s$, then $h_1(t,s) \leq 0$. Then

$$h_2(t,s) = \int_s^t h_1(\tau,s)\,\Delta\tau = -\int_t^s h_1(\tau,s)\,\Delta\tau = \int_t^s (-h_1(\tau,s))\,\Delta\tau \geq 0.$$

That is, $h_2(t,s) \geq 0$, for any $t, s \in \mathbb{T}$. We continue with ($t \leq s$)

$$h_3(t,s) = \int_s^t h_2(\tau,s)\,\Delta\tau = -\int_t^s h_2(\tau,s)\,\Delta\tau \leq 0.$$

Consequently by induction, we obtain ($t \leq s$)

$$|h_k(t,s)| = (-1)^k h_k(t,s), \quad k \in \mathbb{N}_0.$$

Thus $h_k(t,s) \geq 0$, for any $t, s \in \mathbb{T}$, when k is even. Therefore when $a \leq t \leq c$, we derive

$$|f(t) - f(c)| \leq \left(\int_t^c h_{n-1}(t, \sigma(\tau)) \Delta\tau \right) \left\| f^{\Delta^n} \right\|_{\infty, [a,b] \cap \mathbb{T}}.$$

By [6, (1.7), (1.8), (1.9)] and [6, Theorem 1.112], we notice that $(c \leq t \leq b)$

$$\begin{aligned} \int_c^t h_{n-1}(t, \sigma(\tau)) \Delta\tau &= \int_c^t g_{n-1}(\sigma(\tau), t) \Delta\tau \\ &= (-1)^n \int_t^c g_{n-1}(\sigma(\tau), t) \Delta\tau = (-1)^n g_n(c,t) = h_n(t,c). \end{aligned}$$

Also it holds $(a \leq t \leq c)$

$$\begin{aligned} (-1)^{n-1} \int_t^c h_{n-1}(t, \sigma(\tau)) \Delta\tau &= \int_t^c g_{n-1}(\sigma(\tau), t) \Delta\tau \\ &= g_n(c,t) = (-1)^n h_n(t,c). \end{aligned}$$

So we got that ($c \leq t \leq b$)

$$|f(t) - f(c)| \leq h_n(t,c) \left\| f^{\Delta^n} \right\|_{\infty, [a,b] \cap \mathbb{T}},$$

and ($a \leq t \leq c$)

$$|f(t) - f(c)| \leq (-1)^n h_n(t,c) \left\| f^{\Delta^n} \right\|_{\infty, [a,b] \cap \mathbb{T}}.$$

Thus we have

$$\begin{aligned} |E(x)| &\leq \frac{1}{b-a} \left[\int_a^c |f(t) - f(c)| \Delta t + \int_c^b |f(t) - f(c)| \Delta t \right] \\ &\leq \frac{1}{b-a} \left[(-1)^n \int_a^c h_n(t,c) \Delta t + \int_c^b h_n(t,c) \Delta t \right] \left\| f^{\Delta^n} \right\|_{\infty, [a,b] \cap \mathbb{T}} \\ &\leq \frac{\left[\int_c^a h_n(t,c) \Delta t + h_{n+1}(b,c) \right]}{b-a} \left\| f^{\Delta^n} \right\|_{\infty, [a,b] \cap \mathbb{T}} \\ &= \frac{[h_{n+1}(a,c) + h_{n+1}(b,c)]}{b-a} \left\| f^{\Delta^n} \right\|_{\infty, [a,b] \cap \mathbb{T}}, \end{aligned}$$

proving the claim. □

It follows a time scales Hilbert–Pachpatte type inequality.

Theorem 2.6. *Let* $\varepsilon > 0$*,* $i = 1, 2$*;* $f_i \in C_{rd}^n(\mathbb{T}_i)$*,* n *is odd, with* $f_i^{\Delta^k}(a_i) = 0$*,* $k = 0, 1, \ldots, n-1$*;* $a_i \leq b_i$*;* $a_i, b_i \in \mathbb{T}_i$*, time scale. Let also* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*. Call*

$$F(t_1) = \int_{a_1}^{t_1} h_{n-1}^{(1)}(t_1, \sigma_1(\tau_1))^p \, \Delta\tau_1,$$

for all $t_1 \in [a_1, b_1] \cap \mathbb{T}_1$*, and*

$$G(t_2) = \int_{a_2}^{t_2} h_{n-1}^{(2)}(t_2, \sigma_2(\tau_2))^q \, \Delta\tau_2,$$

for all $t_2 \in [a_2, b_2] \cap \mathbb{T}_2$ *(where* $h_{n-1}^{(i)}$*,* $\sigma^{(i)}$ *the corresponding* h_{n-1}*,* σ *to* $\mathbb{T}_i$*,* $i = 1, 2$*). Here* σ_i *is continuous and* $h_{n-1}^{(i)}(t_i, s_i)$ *jointly continuous in* $t_i, s_i \in \mathbb{T}_i$*. We further assume that*

$$\lambda(t_1) = \int_{a_2}^{b_2} \frac{|f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \Delta\tau_2$$

is an rd-continuous function on $\mathbb{T}_1$*. Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)|\,|f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} \Delta t_1 \Delta t_2$$
$$\leq (b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}}$$

(above double time scales integration is considered in the natural iterative way).

