International Journal of Difference Equations
ISSN 0973-6069, Volume 8, Number 1, pp. 77–97 (2013)
http://campus.mst.edu/ijde

Finite Difference Method for the Time-Fractional Thermistor Problem

M. R. Sidi Ammi and **A. Taakili**
University of Moulay Ismaïl
Faculty of Sciences and Technics
Department of Mathematics, AMNEA Group
B.P 509, Errachidia, Morocco
sidiammi@ua.pt, ataakili@gmail.com

Abstract

In this paper we propose a finite difference scheme for temporal discretization of the time-fractional thermistor problem, which is obtained from the so-called thermistor problem by replacing the first-order time derivative with a fractional derivative of order α $(0 \leq \alpha \leq 1)$. An existence result is established for the semi-discrete problem. Stability and error analysis are then provided, showing that the temporal accuracy is of order $2 - \alpha$.

AMS Subject Classifications: 34A08, 34A12, 34D20, 65L12.
Keywords: Fractional differential equations, finite difference method, stability, existence, error analysis.

1 Introduction

Thermistor is a generic name for a device made from materials whose electrical conductivity is highly dependently on temperature. The thermistor problem consists of a system of nonlinear parabolic-elliptic partial differential equations with quadratic growth in the gradient and with appropriate boundary conditions,

$$\frac{\partial u}{\partial t} - \triangle u = \sigma(u)|\nabla \varphi|^2 \quad \text{in } \Omega \times (0, T), \tag{1.1}$$

$$\operatorname{div}(\sigma(u)\nabla \varphi) = 0 \quad \text{in } \Omega \times (0, T), \tag{1.2}$$

Received November 18, 2012; Accepted January 13, 2013
Communicated by Mouffak Benchohra

which model the coupling of the thermistor to its surroundings. To complete the model, we prescribe the boundary conditions and the initial condition for the temperature:

$$\begin{aligned} \frac{\partial u}{\partial n} = 0 \text{ or } u = 0, &\quad \text{on } \partial\Omega \times (0,T), \\ \varphi = \varphi_0 &\quad \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) &\quad \text{in } \Omega, \end{aligned} \tag{1.3}$$

where $\Omega \subseteq \mathbb{R}^n (n \geq 1)$ is a bounded open domain with Lipschitz boundary, $u(x)$ is the temperature inside the conductor, $\varphi(x)$ is the electrical potential, and $\sigma(u)$ is the temperature dependent electrical conductivity. Here n denotes the outward unit normal and $\frac{\partial}{\partial n} = n \cdot \nabla$ is the normal derivative on $\partial\Omega$. The first equation describes the diffusion of heat in the presence of Joule effect due to the electrical current, while the second equation represents the conservation of electrical charges [6,8,17]. Joule heating is generated by the resistance of materials to electrical current and is present in any electrical conductor operating at normal temperatures. The advantages of thermistors as temperatures measurement devices include their low cost, high resolution, and flexibility in size and shape. For their concrete applications we refer the interested reader to [11, 14].

Theoretical analysis of both steady-state and time dependent thermistor equations, with various aspects and with different types of boundary and initial conditions, has received a lot of interest. For existence of weak solutions, uniqueness and related regularity results in several settings with different assumptions on the coefficients, we can see [2,18,19]. Existence of weak solutions to the stationary problem of (1.1)–(1.2) with Dirichlet boundary conditions was proven in [4]. Cimatti [5] was the first to consider the time dependent case in two dimensions. In [17], the problem without restrictions on the space dimension was considered. Asymptotic results for (1.1)–(1.2) were established in [6]. Optimal control problem for the time-dependent thermistor problem can be found in [12], where the source is taken to be the control. A result on optimal control of the thermistor problem for the steady-state case can be found in [9], where a connective boundary coefficient is taken as the control. Recently, [20] was concerned with the optimal control problem of the nonlocal thermistor problem.

In recent years, it has been shown that fractional differential equations can be used successfully to model many phenomena in various fields, such as fluid mechanics, viscoelasticity, chemistry and engineering [1,10,15,16]. In [19], existence and uniqueness of a positive solution to a generalized nonlocal thermistor problem with fractional-order derivatives was proved. Our aim here is to study the time fractional thermistor system. We are not aware of any similar result, and we believe that this work provides the first results on the time-dependent thermistor problem with fractional order derivatives.

The outline of the paper is as follows. In Section 2 we formulate the fractional problem, and we specify the hypotheses under consideration. In Section 3, a finite difference scheme for the temporal discretization of the problem in consideration is given. We obtain existence of weak solutions to the discretized problem. In Section 4, stability results are derived and error estimates are provided for the semi-discrete problem, showing that the temporal accuracy is of order $2 - \alpha$.

2 Formulation and Statement of the Problem

We consider the following time-fractional thermistor problem, which is obtained from (1.1)–(1.2) by replacing the first-order time derivative with a fractional derivative of Caputo type:

$$\begin{gathered} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \triangle u = \sigma(u)|\nabla\varphi|^2 \quad \text{in } \Omega \times (0,T), \\ \operatorname{div}(\sigma(u)\nabla\varphi) = 0 \quad \text{in } \Omega \times (0,T), \end{gathered} \tag{2.1}$$

subject to the initial and boundary conditions (1.3) and where α, $0 \leq \alpha \leq 1$, is the order of the time-fractional derivative, $\dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes the Caputo fractional derivative of order α as defined in [7] and given by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1.$$

Problem (2.1) covers (1.1)–(1.2) and extends it to more general cases. When $\alpha = 1$, the system (2.1) is the classical parabolic-elliptic thermistor problem. In fact the time derivative of integer order in (1.1)–(1.2) can be obtained by taking the limit $\alpha \to 1$ in (2.1). The case $\alpha = 0$ corresponds to the steady state thermistor problem. In the case $0 < \alpha < 1$, the Caputo fractional derivative depends on and uses the information of the solutions at all previous time levels (non-Markovian process). In this case the physical interpretation of fractional derivative is that it represents a degree of memory in the diffusing material [21].

