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# Differential Inclusion Governed by a State Dependent Sweeping Process

**Tahar Haddad**
University of Jijel, Department of Mathematics
Laboratoire de Mathématiques Pures et Appliquées
B.P.98, Jijel, Algeria
haddadtr2000@yahoo.fr

**Abstract**

We establish a result asserting the existence of solutions for a perturbed state dependent sweeping process in an infinite dimensional Hilbert space.

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**Keywords:** Differential inclusion, normal cone, quasi-variational inequality evolution.

## 1 Introduction

Let $H$ be a real separable Hilbert space, $T > 0$ be a real number, and $C : [0,T] \times H \to 2^H$ be a set-valued mapping with nonempty closed convex values moving in a Lipschitz continuous way. The state dependent sweeping process or quasi-variational inequality evolution

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t)) \text{ a.e. on } [0,T]; \\ u(t) \in C(t,u(t)), \text{ for all } t \in [0,T] \\ u(0) = u_0 \in C(0,u_0), \end{cases} \tag{1.1}$$

where $N_{C(t,u(t))}(\cdot)$ denotes a normal cone, has been introduced and studied by Kunze and Monteiro Marques in [5]. They used an implicit projection algorithm based on a fixed point theorem (implicit discretization). Recently, in [1], the authors treated the problem (1.1) in uniformly convex and uniformly smooth Banach spaces.

In this paper, we use a new projection algorithm (semi-implicit discretization) to prove the existence of solutions for the differential inclusion governed by the state dependent sweeping process

$$\begin{cases} -\dot{u}(t) \in N_{C(t,u(t))}(u(t))) + F(t,u(t)) \quad \text{a.e. on } [0,T]; \\ u(t) \in C(t,u(t)), \text{ for all } t \in [0,T] \\ u(0) = u_0 \in C(0,u_0), \end{cases} \tag{1.2}$$

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where $C$ is a Lipschitz continuous set-valued mapping with closed convex values and $F$ is an unbounded upper semicontinuous convex set-valued mapping.

# 2 Notation and Preliminaries

In the sequel, $H$ denotes a real separable Hilbert space. Let $S$ be a closed subset of $H$. We denote by $\mathbb{B}$ the closed unit ball of $H$ and by $d_S(\cdot)$ the usual distance function associated with $S$, i.e., $d_S(x) := \inf_{u \in S} \|x - u\|$ $(x \in H)$. We need first to recall some notations and definitions needed in the paper.

Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semicontinuous (l.s.c.) function and let $x$ be any point where $\varphi$ is finite. We recall that the *subdifferential* $\partial\varphi(x)$ (in the sense of convex analysis) is the set of all $\xi \in H$ such that

$$\langle \xi, x' - x \rangle \leq \varphi(x') - \varphi(x)$$

for all $x' \in H$. By convention we set $\partial\varphi(x) = \emptyset$ if $\varphi(x)$ is not finite. Let $S$ be a nonempty closed convex subset of $H$ and $x$ be a point in $S$. The convex normal cone of $S$ at $x$ is defined by (see, e.g., [4])

$$N_S(x) = \{\xi \in H |\ \langle \xi, x' - x \rangle \leq 0 \quad \text{for all } x^{'} \in S\}.$$

It is well known (see, e.g., [4]) that $N_S(x)$, the normal cone of a closed convex set $S$ at $x \in H$, can be defined in terms of a projection operator $\mathrm{Proj}_S(\cdot)$ as follows:

$$N_S(x) = \{\xi \in H |\ \text{there exists } \ r > 0 \text{ such that } x \in \mathrm{Proj}_S(x + r\xi)\}.$$

Let us recall the two following results. For their proofs we refer to [2,6], respectively.