Proof. Since $f_i^{\Delta^k}(a_i) = 0$, $k = 0, 1, \ldots, n-1$; $i = 1, 2$, by Theorem 1.15 we get

$$f_i(t_i) = \int_{a_i}^{t_i} h_{n-1}^{(i)}(t_i, \sigma_i(\tau_i))\, f_i^{\Delta^n}(\tau_i)\, \Delta\tau_i,$$

$\forall\, t_i \in [a_i, b_i] \cap \mathbb{T}_i$, where $a_i, b_i \in \mathbb{T}_i$. Hence

$$\begin{aligned} |f_1(t_1)| &\leq \left(\int_{a_1}^{t_1} h_{n-1}^{(1)}(t_1, \sigma_1(\tau_1))^p \, \Delta\tau_1\right)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \\ &= F(t_1)^{\frac{1}{p}} \left(\int_{a_1}^{t_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}}, \end{aligned}$$

and

$$\begin{aligned} |f_2(t_2)| &\leq \left(\int_{a_2}^{t_2} h_{n-1}^{(2)}(t_2, \sigma_2(\tau_2))^q \, \Delta\tau_2\right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}} \\ &= G(t_2)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}}. \end{aligned}$$

Young's inequality for $a, b \geq 0$ says that

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}.$$

Therefore we have

$$\begin{aligned}
&|f_1(t_1)| \, |f_2(t_2)| \\
&\leq F(t_1)^{\frac{1}{p}} G(t_2)^{\frac{1}{q}} \left(\int_{a_1}^{t_1} \left| f_1^{\Delta^n}(\tau_1) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| f_2^{\Delta^n}(\tau_2) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}} \\
&\leq \left(\frac{F(t_1)}{p} + \frac{G(t_2)}{q} \right) \left(\int_{a_1}^{t_1} \left| f_1^{\Delta^n}(\tau_1) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| f_2^{\Delta^n}(\tau_2) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}}.
\end{aligned}$$

The last gives ($\varepsilon > 0$)

$$\frac{|f_1(t_1)| \, |f_2(t_2)|}{\varepsilon + \left(\frac{F(t_1)}{p} + \frac{G(t_2)}{q} \right)} \leq \left(\int_{a_1}^{t_1} \left| f_1^{\Delta^n}(\tau_1) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{t_2} \left| f_2^{\Delta^n}(\tau_2) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}},$$

for all $t_i \in [a_i, b_i] \cap \mathbb{T}_i$, $i = 1, 2$. Next we observe that

$$\begin{aligned}
&\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)| \, |f_2(t_2)|}{\varepsilon + \left(\frac{F(t_1)}{p} + \frac{G(t_2)}{q} \right)} \Delta t_1 \Delta t_2 \\
\leq\ & \left(\int_{a_1}^{b_1} \left(\int_{a_1}^{t_1} \left| f_1^{\Delta^n}(\tau_1) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}} \Delta t_1 \right) \left(\int_{a_2}^{b_2} \left(\int_{a_2}^{t_2} \left| f_2^{\Delta^n}(\tau_2) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}} \Delta t_2 \right) \\
\leq\ & \left(\int_{a_1}^{b_1} \left(\int_{a_1}^{t_1} \left| f_1^{\Delta^n}(\tau_1) \right|^q \Delta\tau_1 \right) \Delta t_1 \right)^{\frac{1}{q}} (b_1 - a_1)^{\frac{1}{p}} \times \\
& \times \left(\int_{a_2}^{b_2} \left(\int_{a_2}^{t_2} \left| f_2^{\Delta^n}(\tau_2) \right|^p \Delta\tau_2 \right) \Delta t_2 \right)^{\frac{1}{p}} (b_2 - a_2)^{\frac{1}{q}} \\
\leq\ & \left(\int_{a_1}^{b_1} \left(\int_{a_1}^{b_1} \left| f_1^{\Delta^n}(\tau_1) \right|^q \Delta\tau_1 \right) \Delta t_1 \right)^{\frac{1}{q}} (b_1 - a_1)^{\frac{1}{p}} \times \\
& \times \left(\int_{a_2}^{b_2} \left(\int_{a_2}^{b_2} \left| f_2^{\Delta^n}(\tau_2) \right|^p \Delta\tau_2 \right) \Delta t_2 \right)^{\frac{1}{p}} (b_2 - a_2)^{\frac{1}{q}} \\
=\ & (b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left| f_1^{\Delta^n}(\tau_1) \right|^q \Delta\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left| f_2^{\Delta^n}(\tau_2) \right|^p \Delta\tau_2 \right)^{\frac{1}{p}},
\end{aligned}$$

proving the claim. □

Based on Corollary 1.17 we get the following results. First we present a generalized time scales Poincaré type inequality.