In the analysis of the numerical method, we will assume that problem (2.1) has a unique and sufficiently smooth solution, which can be established by assuming more hypotheses and regularity on the data (see [2,20]). We need the following assumptions:

(H1) $\sigma(\cdot)$ is a continuous function such that there exists $\sigma_2 > \sigma_1 > 0$ such that $0 < \sigma_1 \leq \sigma(\cdot) \leq \sigma_2$.

(H2) $u_0, \varphi_0 \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$.

(H3) $\sigma(\cdot)$ is a continuous Lipschitzian function.

We define some functional spaces endowed with standard norms and inner products that will be used hereafter:

$$H^m(\Omega) = \left\{ v \in L^2(\Omega), \frac{d^k v}{dx^k} \in L^2(\Omega) \text{ for all positive integers } k \leq m \right\},$$

$$H_0^1(\Omega) = \{v \in H^1(\Omega), v/\partial\Omega = 0\}.$$

The inner product of $L^2(\Omega)$ is defined by

$$(u, v) = \int_\Omega uvdx.$$

The norm $\|\cdot\|_m$ of the space $H^m(\Omega)$ is defined by

$$\|v\|_m = \left(\sum_{k=0}^{m} \|\frac{d^k v}{dx^k}\|_2^2\right)^{\frac{1}{2}}.$$

It can be shown that the quantity

$$\|v\|_* = \left(\|v\|_2^2 + \alpha_0 \left\|\frac{du}{dx}\right\|_2^2\right)^{\frac{1}{2}}, \tag{2.2}$$

where α_0 is given below, defines a norm on $H^1(\Omega)$ that is equivalent to the $\|\cdot\|_{H^1(\Omega)}$ norm (see, e.g., [3, 22]).

3 Time Discretization: A Finite Difference Scheme

We introduce a finite difference approximation to discretize the time-fractional derivative. Let $\delta = \dfrac{T}{N}$ be the length of each time step, for some large N, $t_k = k\delta$, $k = 0, 1, \ldots, N$. We use the following formulation: for all $0 \leq k \leq N-1$,

$$\begin{aligned}\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} =& \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t_{k+1}-s)^\alpha} \\ =& \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \frac{u(x,t_{j+1}) - u(x,t_j)}{\delta} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1}-s)^\alpha} + r_\delta^{k+1},\end{aligned} \tag{3.1}$$

where r_δ^{k+1} is the truncation error. It can be seen from [13] that the trunction error verifies

$$r_\delta^{k+1} \lesssim c_u \delta^{2-\alpha}, \tag{3.2}$$

where c_u is a constant depending only on u. On the other hand, by a change of variables, we have

$$
\begin{aligned}
&\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{j+1})-u(x,t_j)}{\delta}\int_{t_j}^{t_{j+1}}\frac{ds}{(t_{k+1}-s)^{\alpha}}\\
&\qquad=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{j+1})-u(x,t_j)}{\delta}\int_{t_{k-j}}^{t_{k+1-j}}\frac{dt}{t^{\alpha}}\\
&\qquad=\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{k+1-j})-u(x,t_{k-j})}{\delta}\int_{t_j}^{t_{j+1}}\frac{dt}{t^{\alpha}}\\
&\qquad=\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{k+1-j})-u(x,t_{k-j})}{\delta^{\alpha}}\left\{(j+1)^{1-\alpha}-(j)^{1-\alpha}\right\}.
\end{aligned}
$$

Let us denote $b_j=(j+1)^{1-\alpha}-j^{1-\alpha}, j=0,1,\dots k$, and define the discrete fractional differential operator L_t^{α} by

$$
L_t^{\alpha}u(x,t_{k+1})=\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k}b_j\frac{u(x,t_{k+1-j})-u(x,t_{k-j})}{\delta^{\alpha}}.
$$

Then (3.1) becomes

$$
\frac{\partial^{\alpha}u(x,t_{k+1})}{\partial t^{\alpha}}=L_t^{\alpha}u(x,t_{k+1})+r_{\delta}^{k+1}.
$$

Using this approximation, we arrive to the following finite difference scheme to (1.1)–(1.2): for $k=1,\dots,N-1$,

$$
\begin{aligned}
L_t^{\alpha}u^{k+1}(x)-\triangle u^{k+1}&=\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2 \quad \text{in } \Omega,\\
\operatorname{div}(\sigma(u^{k+1})\nabla\varphi^{k+1})&=0 \quad \text{in } \Omega,
\end{aligned}
\tag{3.3}
$$

where $u^{k+1}(x)$ and $\varphi^{k+1}(x)$ are approximations to $u(x,t_{k+1})$ and $\varphi(x,t_{k+1})$, respectively. The scheme (3.3) can be reformulated to the form

$$
\begin{aligned}
b_0u^{k+1}-&\Gamma(2-\alpha)\delta^{\alpha}\triangle u^{k+1}\\
&=b_0u^k-\sum_{j=1}^{k}b_j\{u^{k+1-j}-u^{k-j}\}+\Gamma(2-\alpha)\delta^{\alpha}\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2\\
&=b_0u^k-\sum_{j=0}^{k-1}b_{j+1}u^{k-j}+\sum_{j=1}^{k}b_ju^{k-j}+\Gamma(2-\alpha)\delta^{\alpha}\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2\\
&=b_0u^k+\sum_{j=0}^{k-1}(b_j-b_{j+1})u^{k-j}+\Gamma(2-\alpha)\delta^{\alpha}\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2,
\end{aligned}
\tag{3.4}
$$

$$\operatorname{div}(\sigma(u^{k+1})\nabla\varphi^{k+1}) = 0. \tag{3.5}$$

To complete the semi-discrete problem, we consider the boundary conditions

$$\begin{aligned} \frac{\partial u^{k+1}}{\partial n} = 0, \text{ or } u^{k+1} = 0 \quad \text{on } \partial\Omega, \\ \varphi^{k+1} = \varphi_0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.6}$$

and the initial condition $u^0 = u_0$. Note that

$$\begin{aligned} b_j > 0, \quad j = 0, 1, \ldots k, \\ 1 = b_0 > b_1 > \ldots > b_k, b_k \to 0 \text{ as } k \to \infty, \\ \sum_{j=1}^{k}(b_j - b_{j+1}) + b_{k+1} = (1 - b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k = 1. \end{aligned} \tag{3.7}$$