**Proposition 2.1.** *Let $S$ be a nonempty closed subset of $H$ and $x \in S$. Then*

$$\partial d_S(x) = N_S(x) \cap \mathbb{B}.$$

**Proposition 2.2.** *Let $C : [0,T] \times H \to 2^H$ be a Hausdorff-continuous set-valued mapping with nonempty closed convex values. Then the mapping*

$$(t, x, y) \mapsto \partial d_{C(t,x)}(y)$$

*has the following upper semicontinuity property: if $(t_n, x_n)$ is a sequence in $[0,T] \times H$ converging to $(t, x) \in [0,T] \times H$, and $(y_n)$ is a sequence in $H$ with $y_n \in C(t_n, x_n)$ for all $n$, converging to $y \in C(t, x)$, then for any $\xi \in H$, we have*

$$\limsup_n \sigma(\partial d_{C(t_n,x_n)}(y_n), \xi)) \leq \sigma(\partial d_{C(t,x)}(y), \xi),$$

*where*

$$\sigma(\partial d_{C(t,x)}(y), \xi) := \sup_{p \in \partial d_{C(t,x)}(y)} \langle p, \xi \rangle$$

*stands for the support function of $\partial d_{C(t,x)}(y)$ at $\xi$.*

# 3 Main Result

The following existence theorem establishes our main result in this paper.

**Theorem 3.1.** *Let $H$ be a separable Hilbert space and let $C : [0,T] \times H \to 2^H$ be a set-valued mapping with nonempty closed convex values satisfying the following assumptions:*
$(\mathcal{H}_1)$ *$C$ is Lipschitz continuous with constants $L_1 \geq 0$ and $0 < L_2 < 1$, i.e., for all $t, s \in [0,T]$ and $x, u, v \in H$ we have*

$$|d_{C(t,u)}(x) - d_{C(s,v)}(x)| \leq L_1|t-s| + L_2|u-v|;$$

$(\mathcal{H}_2)$ *there exists a convex strongly compact set $S$ such that $C(t,u) \subset S$ for all $(t,u) \in [0,T] \times H$.*
*Let $F : [0,T] \times H \to 2^H$ be an upper semicontinuous set-valued mapping with nonempty convex weakly compact values in $H$. Assume also that $F$ has linear growth, that is, there exists $L > 0$ such that $F(t,u) \subset L(1 + \|u\|)\mathbb{B}$ for all $(t,u) \in [0,T] \times H$. If $u_0 \in C(0,u_0)$, then there exists at least one Lipschitz solution of* (1.2).

*Proof.* We give the proof in four steps.
**Step 1.** Construction of approximants.
Let $\rho > 0$ such that $C(t,u) \subset S \subset \rho\mathbb{B}$ for all $(t,u) \in [0,T] \times H$. For each $n \in N$, we consider the following partition of the interval $I := [0,T]$:

$$I_{i+1}^n := ]t_i^n, t_{i+1}^n], \quad t_i^n := i\mu_n, \quad 0 \leq i \leq n-1, \quad I_0^n := \{t_0^n\}.$$

**Algorithm 1.** Put $\mu_n := \dfrac{T}{n}$. Fix $n \geq 2$. We define by induction:

- $u_0^n = u_0 \in C(0,u_0)$, and $f_0^n \in F(t_0^n, u_0^n)$;
- $0 \leq i \leq n-1 : u_{i+1}^n = \mathrm{Proj}_{C(t_{i+1}^n, u_i^n)}(u_i^n - \mu_n f_i^n)$;
- $0 \leq i \leq n-1 : f_{i+1}^n \in F(t_{i+1}^n, u_{i+1}^n)$.

The existence of the projection is ensured since $C$ has closed convex values, and so the Algorithm 1 is well defined. Using the sequences $(u_i^n)$ and $(f_i^n)$ to construct sequences of mapping $u_n$ and $f_n$ from $[0,T]$ to $H$, we define their restrictions to each interval $I_i^n$ as follows. For $t \in I_0^n$ set $f_n(t) = f_0^n$ and $u_n(t) = u_0$; for $t \in I_{i+1}^n$ $(0 \leq i \leq n-1)$ set $f_n(t) = f_i^n$ and

$$u_n(t) = u_i^n + (u_{i+1}^n - u_i^n)\frac{(t - t_i^n)}{\mu_n}. \tag{3.1}$$

Clearly, $u_n$ is continuous on $[0,T]$ and differentiable on $[0,T]\backslash\{t_i^n\}$ with

$$\dot{u}_n(t) = \frac{u_{i+1}^n - u_i^n}{\mu_n}, \quad \forall t \in [0,T]\backslash\{t_i^n\}. \tag{3.2}$$