Proposition 2.7. *Let* $f \in C_{rd}^{n}(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ *is odd,* $a, b \in \mathbb{T}$*;* $a \leq b$*;* $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$*. Assume* $f^{\Delta^{k+m}}(a) = 0$*,* $k = 0, 1, \ldots, n - m - 1$*. Here* σ *is continuous and* $h_{n-m-1}(t, s)$ *jointly continuous. Then*

$$\int_a^b \left|f^{\Delta^m}(t)\right|^q \Delta t \leq \left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta\tau\right)^{\frac{q}{p}} \Delta t\right) \left(\int_a^b \left|f^{\Delta^n}(\tau)\right|^q \Delta\tau\right).$$

Proof. As in Theorem 2.1. □

It follows a generalized time scales Sobolev type inequality.

Proposition 2.8. *Here all terms and assumptions are as in Proposition 2.7. Let* $r \geq 1$*. Then*

$$\left\|f^{\Delta^m}\right\|_r \leq \left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta\tau\right)^{\frac{r}{p}} \Delta t\right)^{\frac{1}{r}} \left\|f^{\Delta^n}\right\|_q.$$

Proof. As in Theorem 2.2. □

Next comes a generalized time scales Opial type inequality.

Proposition 2.9. *Let* $f \in C_{rd}^{n}(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ *is odd,* $a, b \in \mathbb{T}$; $a \leq b$*;* $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$*. Assume* $f^{\Delta^{k+m}}(a) = 0$*,* $k = 0, 1, \ldots, n - 1$*, and that* $\left|f^{\Delta^n}\right|$ *is increasing on* $[a, b] \cap \mathbb{T}$*. Here* σ *is continuous and* $h_{n-m-1}(t, s)$ *jointly continuous. Then*

$$\int_a^b \left|f^{\Delta^m}(t)\right| \left|f^{\Delta^n}(t)\right| \Delta t \leq (b-a)^{\frac{1}{q}} \left(\int_a^b \left(\int_a^t h_{n-m-1}(t, \sigma(\tau))^p \Delta\tau\right) \Delta t\right)^{\frac{1}{p}} \left(\int_a^b \left(f^{\Delta^n}(t)\right)^{2q} \Delta t\right)^{\frac{1}{q}}.$$

Proof. As in Theorem 2.3. □

We continue with a generalized Ostrowski type inequality over time scales.

Proposition 2.10. *Let* $f \in C_{rd}^{n}(\mathbb{T})$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ *is odd,* $a, b, c \in \mathbb{T}$ *with* $a \leq c \leq b$*. Assume that* $f^{\Delta^{k+m}}(c) = 0$*,* $k = 1, \ldots, n - m - 1$*. Then*

$$\left|\frac{1}{b-a} \int_a^b f^{\Delta^m} \Delta t - f^{\Delta^m}(c)\right| \leq \frac{[h_{n-m+1}(a, c) + h_{n-m+1}(b, c)]}{b-a} \left\|f^{\Delta^n}\right\|_{\infty, [a,b] \cap \mathbb{T}}.$$

Proof. As in Theorem 2.5. □

We finish with the generalized Hilbert–Pachpatte type inequality on time scales.

Proposition 2.11. *Let* $\varepsilon > 0$, $i = 1, 2$; $f_i \in C_{rd}^{n}(\mathbb{T}_i)$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ *is odd, with* $f_i^{\Delta^{k+m}}(a_i) = 0$, $k = 0, 1, \ldots, n - m - 1$; $a_i \le b_i$; $a_i, b_i \in \mathbb{T}_i$, *time scale. Let also* $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *Call*

$$F^*(t_1) = \int_{a_1}^{t_1} h_{n-m-1}^{(1)}(t_1, \sigma_1(\tau_1))^p \, \Delta\tau_1,$$

for all $t_1 \in [a_1, b_1] \cap \mathbb{T}_1$, *and*

$$G^*(t_2) = \int_{a_2}^{t_2} h_{n-m-1}^{(2)}(t_2, \sigma_2(\tau_2))^q \, \Delta\tau_2,$$

for all $t_2 \in [a_2, b_2] \cap \mathbb{T}_2$ *(where* $h_{n-m-1}^{(i)}$, $\sigma^{(i)}$ *the corresponding* h_{n-m-1}, σ *to* $\mathbb{T}_i$, $i = 1, 2$*). Here* σ_i *is continuous and* $h_{n-m-1}^{(i)}(t_i, s_i)$ *jointly continuous in* $t_i, s_i \in \mathbb{T}_i$. *We further assume that*

$$\lambda^*(t_1) = \int_{a_2}^{b_2} \frac{\left|f_2^{\Delta^m}(t_2)\right|}{\left(\varepsilon + \frac{F^*(t_1)}{p} + \frac{G^*(t_2)}{q}\right)} \Delta\tau_2$$

is an rd-continuous function on $\mathbb{T}_1$. *Then*

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{\left|f_1^{\Delta^m}(t_1)\right| \left|f_2^{\Delta^m}(t_2)\right|}{\left(\varepsilon + \frac{F^*(t_1)}{p} + \frac{G^*(t_2)}{q}\right)} \Delta t_1 \Delta t_2$$
$$\le (b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left|f_1^{\Delta^n}(\tau_1)\right|^q \Delta\tau_1\right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left|f_2^{\Delta^n}(\tau_2)\right|^p \Delta\tau_2\right)^{\frac{1}{p}}.$$

Proof. As in Theorem 2.6. □

3 Applications

We need the following remark.

