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Global Attractivity of a Higher-Order Nonlinear Difference Equation

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Abstract

In this paper, we investigate the global attractivity of negative solutions of the nonlinear difference equation

$$x_{n+1} = \frac{1 - x_{n-k}}{A + x_n}, \;\; n = 0, 1, \dots,$$

where $A \in (-\infty, 0)$, k is a positive integer and initial conditions $x_{-k}, \cdots, x_0$ are arbitrary real numbers. We show that the unique negative equilibrium of above-mentioned equation is a global attractor with a basin under certain conditions.

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1 Introduction

Recently, many researchers are interested in the boundedness, invariant intervals, periodic character and global asymptotic stability of positive solutions for nonlinear difference equations, for example, [1, 3–6, 10–12]. Here, we would like to mention the results

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of Li et al. [8], Yan et al. [13] and He et al. [2]. They investigated the global asymptotic stability of positive solutions for the difference equations

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma + x_{n-k}}, \ \ n = 0, 1, \ldots,$$

$$x_{n+1} = \frac{\alpha - \beta x_n}{\gamma - x_{n-k}} \ \ n = 0, 1, \ldots,$$

$$x_{n+1} = \frac{\alpha - b x_{n-k}}{A - x_n}, \ \ n = 0, 1, \ldots,$$

respectively, where the coefficients α, β, γ, A, a and b are nonnegative real numbers, $k \in \{1, 2, \ldots\}$. They established that every positive equilibrium of these equations is a global attractor with a basin under certain conditions. In addition, they obtained some sufficient conditions for global asymptotic stability of positive equilibria of these equations.

In 2005, Li et al. [9] studied the difference equation

$$x_{n+1} = \frac{a + b x_n}{A + x_{n-k}}, \ \ n = 0, 1, \ldots, \tag{1.1}$$

where $a, b, A \in (0, \infty)$, k is a positive integer and the initial conditions $x_{-k}, \ldots, x_0$ are arbitrary positive numbers. They obtained the global attractivity of the unique positive equilibrium for Eq. (1.1).

It is also noted that the above mentioned references [1, 3–6, 10–12] only considered the global attractivity of positive solutions of the difference equation. However, they did not further provide the global attractivity of negative solutions for the difference equation. Hence, in this paper, we deal with the global attractivity of negative solutions for the difference equation

$$x_{n+1} = \frac{1 - x_{n-k}}{A + x_n}, \ \ n = 0, 1, \ldots, \tag{1.2}$$

where $A \in (-\infty, 0)$ is a real number, k is a positive integer and the initial conditions $x_{-k}, \ldots, x_0$ are arbitrary real numbers.

Our aim is to investigate the periodic character, invariant intervals and the global attractivity of negative solutions of Eq. (1.2). It is shown that the unique negative equilibrium of Eq. (1.2) is a global attractor with a basin that depends on certain conditions of the coefficient A.

Now, we present some definitions and results which will be useful in the sequel. Let I be some interval of real numbers and F be a continuous function defined on I^{k+1}. Then, for initial conditions $x_{-k}, \ldots, x_0 \in I$, it is easy to see that the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \ \ n = 0, 1, \ldots \tag{1.3}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$. A point $\bar{x}$ is called an equilibrium of Eq. (1.3) if

$$\bar{x} = F(\bar{x}, \dots, \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0$ is a solution of Eq. (1.3), or equivalently, $\bar{x}$ is a fixed point of F. An interval $J \subseteq I$ is called an invariant interval for Eq. (1.3) if

$$x_{-k}, \dots, x_0 \in J \Rightarrow x_n \in J \text{ for all } n > 0.$$

That is, every solution of Eq. (1.3) with initial conditions in J remains in J. The linearized equation associated with Eq. (1.3) about the equilibrium $\bar{x}$ is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\bar{x}, \cdots, \bar{x}) y_{n-i}, \quad n = 0, 1, \dots,$$

and its characteristic equation is

$$\lambda^{k+1} = \sum_{i=0}^{k} \frac{\partial F}{\partial u_i}(\bar{x}, \cdots, \bar{x}) \lambda^{k-i}.$$

Definition 1.1. Let $\bar{x}$ be an equilibrium point of Eq. (1.3).

(i) The equilibrium point $\bar{x}$ of Eq. (1.3) is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, \dots, x_0 \in I$ with $\sum_{i=-k}^{0} |x_i - \bar{x}| < \delta$, we have $|x_n - \bar{x}| < \varepsilon$ for all $n \geq -k$.

(ii) The equilibrium point $\bar{x}$ of Eq. (1.3) is called locally asymptotically stable if it is locally stable, and if there exist $\gamma > 0$ such that for all $x_{-k}, \dots, x_0 \in I$ with $\sum_{i=-k}^{0} |x_i - \bar{x}| < \gamma$, we have $\lim_{n \to \infty} x_n = \bar{x}$.

(iii) The equilibrium point $\bar{x}$ of Eq. (1.3) is called a global attractor if $x_{-k}, \dots, x_0 \in I$ always implies $\lim_{n \to \infty} x_n = \bar{x}$.

(iv) The equilibrium point $\bar{x}$ of Eq. (1.3) is called global asymptotically stable if it is locally asymptotically stable and a global attractor.

(v) The equilibrium point $\bar{x}$ of Eq. (1.3) is called unstable if it is not locally stable.