If we set

$$\alpha_0 = \Gamma(2-\alpha)\delta^\alpha,$$

then (3.4) can be rewritten in the form

$$u^{k+1} - \alpha_0 \triangle u^{k+1} = (1-b_1)u^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})u^{k-j} + b_k u^0 + \alpha_0 \sigma(u^{k+1})|\nabla\varphi^{k+1}|^2 \tag{3.8}$$

for all $k \geq 1$. When $k = 0$, scheme (3.4) reads

$$u^1 - \alpha_0 \triangle u^1 = u^0 + \alpha_0 \sigma(u^1)|\nabla\varphi^1|^2.$$

When $k = 1$, scheme (3.4) becomes

$$u^2 - \alpha_0 \triangle u^2 = (1 - b_1)u^1 + b_1 u^0 + \alpha_0 \sigma(u^2)|\nabla\varphi^2|^2.$$

We define the error term r^{k+1} by

$$r^{k+1} = \alpha_0 \left\{ \frac{\partial^\alpha u(x, t_{k+1})}{\partial t^\alpha} - L_t^\alpha u(x, t_{k+1}) \right\}.$$

Then we get from (3.2) that

$$|r^{k+1}| = \Gamma(2-\alpha)\delta^\alpha |r_\delta^{k+1}| \leq c_u \delta^2. \tag{3.9}$$

3.1 Existence: The Semi-discrete Scheme

One of the interesting points of the problem is the quadratic term $|\nabla\varphi|^2$. Since two is a critical exponent, this term creates a difficulty and makes the usual compactness arguments to fail, which makes the necessary estimates on the time discretized sequence of solutions, stability results and error analysis technical and somehow delicate. We are able to overcome this difficulty thanks to the following lemma.

Lemma 3.1. *Let $u \in L^1(\Omega)$ and $\varphi - \varphi_0 \in H_0^1(\Omega)$. If the pair (u, φ) satisfies*

$$\int_\Omega \sigma(u)\nabla\varphi \cdot \nabla\psi \; dx = 0 \quad \textit{for all } \psi \in H_0^1(\Omega),$$

then $\forall v \in H^1(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega \sigma(u)|\nabla\varphi|^2 v \, dx = \int_\Omega (\varphi_0 - \varphi)\sigma(u)\nabla\varphi.\nabla v \, dx + \int_\Omega (\sigma(u)\nabla\varphi \cdot \nabla\varphi_0) \, v \, dx. \quad (3.10)$$

Moreover, $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$*, and the following estimates hold:*

$$\|\varphi\|_{L^\infty(\Omega)} \le \sup_{x\in\partial\Omega} |\varphi_0|, \quad (3.11)$$

$$\|\nabla\varphi\|_2 \le \left(\frac{\sigma_2}{\sigma_1}\right)^{\frac{1}{2}} \|\nabla\varphi_0\|_2. \quad (3.12)$$

Proof. Equation (3.10) follows by choosing $\psi = (\varphi - \varphi_0)v, v \in C^1(\overline{\Omega})$ as the test function. The estimate (3.11) comes from the weak maximum principle [17]. On the other hand, (3.12) is obtained by choosing $\psi = \varphi - \varphi_0 \in H_0^1(\Omega)$. By the Cauchy–Schwartz inequality,

$$\begin{aligned}
\sigma_1 \int_\Omega |\nabla\varphi|^2 dx &\le \int_\Omega \sigma(u)|\nabla\varphi|^2 dx \\
&\le \left| \int_\Omega \sigma(u)\nabla\varphi\nabla\varphi_0 dx \right| \\
&\le \sigma_2 \int_\Omega |\nabla\varphi||\nabla\varphi_0| dx \\
&\le \sigma_2 \|\nabla\varphi\|_2 \|\nabla\varphi_0\|_2.
\end{aligned}$$

Then,

$$\|\nabla\varphi\|_2 \le \left(\frac{\sigma_2}{\sigma_1}\right)^{\frac{1}{2}} \|\nabla\varphi_0\|_2.$$

This concludes the proof. □

Definition 3.2. We say that a couple (u^{k+1}, φ^{k+1}) is a weak solution of (3.3) if

$$\begin{aligned}
&\langle u^{k+1}, v\rangle + \alpha_0 \int_\Omega \nabla u^{k+1} \nabla v \, dx \\
&\qquad = (f^k, v) + \alpha_0 \int_\Omega \sigma(u^{k+1})\nabla\varphi^{k+1}\nabla\varphi_0 v \, dx \qquad (3.13) \\
&\qquad - \alpha_0 \int_\Omega \sigma(u^{k+1})(\varphi^{k+1} - \varphi_0)\nabla\varphi^{k+1}\nabla v \, dx, \quad \text{for all } v \in V \cap C^1(\overline{\Omega}),
\end{aligned}$$

and

$$\int_\Omega \sigma(u^{k+1})\nabla\varphi^{k+1}\nabla\psi\ dx = 0, \quad \text{for all } \psi \in H_0^1(\Omega), \tag{3.14}$$

where $f^k = (1-b_1)u^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})u^{k-j} + b_k u^0$.