Remark 3.1 *(see [6]).* i) When $\mathbb{T} = \mathbb{R}$, then $h_k(t, s) = \frac{(t-s)^k}{k!}$, $\forall\, k \in \mathbb{N}_0$, $\forall\, t, s \in \mathbb{R}$, $\sigma(t) = t$, $\int_a^b f(t)\, \Delta t = \int_a^b f(t)\, dt$, $f^{\Delta}(t) = f'(t)$, $f^{\Delta^k} = f^{(k)}$; rd-continuous corresponds to f continuous.

ii) When $\mathbb{T} = \mathbb{Z}$, $h_k(t,s) = \dfrac{(t-s)^{(k)}}{k!}$, $\forall\, k \in \mathbb{N}_0$, $\forall\, t,s \in \mathbb{Z}$, where $t^{(0)} = 1$, $t^{(k)} = \prod_{i=0}^{k-1}(t-i)$ for $k \in \mathbb{N}$, $\sigma(t) = t+1$,

$$\int_a^b f(t)\,\Delta t = \sum_{t=a}^{b-1} f(t), \quad a < b,$$

$$f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t),$$

$$f^{\Delta^k}(t) = \Delta^k f(t) = \sum_{l=0}^{k} \begin{pmatrix} k \\ l \end{pmatrix} (-1)^{k-l} f(t+l),$$

rd-continuous f corresponds to any f.

Next we present a Poincaré inequality.

Corollary 3.2. *Let* $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $a,b \in \mathbb{R}$; $a \leq b$; $p,q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume* $f^{(k)}(a) = 0$, $k = 0,1,\ldots,n-1$. *Then*

$$\int_a^b |f(t)|^q\,dt \leq \frac{(b-a)^{nq}}{((n-1)!)^q\,(p(n-1)+1)^{(q-1)}\,nq} \left(\int_a^b \left|f^{(n)}(t)\right|^q dt\right). \tag{3.1}$$

Proof. Based on Theorem 2.1 and Remark 3.1 (i). □

A discrete Poincaré inequality follows.

Corollary 3.3. *Let* $f : \mathbb{Z} \to \mathbb{R}$, n *is odd*, $a,b \in \mathbb{Z}$; $a \leq b$; $p,q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume* $\Delta^k f(a) = 0$, $k = 0,1,\ldots,n-1$. *Then*

$$\sum_{t=a}^{b-1} |f(t)|^q \leq \frac{1}{((n-1)!)^q} \left(\sum_{t=a}^{b-1} \left(\sum_{\tau=a}^{t-1} \left((t-\tau-1)^{(n-1)}\right)^p\right)^{\frac{q}{p}}\right) \left(\sum_{\tau=a}^{b-1} |\Delta^n f(\tau)|^q\right).$$

Proof. Based on Theorem 2.1 and Remark 3.1 (ii). □

A Sobolev inequality is presented next.

Corollary 3.4. *All as in Corollary 3.2. Let* $r \geq 1$. *Then*

$$\left(\int_a^b |f(t)|^r\,dt\right)^{\frac{1}{r}} \leq \frac{(b-a)^{\left(n-1+\frac{1}{p}+\frac{1}{r}\right)}}{(n-1)!\,((n-1)p+1)^{\frac{1}{p}} \left(\left(n-1+\frac{1}{p}\right)r+1\right)^{\frac{1}{r}}} \left(\int_a^b \left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}.$$

Proof. Based on Theorem 2.2 and Remark 3.1 (i). □

A discrete Sobolev inequality follows.

Corollary 3.5. *All as in Corollary 3.3 and let* $r \geq 1$. *Then*

$$\left(\sum_{t=a}^{b-1} |f(t)|^r \right)^{\frac{1}{r}} \leq \frac{1}{(n-1)!} \left(\sum_{t=a}^{b-1} \left(\sum_{\tau=a}^{t-1} \left((t-\tau-1)^{(n-1)} \right)^p \right)^{\frac{r}{p}} \right)^{\frac{1}{r}} \left(\sum_{t=a}^{b-1} |\Delta^n f(t)|^q \right)^{\frac{1}{q}}.$$

Proof. Based on Theorem 2.2 and Remark 3.1 (ii). □

An Opial inequality follows.