By Algorithm 1, we have

$$u_{i+1}^n = \mathrm{Proj}_{C(t_{i+1}^n, u_i^n)}(u_i^n - \mu_n f_i^n).$$

Using the characterization of the normal cone in terms of the projection operator, we can write for a.a. $t \in [0, T]$ that

$$-\frac{u_{i+1}^n - u_i^n}{\mu_n} - f_i^n \in N_{C(t_{i+1}^n, u_i^n)}(u_{i+1}^n). \tag{3.3}$$

Let us find an upper bound estimate for the expression $\left\| -\frac{u_{i+1}^n - u_i^n}{\mu_n} - f_i^n \right\|$. By the Algorithm, we have $u_{i+1}^n \in C(t_{i+1}^n, u_i^n) \subset \rho\mathbb{B}$, that is, $\|u_i^n\| \leq \rho$, for all $i \geq 0$. Now by Algorithm and by the fact that $F$ has linear growth we obtain that

$$\|f_i^n\| \leq L(1 + \|u_i^n\|) \leq L(1 + \rho) = \beta.$$

Therefore, the Lipschitz property of $C$ ensures that

$$\begin{aligned} \|u_i^n - u_{i+1}^n - \mu_n f_i^n\| &= d_{C(t_{i+1}^n, u_i^n)}(u_i^n - \mu_n f_i^n) \\ &\leq d_{C(t_{i+1}^n, u_i^n)}(u_i^n) - d_{C(t_i^n, u_{i-1}^n)}(u_i^n) + \mu_n \|f_i^n\| \\ &\leq L_1 |t_{i+1}^n - t_i^n| + L_2 \|u_i^n - u_{i-1}^n\| + \mu_n \beta. \end{aligned} \tag{3.4}$$

By construction we have

$$\begin{aligned} \|u_i^n - u_{i-1}^n\| &= \|u_i^n - u_{i-1}^n + \mu_n f_{i-1}^n - \mu_n f_{i-1}^n\| \\ &\leq \|u_{i-1}^n - u_i^n + \mu_n f_{i-1}^n\| + \mu_n \|f_{i-1}^n\| \\ &= d_{C(t_i^n, u_{i-1}^n)}(u_{i-1}^n - \mu_n f_{i-1}^n) + \mu_n \|f_{i-1}^n\| \\ &\leq d_{C(t_i^n, u_{i-1}^n)}(u_{i-1}^n) - d_{C(t_{i-1}^n, u_{i-2}^n)}(u_{i-1}^n) + 2\mu_n \|f_{i-1}^n\| \\ &\leq L_1 |t_i^n - t_{i-1}^n| + L_2 \|u_{i-1}^n - u_{i-2}^n\| + 2\mu_n \beta \end{aligned}$$

and by induction we obtain

$$\begin{aligned} \|u_i^n - u_{i-1}^n\| &\leq L_1 \mu_n + 2\mu_n \beta + L_2 \Big( 2\mu_n \beta + L_1 \mu_n + L_2 \|u_{i-2}^n - u_{i-3}^n\| \Big) \\ &= L_1 \mu_n \big(1 + L_2\big) + 2\mu_n \beta \big(1 + L_2\big) + L_2^2 \|u_{i-2}^n - u_{i-3}^n\| \\ &\vdots \\ &\leq L_1 \mu_n \big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2}\big) \\ &\quad + 2\mu_n \beta \big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2}\big) + L_2^{i-1} \|u_1^n - u_0^n\|. \end{aligned}$$

The initial condition $u_0 \in C(0, u_0)$ entails

$$\begin{aligned} \|u_1^n - u_0^n\| &\leq \|u_1^n - u_0^n - \mu_n f_0^n\| + \mu_n \|f_0^n\| \\ &\leq d_{C(t_1^n, u_0^n)}(u_0^n - \mu_n f_0^n) + \mu_n \beta \\ &\leq d_{C(t_1^n, u_0^n)}(u_0^n) - d_{C(t_0^n, u_0)}(u_0) + 2\mu_n \beta \\ &\leq L_1 \mu_n + 2\mu_n \beta. \end{aligned}$$