Theorem 1.2 (See [5]). *Assume that* $p, q \in \mathbb{R}$ *and* $k \in \{0, 1, \dots\}$. *Then*

$$|p| + |q| < 1$$

is a sufficient condition for asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, \dots .$$

Theorem 1.3 (See [7]). *Consider the difference equation*

$$x_{n+1} = f(x_n, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.4}$$

where $k \geq 1$ is an integer. Let $I = [a, b]$ be some interval of real numbers, and assume that $f : [a, b] \times [a, b] \to [a, b]$ is a continuous function satisfying the following properties:

(a) *$f(u, v)$ is a nonincreasing function in u, and a nondecreasing function in v;*

(b) *If $(m, M) \in [a, b] \times [a, b]$ is a solution of*

$$m = f(M, m) \text{ and } M = f(m, M),$$

then $m = M$.

Then Eq. (1.4) *has a unique equilibrium $\bar{x}$ and every solution of Eq.* (1.4) *converges to $\bar{x}$.*

2 Main Results

In this section, we are concerned with the global attractivity of Eq. (1.2) and obtain that the unique negative equilibrium of equation Eq. (1.2) is a global attractor with a basin under certain conditions. The unique negative equilibrium of Eq. (1.2) is

$$\bar{x} = \frac{-(A+1) - \sqrt{(A+1)^2 + 4}}{2}.$$

The linearized equation associated with Eq. (1.2) about the equilibrium $\bar{x}$ is

$$y_{n+1} + \frac{\bar{x}}{A + \bar{x}} y_n + \frac{1}{A + \bar{x}} y_{n-k} = 0, \quad n = 0, 1, \ldots,$$

and its characteristic equation is

$$\lambda^{k+1} + \frac{\bar{x}}{A + \bar{x}} \lambda^k + \frac{1}{A + \bar{x}} = 0.$$

By Theorem 1.2, it is easy to obtain the following result.

Theorem 2.1. *Assume that $A < -1$. Then the unique negative equilibrium $\bar{x}$ of Eq.* (1.2) *is locally asymptotically stable.*

Theorem 2.2. *Assume that $A < -1$. Then Eq.* (1.2) *has no negative prime period two solution.*

Proof. For the sake of contradiction, assume that there exist distinct negative real numbers ϕ and ψ, such that

$$\dots, \phi, \psi, \phi, \psi, \dots,$$

is a prime period two solution of Eq. (1.2). Then, there are two cases to be considered.

Case (i): If k is odd, then $x_{n+1} = x_{n-k}$, and ϕ, ψ satisfy the system

$$\phi = \frac{1-\phi}{A+\psi}, \quad \psi = \frac{1-\psi}{A+\phi}.$$

Thus

$$(\phi - \psi)(A+1) = 0.$$

In view of $A + 1 < 0$, we get $\phi = \psi$, which contradicts the hypothesis that $\phi \neq \psi$.

Case (ii): If k is even, then ϕ, ψ satisfy the system

$$\phi = \frac{1-\psi}{A+\psi}, \quad \psi = \frac{1-\phi}{A+\phi}.$$

Thus, we have $(\phi - \psi)(A - 1) = 0$. Clearly, $A - 1 < 0$, so $\phi = \psi$, which is also a contradiction. The proof is complete. □

Lemma 2.3. *If $A < -1$ and $f(u, v) = \dfrac{1-v}{A+u}$, then the following statements are true.*

(i) $-1 < \bar{x} = \dfrac{-(1+A) - \sqrt{(1+A)^2 + 4}}{2} < 0.$

(ii) *If $u, v \in (-\infty, 0]$, then $f(u, v)$ is a strictly decreasing function in u, and a strictly increasing function in v.*

Proof. The proof is simple and omitted. □

Theorem 2.4. *Assume that $A \in \left(-\infty, \dfrac{-1-\sqrt{5}}{2}\right]$. Then $[A, 0]$ is an invariant interval of Eq.* (1.2).

Proof. Let $\{x_n\}$ be a solution of Eq. (1.2) with initial conditions $x_k, \dots, x_{-1}, x_0 \in [A, 0]$. By Lemma 2.3, the function $f(u, v)$ is strictly decreasing in u, and strictly increasing in v for each fixed $u, v \in (-\infty, 0]$, we obtain that

$$x_1 = f(x_0, x_{-k}) < f(A, 0) = \frac{1}{2A} < 0$$

and

$$x_1 = f(x_0, x_{-k}) > f(0, A) = \frac{1-A}{A} \geq A,$$

which implies that $x_1 \in [A, 0]$. By induction, it follows that $x_n \in [A, 0]$ for $n \geq 1$. The proof is complete. □

Theorem 2.5. *Assume that* $A \in \left(-\infty, \frac{-1-\sqrt{5}}{2}\right]$. *Then the unique negative equilibrium* $\bar{x}$ *of Eq.* (1.2) *is a global attractor with a basin*

$$S = [A, 0]^{k+1}.$$

Proof. Set

$$f(u, v) = \frac{1-v}{A+u}.$$

Then $f : [A, 0] \times [A, 0] \to [A, 0]$ is a continuous function and nonincreasing in u and nondecreasing in v. Let $\{x_n\}$ be a solution of Eq. (1.2) with initial conditions $x_k, \ldots, x_{-1}, x_0 \in S$ and $m, M \in [A, 0]$ be a solution of the system

$$m = f(M, m), \quad M = f(m, M).$$

Then $(m - M)(A + 1) = 0$. Since $A + 1 < 0$, we get $m = M$. Applying Theorem 1.3, we get $\lim_{n\to\infty} x_n = \bar{x}$. The proof is complete. □

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