Corollary 3.6. *Let* $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $a, b \in \mathbb{R}$; $a \leq b$; $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume* $f^{(k)}(a) = 0$, $k = 0, 1, \ldots, n-1$, *and* $\left| f^{(n)} \right|$ *is increasing on* $[a, b]$. *Then*

$$\int_a^b |f(t)| \left| f^{(n)}(t) \right| dt \leq \frac{(b-a)^{n+\frac{1}{p}}}{(n-1)! \left[((n-1)p+1)((n-1)p+2) \right]^{\frac{1}{p}}} \left(\int_a^b \left(f^{(n)}(t) \right)^{2q} dt \right)^{\frac{1}{q}}.$$

Proof. Based on Theorem 2.3 and Remark 3.1 (i). □

A discrete Opial inequality follows.

Corollary 3.7. *Let* $f : \mathbb{Z} \to \mathbb{R}$, n *is odd,* $a, b \in \mathbb{Z}$; $a \leq b$; $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume* $\Delta^k f(a) = 0$, $k = 0, 1, \ldots, n-1$, *and that* $|\Delta^n f|$ *is increasing on* $[a, b]$. *Then*

$$\sum_{t=a}^{b-1} |f(t)| |\Delta^n f(t)| \leq \frac{(b-a)^{\frac{1}{q}}}{(n-1)!} \left(\sum_{t=a}^{b-1} \left(\sum_{\tau=a}^{t-1} \left((t-\tau-1)^{(n-1)} \right)^p \right)^{\frac{1}{p}} \right) \left(\sum_{t=a}^{b-1} (\Delta^n f(t))^{2q} \right)^{\frac{1}{q}}.$$

Proof. By Theorem 2.3 and Remark 3.1 (ii). □

An Ostrowski inequality follows.

Corollary 3.8. *Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, $a, b, c \in \mathbb{R}$ with $a \leq c \leq b$. Assume that $f^{(k)}(c) = 0$, $k = 1, \dots, n-1$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(c) \right| \leq \left[\frac{\left((c-a)^{n+1} + (b-c)^{n+1}\right)}{(n+1)!\,(b-a)} \right] \left\| f^{(n)} \right\|_{\infty,[a,b]}.$$

Proof. Based on Theorem 2.5 and Remark 3.1 (i). □

A discrete Ostrowski inequality follows.

Corollary 3.9. *Let $f : \mathbb{Z} \to \mathbb{R}$, n is odd, $a, b, c \in \mathbb{Z}$ with $a \leq c \leq b$. Assume that $\Delta^k f(c) = 0$, $k = 1, \dots, n-1$. Then*

$$\left| \frac{1}{b-a} \sum_{t=a}^{b-1} f(t) - f(c) \right| \leq \left[\frac{(a-c)^{(n+1)} + (b-c)^{(n+1)}}{(n+1)!\,(b-a)} \right] \|\Delta^n f\|_{\infty,[a,b]}.$$

Proof. By Theorem 2.5 and Remark 3.1 (ii). □

A Hilbert–Pachpatte inequality is next.

Corollary 3.10. *Let $\varepsilon > 0$, $i = 1, 2$; $f_i \in C^n(\mathbb{R})$, $n \in \mathbb{N}$, with $f_i^{(k)}(a_i) = 0$, $k = 0, 1, \dots, n-1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{R}$. Let $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Call*

$$F(t_1) = \frac{1}{((n-1)!)^p} \frac{(t_1 - a_1)^{p(n-1)+1}}{(p(n-1)+1)}, \quad \forall\, t_1 \in [a_1, b_1],$$

$$G(t_2) = \frac{1}{((n-1)!)^q} \frac{(t_2 - a_2)^{q(n-1)+1}}{(q(n-1)+1)}, \quad \forall\, t_2 \in [a_2, b_2].$$

Then

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{|f_1(t_1)|\,|f_2(t_2)|}{\left(\varepsilon + \frac{F(t_1)}{p} + \frac{G(t_2)}{q}\right)} dt_1 dt_2$$
$$\leq (b_1 - a_1)(b_2 - a_2) \left(\int_{a_1}^{b_1} \left| f_1^{(n)}(\tau_1) \right|^q d\tau_1 \right)^{\frac{1}{q}} \left(\int_{a_2}^{b_2} \left| f_2^{(n)}(\tau_2) \right|^p d\tau_2 \right)^{\frac{1}{p}}.$$

Proof. Based on Theorem 2.6 and Remark 3.1 (i). Notice here that $\lambda(t_1)$ is a continuous function on $[a_1, b_1]$ by bounded convergence theorem. □

It follows a discrete Hilbert–Pachpatte inequality.