At each time step we solve a discretized fractional thermistor problem.

Theorem 3.3. *If hypotheses (H1)–(H2) are satisfied, then there exists at least a weak solution* (u^k, φ^k) *of* (3.4)–(3.5) *such that*

$$u^k \in H^1(\Omega), \varphi^k \in H^1(\Omega) \cap L^\infty(\Omega).$$

Before beginning the proof of Theorem 3.3, we proceed with the derivation of a priori estimates. From now on we denote by c a generic constant, which may not be the same at different occurrences.

3.2 A Priori Estimates

We search a priori estimates for solutions.

Lemma 3.4. *One has*

$$\|u^{k+1}\|_{H^1(\Omega)} \le c,$$

where c is a positive constant independent of k.

Proof. We prove this result by recurrence. When $k = 0$, we have for $v \in H_0^1(\Omega)$ that

$$\int_\Omega u^1 v dx + \alpha_0 \int_\Omega \nabla u^1 \nabla v dx = \int_\Omega u^0 v dx + \alpha_0 \int_\Omega \sigma(u^1)|\nabla\varphi^1|^2 v dx.$$

Note that $u^0 \in L^\infty(\Omega) \subset L^2(\Omega), \varphi_0 \in W^{1,\infty}(\Omega)$ and by Lemma 3.1 we have $\varphi^k \in H^1(\Omega) \cap L^\infty(\Omega)\ \forall\ k \ge 1$. Taking $v = u^1$ and using Lemma 3.1,

$$\begin{aligned}
\|u^1\|_2^2 + \alpha_0\|\nabla u^1\|_2^2 &= \int_\Omega u^0 u^1 dx + \alpha_0 \int_\Omega \sigma(u^1)|\nabla\varphi^1|^2 u^1 dx \\
&\le \|u^1\|_2\|u^0\|_2 + \alpha_0 \int_\Omega (\varphi_0 - \varphi^1)\sigma(u^1)\nabla\varphi^1.\nabla u^1 dx \\
&+ \alpha_0 \int_\Omega \sigma(u^1)\nabla\varphi^1.\nabla\varphi_0\, u^1 dx \\
&\le c\|u^1\|_2 + c\|\nabla\varphi^1\|_2\|\nabla u^1\|_2 + +c\|\nabla\varphi^1\|_2\|u^1\|_2 \\
&\le c\|u^1\|_2 + c\|\nabla u^1\|_2 + c\|u^1\|_2 \\
&\le c\|u^1\|_2 + c\|\nabla u^1\|_2 \\
&\le \frac{1}{2}\|u^1\|_2^2 + c + \frac{\alpha_0}{2}\|\nabla u^1\|_2^2 + c.
\end{aligned}$$

Then,

$$\|u^1\|_2^2 + \alpha_0 \|\nabla u^1\|_2^2 \leq c.$$

Hence, since the standard H^1-norm and the norm $\|\cdot\|_*$ defined by (2.2) are equivalent, we have

$$\|u^1\|_{H^1(\Omega)} \leq c.$$

Suppose now that we have

$$\|u^j\|_{H^1(\Omega)} \leq c, \quad j = 1, 2, \ldots, k,$$

and prove that $\|u^{k+1}\|_{H^1(\Omega)} \leq c$. Multiplying (3.8) by $v = u^{k+1}$, and using the fact that $f^k \in H^1(\Omega)$, we obtain

$$\begin{aligned}
\|u^{k+1}\|_2^2 + \alpha_0 \|\nabla u^{k+1}\|_2^2 &= \int_\Omega f^k u^{k+1} dx + \alpha_0 \int_\Omega \sigma(u^{k+1}) |\nabla \varphi^{k+1}|^2 u^{k+1} dx \\
&\leq \|f^k\|_2 \|u^{k+1}\|_2 + \alpha_0 \int_\Omega (\varphi_0 - \varphi^{k+1}) \sigma(u^{k+1}) \nabla \varphi^{k+1} . \nabla u^{k+1} dx \\
&+ \alpha_0 \int_\Omega (\sigma(u^{k+1}) \nabla \varphi^{k+1} \cdot \nabla \varphi_0) u^{k+1} dx \\
&\leq c\|u^{k+1}\|_2 + c\|\nabla \varphi^{k+1}\|_2 \|\nabla u^{k+1}\|_2 + c\|\nabla \varphi^{k+1}\|_2 \|u^{k+1}\|_2 \\
&\leq c\|u^{k+1}\|_2 + c\|\nabla u^{k+1}\|_2 + c\|u^{k+1}\|_2 \\
&\leq c\|u^{k+1}\|_2 + c\|\nabla u^{k+1}\|_2 \\
&\leq \frac{1}{2}\|u^{k+1}\|_2^2 + c + \frac{\alpha_0}{2}\|\nabla u^{k+1}\|_2^2 + c.
\end{aligned}$$

Then,

$$\|u^{k+1}\|_2^2 + \alpha_0 \|\nabla u^{k+1}\|_2^2 \leq c$$

and therefore

$$\|u^{k+1}\|_{H^1(\Omega)} \leq c.$$

This concludes the proof. □

3.3 Proof of Theorem 3.3

The proof uses Schauder's fixed point theorem. We construct an appropriate mapping whose fixed points will be solutions to (3.13)–(3.14). Let $z \in H^1(\Omega)$ and let φ_z^{k+1} be the unique solution of

$$\int_\Omega \sigma(z) \nabla \varphi_z^{k+1} \nabla \psi \; dx = 0 \quad \text{for all } \psi \in H_0^1(\Omega).$$