Thus

$$\begin{aligned}\|u_i^n - u_{i-1}^n\| &\leq L_1\mu_n\Big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2} + L_2^{i-1}\Big)\\ &+ 2\mu_n\beta\Big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2} + L_2^{i-1}\Big)\\ &\leq \mu_n\big(L_1 + 2\beta\big)\Big(1 + L_2 + L_2^2 + \cdots + L_2^{i-2} + L_2^{i-1}\Big)\end{aligned} \tag{3.5}$$

and (3.4) and (3.5) imply that

$$\begin{aligned}\|u_i^n - u_{i+1}^n - \mu_n f_i^n\| &\leq \mu_n\big(L_1 + 2\beta\big) + \mu_n\beta\\ &+ \mu_n\big(L_1 + 2\beta\big)\Big(L_2 + L_2^2 + \cdots + L_2^{i-1} + L_2^i\Big)\\ &= \mu_n\big(L_1 + 2\beta\big)\Big(1 + L_2 + L_2^2 + \cdots + L_2^{i-1} + L_2^i\Big) + \mu_n\beta.\end{aligned}$$

Using the fact that $L_2 < 1$, we get

$$\begin{aligned}\|u_i^n - u_{i+1}^n - \mu_n f_i^n\| &\leq \mu_n\big(L_1 + 2\beta\big)\left(\frac{1 - L_2^{i+1}}{1 - L_2}\right) + \mu_n\beta\\ &\leq \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right)\mu_n\end{aligned}$$

or

$$\left\|-\frac{u_{i+1}^n - u_i^n}{\mu_n} - f_i^n\right\| \leq \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right). \tag{3.6}$$

The inclusion (3.3) and Proposition 2.1 give

$$-\frac{u_{i+1}^n - u_i^n}{\mu_n} - f_i^n \in \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right)\partial d_{C(t_{i+1}^n, u_i^n)}(u_{i+1}^n). \tag{3.7}$$

Now let us define the step functions from $[0, T]$ to $[0, T]$ by

$$\begin{aligned}&\theta_n(t) = t_i^n; && t \in I_{i+1}^n,\\ &\eta_n(t) = t_{i+1}^n; && t \in I_{i+1}^n,\\ &\theta_n(0) = \eta_n(0) = 0.\end{aligned} \tag{3.8}$$

Then (3.1), (3.2), (3.7) and (3.8) yield that

$$-\dot{u}_n(t) - f_n(t) \in \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right)\partial d_{C(\eta_n(t), u_n(\theta_n(t)))}(u_n(\eta_n(t))) \text{ a.e. on } [0, T]. \tag{3.9}$$

**Step 2.** The convergence of $(u_n)$ and $(f_n)$.

By (3.2) and (3.6) we have

$$\|\dot{u}_n(t)\| \leq \left(\frac{L_1 + 2\beta}{1 - L_2}\right) := \gamma, \tag{3.10}$$

and it is clear that the sequence $(u_n(t))$ is equi-Lipschitz with constant $\gamma$. From the definition of $(u_n)$, we have $u_i^n, u_{i+1}^n \in S$. Since

$$\frac{(t-t_i^n)}{\mu_n} \leq 1 \text{ for all } t \in I_{i+1}^n$$

and $S$ is a convex compact set in $H$, one gets that $u_n(t) \in S$ for all $t \in [0,T]$ so that the set $\mathcal{X}(t) = \{u_n(t) | n \geq 2\}$ is relatively compact in $H$ for every $t \in [0,T]$. Then all assumptions of the Arzela–Ascoli theorem are satisfied and hence there exists a Lipschitz mapping $u : [0,T] \to H$ with ratio $\gamma$ such that

- $(u_n)$ converges uniformly to $u$ on $[0,T]$, that is,

$$\lim_{n \to +\infty} \max_{t \in [0,T]} \|u_n(t) - u(t)\| = 0;$$

- $(\dot{u}_n)$ weakly converges to $\dot{u}$ in $L^1([0,T], H)$.

Now, as $\|f_n(t)\| \leq \beta$ for all $t \in [0,T]$, we see that $(f_n)$ is a bounded sequence in $L^\infty([0,T], H)$. By extracting a subsequence we can consider, without loss of generality, that $(f_n)$ weakly-star converges in $L^\infty([0,T], H)$ to some mapping $f : [0,T] \to H$.