Corollary 3.11. *Let* $\varepsilon > 0$*,* $i = 1, 2$*;* $f_i : \mathbb{Z} \to \mathbb{R}$*,* n *is odd, with* $\Delta^k f_i(a_i) = 0$*,* $k = 0, 1, \ldots, n-1$*;* $a_i \leq b_i$*;* $a_i, b_i \in \mathbb{Z}$*. Let* $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$*. Call*

$$\overline{F}(t_1) = \frac{\sum_{\tau_1=a_1}^{t_1-1} \left((t_1 - \tau_1 - 1)^{(n-1)} \right)^p}{((n-1)!)^p}, \quad \forall\, t_1 \in [a_1, b_1] \cap \mathbb{Z},$$

and

$$\overline{G}(t_2) = \frac{\sum_{\tau_2=a_2}^{t_2-1} \left((t_2 - \tau_2 - 1)^{(n-1)} \right)^q}{((n-1)!)^q}, \quad \forall\, t_2 \in [a_2, b_2] \cap \mathbb{Z}.$$

Then

$$\sum_{t_1=a_1}^{b_1-1} \sum_{t_2=a_2}^{b_2-1} \frac{|f_1(t_1)|\,|f_2(t_2)|}{\left(\varepsilon + \frac{\overline{F}(t_1)}{p} + \frac{\overline{G}(t_2)}{q} \right)}$$
$$\leq (b_1 - a_1)(b_2 - a_2) \left(\sum_{\tau_1=a_1}^{b_1-1} |\Delta^n f_1(\tau_1)|^q \right)^{\frac{1}{q}} \left(\sum_{\tau_2=a_2}^{b_2-1} |\Delta^n f_2(\tau_2)|^p \right)^{\frac{1}{p}}.$$

Proof. By Theorem 2.6 and Remark 3.1 (ii). □

Another generalized Poincaré inequality is next.

Corollary 3.12. *Let* $f \in C^n(\mathbb{R})$*,* $m, n \in \mathbb{N}$*,* $m < n$*,* $a, b \in \mathbb{R}$*;* $a \leq b$*;* $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$*. Assume* $f^{(k+m)}(a) = 0$*,* $k = 0, 1, \ldots, n-m-1$*. Then*

$$\int_a^b \left| f^{(m)}(t) \right|^q dt$$
$$\leq \frac{(b-a)^{(n-m)q}}{((n-m-1)!)^q (p(n-m-1)+1)^{(q-1)} (n-m)\, q} \left(\int_a^b \left| f^{(n)}(t) \right|^q dt \right).$$

Proof. By Corollary 3.2, $n \mapsto n-m$, $f \mapsto f^{(m)}$, $f^{(k)} \mapsto f^{(k+m)}$ into (3.1). □

A generalized discrete Poincaré inequality follows.

Corollary 3.13. *Let* $f : \mathbb{Z} \to \mathbb{R}$*,* $m, n \in \mathbb{N}$*,* $m < n$*,* $n - m$ *is odd,* $a, b \in \mathbb{Z}$*;* $a \leq b$*;* $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$*. Assume* $\Delta^{k+m} f(a) = 0$*,* $k = 0, 1, \ldots, n-m-1$*. Then*

$$\sum_{t=a}^{b-1} |\Delta^m f(t)|^q$$
$$\leq \frac{1}{((n-m-1)!)^q} \left(\sum_{t=a}^{b-1} \left(\sum_{\tau=a}^{t-1} \left((t - \tau - 1)^{(n-m-1)} \right)^p \right)^{\frac{q}{p}} \right) \left(\sum_{\tau=a}^{b-1} |\Delta^n f(\tau)|^q \right).$$

Proof. By Corollary 3.3. □

A generalized Sobolev inequality is next.

Corollary 3.14. *All as in Corollary 3.12, $r \geq 1$. Then*

$$\left(\int_a^b \left|f^{(m)}(t)\right|^r dt\right)^{\frac{1}{r}} \leq \frac{(b-a)^{\left(n-m-1+\frac{1}{p}+\frac{1}{r}\right)} \left(\int_a^b \left|f^{(n)}(t)\right|^q dt\right)^{\frac{1}{q}}}{(n-m-1)!\,((n-m-1)p+1)^{\frac{1}{p}} \left(\left(n-m-1+\frac{1}{p}\right)r+1\right)^{\frac{1}{r}}}.$$

Proof. By Corollary 3.4. □

A generalized discrete Sobolev inequality follows.

Corollary 3.15. *All as in Corollary 3.13, $r \geq 1$. Then*

$$\left(\sum_{t=a}^{b-1} \left|\Delta^m f(t)\right|^r\right)^{\frac{1}{r}} \leq \frac{1}{(n-m-1)!} \left(\sum_{t=a}^{b-1} \left(\sum_{\tau=a}^{t-1} \left((t-\tau-1)^{(n-m-1)}\right)^p\right)^{\frac{r}{p}}\right)^{\frac{1}{r}} \left(\sum_{t=a}^{b-1} \left|\Delta^n f(t)\right|^q\right)^{\frac{1}{q}}.$$

Proof. By Corollary 3.5. □

A generalized Opial inequality follows.