Recall that by Lemma 3.1 we have $\varphi_z^{k+1} \in L^\infty(\Omega)$ and $\|\varphi_z^{k+1}\|_{H^1(\Omega)} \le c$, independent of z. For the sake of simplicity, let us define the functional $F_z^k \in H^{-1}(\Omega)$ by

$$\langle F_z^k, v\rangle = (1-b_1)(u^k, v) + \sum_{j=1}^{k-1}(b_j - b_{j+1})(u^{k-j}, v) + b_k(u^0, v) + \alpha_0 \left(\sigma(z)|\nabla\varphi_z^{k+1}|^2, v\right).$$

For the special $k = 0$, we have

$$\langle F_z^0, v\rangle = (u^0, v) + \alpha_0 \left(\sigma(z)|\nabla\varphi_z^1|^2, v\right).$$

When $k = 1$,

$$\langle F_z^1, v\rangle = (1 - b_1)(u^1, v) + b_1(u^0, v) + \alpha_0(\sigma(z)|\nabla\varphi_z^2|^2, v).$$

We show that F_z^k is well defined. Indeed, by Lemma 3.1, we easily show that there exists a constant $c > 0$, independent of z and k, such that

$$\|F_z^k\|_{H^1(\Omega)} \le c, \forall z \in H^1(\Omega).$$

Then we define the operator $A : H^1(\Omega) \to H^1(\Omega)$ as follows: $Az = w_z = u^{k+1}$ if w_z is the unique solution in $H^1(\Omega)$ of

$$\int_\Omega w_z v + \alpha_0 \int_\Omega \nabla w_z \nabla v dx = \langle F_z^k, v\rangle. \tag{3.15}$$

Similarly, there exists a constant $c > 0$, independent of z and k, such that

$$\|Az\|_{H^1(\Omega)} = \|w_z\|_{H^1(\Omega)} \le c, \forall z \in H^1(\Omega). \tag{3.16}$$

Then the operator A is also well defined. In order to prove that A has a fixed point w_z in the ball B_c of center 0 and radius c in $H^1(\Omega)$ defined by

$$B_c = \{z \in H^1(\Omega); \|z\|_{H^1(\Omega)} \le c\},$$

for c large enough, it remains to prove that the operator A is continuous in the weak topology of $H^1(\Omega)$. Then it is sufficient to show that

$$z \to F_z^k \text{ is weakly continuous from } B_c \text{ to } H^{-1}(\Omega) \tag{3.17}$$

and

$$F_z^k \to w_z \text{ is weakly continuous from } H^{-1}(\Omega) \text{ to } B_c. \tag{3.18}$$

To proceed with the proof of (3.17), we assume that

$$(z_n) \subset B_c \text{ and } z_n \to z \text{ weakly in } H^1(\Omega).$$

Since B_c is bounded, there exists a $2 < p < \dfrac{2N}{N-2}$, $N \geq 2$, such that

$$z_n \to z \text{ strongly in } L^p(\Omega) \tag{3.19}$$

on a subsequence. By Lemma 3.1, we have that $(\varphi_{z_n}^{k+1})$ is bounded in $H^1(\Omega)$ independently of z. Then, for a subsequence of (z_n), we have

$$\varphi_{z_n}^{k+1} \to \varphi^{k+1} \text{ in } H^1(\Omega) \text{ and } \varphi_{z_n}^{k+1} \to \varphi^{k+1} \text{ in } L^p(\Omega), 2 < p < \frac{2N}{N-2}\,(N > 2)\,. \tag{3.20}$$

On the other hand, since σ is continuous and bounded, it follows from (3.19) a subsequence (z_{n_k}) such that

$$\sigma(z_{n_j}) \to \sigma(z) \text{ strongly in } L^p(\Omega), \forall p \geq 1. \tag{3.21}$$

Passing to the limit as $j \to \infty$ in

$$\int_\Omega \sigma(z_{n_j}) \nabla \varphi_{z_{n_j}}^{k+1} \nabla \psi \, dx = 0 \quad \text{ for all } \psi \in C^1(\overline{\Omega}) \cap H_0^1(\Omega),$$

and using (3.20)–(3.21), it follows that

$$\int_\Omega \sigma(z) \nabla \varphi^{k+1} \nabla \psi \, dx = 0 \quad \text{ for all } \psi \in H_0^1(\Omega).$$

Since φ_z^{k+1} is unique, we conclude that

$$\varphi^{k+1} = \varphi_z^{k+1}.$$

This implies that

$$z \to \varphi_z^{k+1} \text{ is weakly continuous from } H^1(\Omega) \text{ to } H^1(\Omega).$$

To prove (3.18), it is sufficient to show that $F_{z_n}^k \to F_z^k$ weakly in $H^{-1}(\Omega)$. For $v \in C^1(\overline{\Omega}) \cap H^1(\Omega)$, we have

$$\begin{aligned}\langle F_{z_n}^k, v \rangle = & \int_\Omega \sigma(z_n)(\varphi_0 - \varphi_{z_n}^{k+1}) \nabla \varphi_{z_n}^{k+1} \cdot \nabla v dx + \int_\Omega \sigma(z_n) \nabla \varphi_{z_n}^{k+1} \cdot \nabla \varphi_0 v dx \\ & + (1 - b_1)(u^k, v) + \sum_{j=1}^{k-1} (b_j - b_{j+1})(u^{k-j}, v) + b_k \left(u^0, v \right).\end{aligned}$$

Using again (3.20)–(3.21), we may pass to the limit as $n \to \infty$ and obtain

$$\lim_{n \to \infty} \langle F_{z_n}^k - F_z^k, v \rangle = 0, \forall v \in C^1(\overline{\Omega}) \cap H^1(\Omega).$$

It follows from the boundedness of $F^k_{z_n}$ and the density of $C^1(\overline{\Omega}) \cap H^1(\Omega)$ in $H^1(\Omega)$ that

$$F^k_{z_n} \to F^k_z \text{ in } H^{-1}(\Omega).$$

We have from (3.16) and the weak compactness of B_c in $H^1(\Omega)$ that

$$Az_n \to w_z \text{ in } H^1(\Omega),$$

which proves the continuity of A. Then

$$\int_\Omega w_z v dx + \alpha_0 \int_\Omega \nabla w_z \nabla v dx = \langle F^k_z, v\rangle, \quad \forall v \in H^1(\Omega).$$

By the unique solvability of (3.15), we obtain that $Az = w_z = u^{k+1}$. This ends the proof. We point out that uniqueness can be shown by strengthening the hypotheses on the data (see, e.g., [2, 20]).