**Step 3.** Existence of solutions.

Since $\lim_{n \to +\infty} \theta_n(t) = \lim_{n \to +\infty} \eta_n(t) = t$, we have

$$\begin{aligned} \lim_{n \to +\infty} u_n(\theta_n(t)) &= \lim_{n \to +\infty} u_n(\eta_n(t)) \\ &= \lim_{n \to +\infty} u_n(t) \\ &= u(t) \end{aligned}$$

uniformly on $[0,T]$. Using now the Lipschitz property of $C$, and the fact that

$$u_n(\eta_n(t)) \in C(\eta_n(t), u_n(\theta_n(t)))$$

for all $t \in [0,T]$ and for all $n \geq 2$, we get

$$\begin{aligned} d(u(t), C(t, u(t))) &= d_{C(t,u(t))}(u(t)) - d_{C(\eta_n(t), u_n(\theta_n(t)))}(u_n(\eta_n(t))) \\ &= d_{C(t,u(t))}(u(t)) - d_{C(\eta_n(t), u_n(\theta_n(t)))}(u_n(\eta_n(t)) - u(t) + u(t)) \\ &\leq \|u_n(\eta_n(t)) - u(t)\| + L_1|t - \eta_n(t)| \\ &+ L_2\|u_n(\theta_n(t)) - u(t)\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

and so the closedness of the set $C(t, u(t))$ ensures that

$$u(t) \in C(t, u(t)) \text{ for all } t \in [0,T].$$

We proceed now to prove that

$$-\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + f(t) \quad \text{for almost all } t \in [0,T].$$

Integrating we have

$$\begin{aligned}
\int_\Omega \langle \xi, f(t)\rangle dt &= \lim_n \int_\Omega \langle \xi, f_n(t)\rangle dt \\
&\leq \limsup_n \int_\Omega \sigma\left(\xi, F\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)\right) dt \\
&\leq \int_\Omega \limsup_n \sigma\left(\xi, F\left(\theta_n(t), u_n\left(\theta_n(t)\right)\right)\right) dt \\
&\leq \int_\Omega \sigma\left(\xi, F\left(t, u(t)\right)\right) dt
\end{aligned}$$

for every measurable set $\Omega$ in $[0,T]$ and every $\xi \in H$. As the set-valued mapping $t \to F(t,u(t))$ is measurable and convex weakly with compact values, from [3, Proposition III.35] it follows that

$$f(t) \in F(t,u(t)), \text{ a.e. on } [0,T].$$

Using the same technique we get

$$-\dot{u}(t) - f(t) \in \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right) \partial d_{C(t,u(t))}(u(t)).$$

Indeed, for every measurable set $\Omega$ in $[0,T]$ and every $\xi \in H$, we have

$$\begin{aligned}
\int_\Omega \langle \xi, -\dot{u}(t) - f(t)\rangle dt &= \lim_n \int_\Omega \langle \xi, -\dot{u}_n(t) - f_n(t)\rangle dt \\
&\leq \limsup_n \int_\Omega \sigma\left(\xi, \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right) \partial d_{C(\eta_n(t), u_n(\theta_n(t)))}\left(u_n\left(\eta_n(t)\right)\right)\right) dt \\
&\leq \int_\Omega \limsup_n \sigma\left(\xi, \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right) \partial d_{C(\eta_n(t), u_n(\theta_n(t)))}\left(u_n\left(\eta_n(t)\right)\right)\right) dt \\
&\leq \int_\Omega \sigma\left(\xi, \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right) \partial d_{C(t,u(t))}(u(t))\right) dt,
\end{aligned}$$

where the last inequality follows from the upper semicontinuity property given in Proposition 2.2. Thus, as the set-valued mapping $t \to \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right) \partial d_{C(t,u(t)}(u(t))$ is measurable with convex weakly compact values (see [6]), from [3, Proposition III.35] it follows that

$$-\dot{u}(t) - f(t) \in \left(\frac{L_1 + 2\beta}{1 - L_2} + \beta\right) \partial d_{C(t,u(t)}(u(t)).$$

Since

$$u(t) \in C(t,u(t)) \text{ for all } t \in [0,T],$$

we get a.e. in $t \in [0, T]$ that

$$-\dot{u}(t) \in N_{C(t,u(t))}(u(t)) + f(t) \subset N_{C(t,u(t))}(u(t)) + F(t, u(t)).$$

This completes the proof of the theorem. □


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