Corollary 3.16. *Let $f \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$, $a, b \in \mathbb{R}$; $a \leq b$; $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Assume $f^{(k+m)}(a) = 0$, $k = 0, 1, \ldots, n-m-1$, and $\left|f^{(n)}\right|$ is increasing on $[a, b]$. Then*

$$\int_a^b \left|f^{(m)}(t)\right| \left|f^{(n)}(t)\right| dt \leq \frac{(b-a)^{n-m+\frac{1}{p}} \left(\int_a^b \left(f^{(n)}(t)\right)^{2q} dt\right)^{\frac{1}{q}}}{(n-m-1)!\,\left[((n-m-1)p+1)((n-m-1)p+2)\right]^{\frac{1}{p}}}.$$

Proof. By Corollary 3.6. □

A generalized discrete Opial inequality follows.

Corollary 3.17. *Let* $f : \mathbb{Z} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ *is odd,* $a, b \in \mathbb{Z}$; $a \leq b$; $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *Assume* $\Delta^{k+m} f(a) = 0$, $k = 0, 1, \ldots, n - m - 1$, *and that* $|\Delta^n f|$ *is increasing on* $[a, b]$. *Then*

$$\sum_{t=a}^{b-1} |\Delta^m f(t)| |\Delta^n f(t)|$$
$$\leq \frac{(b-a)^{\frac{1}{q}}}{(n-m-1)!} \left(\sum_{t=a}^{b-1} \left(\sum_{\tau=a}^{t-1} \left((t-\tau-1)^{(n-m-1)} \right)^p \right) \right)^{\frac{1}{p}} \left(\sum_{t=a}^{b-1} (\Delta^n f(t))^{2q} \right)^{\frac{1}{q}}.$$

Proof. By Corollary 3.7. □

A generalized Ostrowski inequality follows.

Corollary 3.18. *Let* $f \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$, $a, b, c \in \mathbb{R}$ *with* $a \leq c \leq b$. *Assume that* $f^{(k+m)}(c) = 0$, $k = 1, \ldots, n - m - 1$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f^{(m)}(t)\, dt - f^{(m)}(c) \right|$$
$$\leq \left[\frac{\left((c-a)^{n-m+1} + (b-c)^{n-m+1} \right)}{(n-m+1)!\,(b-a)} \right] \left\| f^{(n)} \right\|_{\infty,[a,b]}.$$

Proof. By Corollary 3.8. □

A generalized discrete Ostrowski inequality follows.

Corollary 3.19. *Let* $f : \mathbb{Z} \to \mathbb{R}$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ *is odd,* $a, b, c \in \mathbb{Z}$ *with* $a \leq c \leq b$. *Assume that* $\Delta^{k+m} f(c) = 0$, $k = 1, \ldots, n - m - 1$. *Then*

$$\left| \frac{1}{b-a} \sum_{t=a}^{b-1} \Delta^m f(t) - \Delta^m f(c) \right|$$
$$\leq \left[\frac{(a-c)^{(n-m+1)} + (b-c)^{(n-m+1)}}{(n-m+1)!\,(b-a)} \right] \|\Delta^n f\|_{\infty,[a,b]}.$$

Proof. By Corollary 3.9. □

A generalized Hilbert–Pachpatte is next.

Corollary 3.20. *Let* $\varepsilon > 0$, $i = 1, 2$; $f_i \in C^n(\mathbb{R})$, $m, n \in \mathbb{N}$, $m < n$, *with* $f_i^{(k+m)}(a_i) = 0$, $k = 0, 1, \ldots, n - m - 1$; $a_i \leq b_i$; $a_i, b_i \in \mathbb{R}$. *Let* $p, q > 1$ *satisfy* $\frac{1}{p} + \frac{1}{q} = 1$. *Call*

$$F^*(t_1) = \frac{1}{((n-m-1)!)^p} \frac{(t_1 - a_1)^{p(n-m-1)+1}}{(p(n-m-1)+1)}, \quad \forall\, t_1 \in [a_1, b_1],$$

$$G^{*}\left(t_{2}\right)=\frac{1}{((n-m-1)!)^{q}}\frac{\left(t_{2}-a_{2}\right)^{q(n-m-1)+1}}{\left(q\left(n-m-1\right)+1\right)},\quad \forall\, t_{2}\in\left[a_{2},b_{2}\right].$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\left|f_1^{(m)}\left(t_1\right)\right|\left|f_2^{(m)}\left(t_2\right)\right|}{\left(\varepsilon+\frac{F^{*}(t_1)}{p}+\frac{G^{*}(t_2)}{q}\right)}dt_1dt_2$$
$$\leq\left(b_1-a_1\right)\left(b_2-a_2\right)\left(\int_{a_1}^{b_1}\left|f_1^{(n)}\left(\tau_1\right)\right|^{q}d\tau_1\right)^{\frac{1}{q}}\left(\int_{a_2}^{b_2}\left|f_2^{(n)}\left(\tau_2\right)\right|^{p}d\tau_2\right)^{\frac{1}{p}}.$$

Proof. By Corollary 3.10. □

It follows a generalized discrete Hilbert–Pachpatte inequality.