In the next section we prove a stability result and obtain some error estimates.

4 Stability and Error Analysis

The weak formulation of equation (3.8) is, $\forall k \geq 1$ and $v \in H^1(\Omega)$,

$$\begin{aligned} &(u^{k+1}, v) + \alpha_0(-\triangle u^{k+1}, v) \\ &= (1-b_1)(u^k, v) + \sum_{j=1}^{k-1}(b_j - b_{j+1})(u^{k-j}, v) + b_k(u^0, v) + \alpha_0\left(\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2, v\right). \end{aligned} \tag{4.1}$$

We have the following unconditionally stability result.

Theorem 4.1. *The semi-discretized problem* (3.13)–(3.14) *is stable, in the sense that for all* $\delta > 0$ *the following inequality holds:*

$$\|u^{k+1}\|_{H^1(\Omega)} \leq \|u^0\|_2 + c.$$

Proof. We prove the result by recurrence. When $k = 0$, we have for $v \in H^1(\Omega)$ that

$$(u^1, v) + \alpha_0(-\triangle u^1, v) = (u^0, v) + \alpha_0\left(\sigma(u^1)|\nabla\varphi^1|^2, v\right).$$

In other terms,

$$\int_\Omega u^1 v dx + \alpha_0 \int_\Omega \nabla u^1 \nabla v dx = \int_\Omega u^0 v dx + \alpha_0 \int_\Omega \sigma(u^1)|\nabla\varphi^1|^2 v dx. \tag{4.2}$$

Taking $v = u^1$ in (4.2), and using Lemma 3.1, we have

$$\begin{aligned}
&\alpha_0 \int_\Omega \sigma(u^1)|\nabla\varphi^1|^2 u^1 dx \\
&\quad = \alpha_0 \int_\Omega (\varphi_0 - \varphi^1)\sigma(u^1)\nabla\varphi^1\nabla u^1 dx + \alpha_0 \int_\Omega \sigma(u^1)\nabla\varphi^1\nabla\varphi_0 u^1 dx \\
&\quad \le c\|\varphi_0 - \varphi^1\|_\infty \|\nabla\varphi^1\|_2 \|\nabla u^1\|_2 + c\|\nabla\varphi_0\|_\infty \|\nabla\varphi^1\|_2 \|u^1\|_2 \\
&\quad \le c\|\nabla u^1\|_2 + c\|u^1\|_2 \\
&\quad \le c\|u^1\|_{H^1(\Omega)}.
\end{aligned}$$

We also have

$$\int_\Omega u^0 u^1 dx \le \|u^0\|_2 \|u^1\|_2 \le \|u^0\|_2 \|u^1\|_2 \le \|u^0\|_2 \|u^1\|_{H^1(\Omega)}.$$

We then obtain by (2.2) and (4.2) that

$$\left\|u^1\right\|^2_{H^1(\Omega)} \le \left(\left\|u^0\right\|_2 + c\right) \|u^1\|_{H^1(\Omega)}. \tag{4.3}$$

Dividing both sides of the above inequality (4.3) by $\|u^1\|_{H^1(\Omega)}$, we get

$$\|u^1\|_{H^1(\Omega)} \le \|u^0\|_2 + c.$$

Suppose now that we have

$$\|u^j\|_{H^1(\Omega)} \le \|u^0\|_2 + c, \quad j = 1, 2, \ldots, k, \tag{4.4}$$

and prove that $\|u^{k+1}\|_{H^1(\Omega)} \le \|u^0\|_2 + c$. Choosing $v = u^{k+1}$ in (4.1), we obtain

$$\begin{aligned}
&(u^{k+1}, u^{k+1}) + \alpha_0(-\triangle u^{k+1}, u^{k+1}) \\
&\quad = (1 - b_1)(u^k, u^{k+1}) + \sum_{j=1}^{k-1}(b_j - b_{j+1})(u^{k-j}, u^{k+1}) + b_k(u^0, u^{k+1}) \\
&\qquad + \alpha_0 \left(\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2, u^{k+1}\right).
\end{aligned}$$

Then, using the recurrence hypothesis (4.4), we obtain

$$\begin{aligned}\|u^{k+1}\|^2_{H^1(\Omega)} &\leq (1-b_1)\|u^k\|_2\|u^{k+1}\|_2 + \sum_{j=1}^{k-1}(b_j - b_{j+1})\|u^{k-j}\|_2\|u^{k+1}\|_2 \\ &+ b_k\|u^0\|_2\|u^{k+1}\|_2 + \alpha_0(\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2, u^{k+1}) \\ &\leq \left\{(1-b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k\right\}\left(\|u^0\|_2 + c\right)\|u^{k+1}\|_2 \\ &+ \alpha_0\left(\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2, u^{k+1}\right) \\ &\leq \left\{(1-b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k\right\}\left(\|u^0\|_2 + c\right)\|u^{k+1}\|_{H^1(\Omega)} \\ &+ \alpha_0\left(\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2, u^{k+1}\right) \\ &\leq \left(\|u^0\|_2 + c\right)\|u^{k+1}\|_{H^1(\Omega)} + \alpha_0\left(\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2, u^{k+1}\right),\end{aligned}$$

since $(1-b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k = 1$. Similarly to the case $k = 0$, we have

$$\begin{aligned}\int_\Omega \sigma(u^{k+1})|\nabla\varphi^{k+1}|^2 u^{k+1}dx &= \int_\Omega (\varphi_0 - \varphi^{k+1})\sigma(u^{k+1})\nabla\varphi^{k+1}\nabla u^{k+1}dx \\ &+ \int_\Omega \sigma(u^{k+1})\nabla\varphi^{k+1}\nabla\varphi_0 u^{k+1}dx \\ &\leq c\|u^{k+1}\|_{H^1(\Omega)}.\end{aligned}$$

Then,

$$\|u^{k+1}\|_{H^1(\Omega)} \leq \|u^0\|_2 + c.$$

This concludes the proof. □

We have the following error analysis for the solution of the semi-discretized problem (3.13)–(3.14).

Theorem 4.2. *Let u be the exact solution of* (2.1)*, $(u^j)_j$ be the time-discrete solution of problem* (3.13) *with the initial condition $u^0(x) = u(x,0)$. If we suppose further to hypotheses $(H1)$–$(H3)$ that*

(H4) $\nabla u(x, t_{k+1}), \nabla\varphi(x, t_{k+1})$ for $q > max(N, 2)$,

then we have the following error estimates:

(a) $\|u(t_j) - u^j\|_{H^1(\Omega)} \leq c_{u,\alpha}T^\alpha\delta^{2-\alpha}$, $j = 1, \ldots, N$, where $0 \leq \alpha < 1$ and

$$c_{u,\alpha} = \frac{c_u}{1-\alpha}$$

with c_u a constant depending on u.

(b) when $\alpha \to 1$,

$$\|u(t_j) - u^j\|_{H^1(\Omega)} \leq c_u T\delta, \quad j = 1, \ldots, N.$$

Proof. Let $e^k = u(x,t_k) - u^k(x)$ be the difference between the exact solution of (2.1) and u^k, the time-discrete solution of (3.13). Obviously, $e^0 = 0$.

(a) We will prove the result by induction. We begin with $0 \leq \alpha < 1$. For $j = 1$, by gathering (2.1) and (3.13), the error equation reads:

$$\begin{aligned}(e^1,v) &+ \alpha_0 \int_\Omega \nabla e^1 \nabla v dx \\ &= (e^0, v) + (r^1, v) + \alpha_0(\sigma(u(x,t_2))|\nabla\varphi(x,t_2)|^2, v) - \alpha_0(\sigma(u^2)|\nabla\varphi^2|^2, v \\ &= (r^1, v) + \alpha_0\left(\sigma(u(x,t_2))|\nabla\varphi(x,t_2)|^2, v\right) - \alpha_0\left(\sigma(u^2)|\nabla\varphi^2|^2, v\right).\end{aligned}$$

Choosing $v = e^1$ in the above equation, it yields that

$$\begin{aligned}\|e^1\|_2^2 + \alpha_0\|\nabla e^1\|_2^2 &\leq \|r^1\|_2\|e^1\|_2 \\ &\quad + \alpha_0\left(\sigma(u(x,t_2))|\nabla\varphi(x,t_2)|^2 - \sigma(u^2)|\nabla\varphi^2|^2, e^2\right). \quad (4.5)\end{aligned}$$

To continue the proof, we shall need the following lemma.

Lemma 4.3. *Let* (u_i, φ_i), $i = 1, 2$, *be two weak solutions of* (1.1)–(1.2). *Assume that* $(H1)$–$(H4)$ *hold. Then,*

$$\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2 \leq (\varepsilon + cc_\varepsilon)\|w\|_{H^1(\Omega)}^2 + \frac{1}{2}\|\nabla w\|_2^2,$$

where $w = u_1 - u_2$ *and* $\varepsilon, c, c_\varepsilon$ *are positive constants.*

Proof. Set $w = u_1 - u_2$ and $\varphi = \varphi_1 - \varphi_2$. It is easy to see that $\sigma(u)|\nabla\varphi|^2 = \mathrm{div}(\sigma(u)\varphi\nabla\varphi)$ (see [2, 17]). Then, we have

$$\begin{aligned}&\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2 \\ &= \mathrm{div}\,(\sigma(u_1)\varphi_1\nabla\varphi_1) - \mathrm{div}\,(\sigma(u_2)\varphi_2\nabla\varphi_2) \\ &= \mathrm{div}\,(\sigma(u_1)\varphi_1\nabla\varphi_1 - \sigma(u_2)\varphi_2\nabla\varphi_2) \\ &= \mathrm{div}\,((\sigma(u_1) - \sigma(u_2))\varphi_1\nabla\varphi_1 + \sigma(u_2)(\varphi_1 - \varphi_2)\nabla\varphi_1 + \sigma(u_2)\varphi_2(\nabla\varphi_1 - \nabla\varphi_2)) \\ &= \mathrm{div}\,((\sigma(u_1) - \sigma(u_2))\varphi_1\nabla\varphi_1 + \sigma(u_2)\varphi\nabla\varphi_1 + \sigma(u_2)\varphi_2\nabla\varphi).\end{aligned}$$

If we multiply by w and integrate over Ω, we get

$$(\mathrm{div}\,(\sigma(u_1)\varphi_1\nabla\varphi_1) - \mathrm{div}\,(\sigma(u_2)\varphi_2\nabla\varphi_2)\,, w) \leq I_1 + I_2 + I_3, \quad (4.6)$$

where

$$I_1 = -\int_\Omega (\sigma(u_1) - \sigma(u_2))\varphi_1 \nabla\varphi_1 \cdot \nabla w dx,$$
$$I_2 = -\int_\Omega \sigma(u_2)\varphi \nabla\varphi_1 \cdot \nabla w dx,$$
$$I_3 = -\int_\Omega \sigma(u_2)\varphi_2 \nabla\varphi \cdot \nabla w dx.$$