Corollary 3.21. *Let* $\varepsilon>0$*,* $i=1,2$*;* $f_i:\mathbb{Z}\to\mathbb{R}$*,* $m,n\in\mathbb{N}$*,* $m<n$*,* $n-m$ *is odd, with* $\Delta^{k+m}f_i\left(a_i\right)=0$*,* $k=0,1,\ldots,n-m-1$*;* $a_i\leq b_i$*;* $a_i,b_i\in\mathbb{Z}$*. Let* $p,q>1$ *satisfy* $\frac{1}{p}+\frac{1}{q}=1$*. Call*

$$\overline{F}^{*}\left(t_{1}\right)=\frac{\sum_{\tau_1=a_1}^{t_1-1}\left(\left(t_1-\tau_1-1\right)^{(n-m-1)}\right)^{p}}{((n-m-1)!)^{p}},\quad \forall\, t_1\in\left[a_1,b_1\right]\cap\mathbb{Z},$$

and

$$\overline{G}^{*}\left(t_{2}\right)=\frac{\sum_{\tau_2=a_2}^{t_2-1}\left(\left(t_2-\tau_2-1\right)^{(n-m-1)}\right)^{q}}{((n-m-1)!)^{q}},\quad \forall\, t_2\in\left[a_2,b_2\right]\cap\mathbb{Z}.$$

Then

$$\sum_{t_1=a_1}^{b_1-1}\sum_{t_2=a_2}^{b_2-1}\frac{\left|\Delta^{m}f_1\left(t_1\right)\right|\left|\Delta^{m}f_2\left(t_2\right)\right|}{\left(\varepsilon+\frac{\overline{F}^{*}(t_1)}{p}+\frac{\overline{G}^{*}(t_2)}{q}\right)}$$
$$\leq\left(b_1-a_1\right)\left(b_2-a_2\right)\left(\sum_{\tau_1=a_1}^{b_1-1}\left|\Delta^{n}f_1\left(\tau_1\right)\right|^{q}\right)^{\frac{1}{q}}\left(\sum_{\tau_2=a_2}^{b_2-1}\left|\Delta^{n}f_2\left(\tau_2\right)\right|^{p}\right)^{\frac{1}{p}}.$$

Proof. By Corollary 3.11. □

Remark 3.22 *(see [2, 6]).* Consider $q>1$, $q^{\mathbb{Z}}=\{q^{k}:k\in\mathbb{Z}\}$, and the time scale $\mathbb{T}=q^{\overline{\mathbb{Z}}}=q^{\mathbb{Z}}\cup\{0\}$, which is very important in q-difference equations. It holds [2, 6] that

$$h_k\left(t,s\right)=\prod_{\nu=0}^{k-1}\frac{t-q^{\nu}s}{\sum_{\mu=0}^{\nu}q^{\mu}},\quad \forall\, s,t\in\mathbb{T};$$

$$\sigma(t) = qt, \;\; \rho(t) = \frac{t}{q}, \;\; \forall\, t \in \mathbb{T},$$

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)\,t}, \;\; \forall\, t \in \mathbb{T} - \{0\},$$

$$f^{\Delta}(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s}.$$

We give a related q-Ostrowski type inequality.

Corollary 3.23. *Let* $f \in C^n_{rd}\left(q^{\overline{\mathbb{Z}}}\right)$, n *is odd,* $a, b, c \in q^{\overline{\mathbb{Z}}}$ *with* $a \le c \le b$. *Assume that* $f^{\Delta^k}(c) = 0$, $k = 1, \ldots, n-1$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\Delta t - f(c) \right| \le \left[\frac{\prod_{\nu=0}^{n} \frac{a - q^{\nu} c}{\sum_{\mu=0}^{\nu} q^{\mu}} + \prod_{\nu=0}^{n} \frac{b - q^{\nu} c}{\sum_{\mu=0}^{\nu} q^{\mu}}}{b-a} \right] \left\| f^{\Delta^n} \right\|_{\infty, [a,b] \cap q^{\overline{\mathbb{Z}}}}.$$

Proof. By Theorem 2.5. □

We finish with a generalized q-Ostrowski type inequality.

Corollary 3.24. *Let* $f \in C^n_{rd}\left(q^{\overline{\mathbb{Z}}}\right)$, $m, n \in \mathbb{N}$, $m < n$, $n - m$ *is odd,* $a, b, c \in q^{\overline{\mathbb{Z}}}$ *with* $a \le c \le b$. *Assume that* $f^{\Delta^{k+m}}(c) = 0$, $k = 1, \ldots, n-m-1$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f^{\Delta^m}(t)\,\Delta t - f^{\Delta^m}(c) \right| \le \left[\frac{\prod_{\nu=0}^{n-m} \frac{a - q^{\nu} c}{\sum_{\mu=0}^{\nu} q^{\mu}} + \prod_{\nu=0}^{n-m} \frac{b - q^{\nu} c}{\sum_{\mu=0}^{\nu} q^{\mu}}}{b-a} \right] \left\| f^{\Delta^n} \right\|_{\infty, [a,b] \cap q^{\overline{\mathbb{Z}}}}.$$

Proof. By Corollary 3.23. □

One can give many similar applications for other time scales.

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