Using $(H3)$, since σ and the φ_i's are bounded, we obtain by Hölder's inequality

$$|I_1| \le c\int_\Omega |\nabla\varphi_1||\nabla w||w| ds \le c\|\nabla\varphi_1\|_q \|\nabla w\|_2 \|w\|_{\frac{2q}{q-2}},$$
$$|I_2| \le c\int_\Omega |\varphi||\nabla\varphi_1||\nabla w| dx \le c\|\nabla\varphi_1\|_q \|\nabla w\|_2 \|\varphi\|_{\frac{2q}{q-2}},$$
$$|I_3| \le c\int_\Omega |\varphi_2||\nabla\varphi||\nabla w| dx \le c\|\nabla\varphi\|_2 \|\nabla w\|_2.$$

Since $q > N$ from the Sobolev imbedding theorem, we have

$$|I_2| \le c\|\nabla\varphi_1\|_q \|\nabla w\|_2 \|\nabla\varphi\|_2.$$

Note that from the equation satisfied by φ_1, φ_2 we have

$$0 = \operatorname{div}(\sigma(u_2)\nabla\varphi_2) = \operatorname{div}(\sigma(u_2)\nabla(\varphi_2 - \varphi_1)) + \operatorname{div}((\sigma(u_2) - \sigma(u_1))\nabla\varphi_1).$$

Then,

$$\operatorname{div}(\sigma(u_2)\nabla\varphi) = \operatorname{div}((\sigma(u_1) - \sigma(u_2))\nabla\varphi_1).$$

If we multiply by φ and integrate on Ω, we get

$$\sigma_1\|\nabla\varphi\|_2^2 \le \int_\Omega \sigma(u_2)|\nabla\varphi|^2 dx = \int_\Omega (\sigma(u_1) - \sigma(u_2))\nabla\varphi_1 \cdot \nabla\varphi dx.$$

Using again Hölder's inequality and $(H3)$, we obtain

$$\|\nabla\varphi\|_2^2 \le c\int_\Omega |w||\nabla\varphi_1||\nabla\varphi|| dx \le c\|\nabla\varphi_1\|_q \|\nabla\varphi\|_2 \|w\|_{\frac{2q}{q-2}}.$$

Thus,

$$\|\nabla\varphi\|_2 \le c\|\nabla\varphi_1\|_q \|w\|_{\frac{2q}{q-2}}.$$

From (4.6) we have

$$\sigma(u_1)|\nabla\varphi_1|^2 - \sigma(u_2)|\nabla\varphi_2|^2 \le c\left\{\|\nabla\varphi_1\|_q \|\nabla w\|_2 \|w\|_{\frac{2q}{q-2}} + \|\nabla\varphi_1\|_q^2 \|\nabla w\|_2 \|w\|_{\frac{2q}{q-2}}\right\}.$$

A use of Young's inequality in the right-hand side of the above inequality allows us to obtain

$$\left(\sigma(u_1)|\nabla\varphi_1|^2-\sigma(u_2)|\nabla\varphi_2|^2,w\right)\leq c\left\{\|\nabla\varphi_1\|_q^2+\|\nabla\varphi_1\|_q^4\right\}\|w\|^2_{\frac{2q}{q-2}}+\frac{1}{2}\|\nabla w\|_2^2.$$

Applying Gagliardo's inequality [2],

$$\|w\|^2_{\frac{2q}{q-2}}\leq\|w\|_2^{1-\frac{n}{q}}\left(\|w\|_2^2+\|\nabla w\|_2^2\right)^{\frac{n}{q}},$$

and Young's inequality, we obtain

$$\begin{aligned}
&(\sigma(u_1)|\nabla\varphi_1|^2-\sigma(u_2)|\nabla\varphi_2|^2,w)\\
&\quad\leq c\left\{\|\nabla\varphi_1\|_q^2+\|\nabla\varphi_1\|_q^4\right\}\|w\|_2^{1-\frac{n}{q}}\cdot\left(\|w\|_2^2+\|\nabla w\|_2^2\right)^{\frac{n}{q}}+\frac{1}{2}\|\nabla w\|_2^2\\
&\quad\leq\varepsilon\left(\|w\|_2^2+\|\nabla w\|_2^2\right)+c_\varepsilon\left\{\|\nabla\varphi_1\|_q^{\frac{2q}{q-n}}+\|\nabla\varphi_1\|_q^{\frac{4q}{q-n}}\right\}\|w\|_2^2+\frac{1}{2}\|\nabla w\|_2^2\\
&\quad\leq(\varepsilon+cc_\varepsilon)\left(\|w\|_2^2+\|\nabla w\|_2^2\right)+\frac{1}{2}\|\nabla w\|_2^2\\
&\quad=(\varepsilon+cc_\varepsilon)\|w\|^2_{H^1(\Omega)}+\frac{1}{2}\|\nabla w\|_2^2.
\end{aligned}$$

This concludes the proof of Lemma 4.3. □

Now, we continue the proof of Theorem 4.2. Using (4.5), it follows that

$$\|e^1\|_2^2+\alpha_0\|\nabla e^1\|_2^2\leq\|r^1\|_2\|e^1\|_2+\alpha_0(\varepsilon+cc_\varepsilon)\|e^1\|^2_{H^1(\Omega)}+\frac{\alpha_0}{2}\|\nabla e^1\|_2^2.$$

Then, by (2.2), we have

$$\|e^1\|^2_{H^1(\Omega)}\leq\|r^1\|_2\|e^1\|_2+\alpha_0(\varepsilon+cc_\varepsilon)\|e^1\|^2_{H^1(\Omega)}.$$

It follows that

$$(1-\alpha_0(\varepsilon+cc_\varepsilon))\|e^1\|^2_{H^1(\Omega)}\leq\|r^1\|_2\|e^1\|_2\leq\|r^1\|_2\|e^1\|_{H^1(\Omega)}.$$

For a good choice of ε and dividing both sides by $\|e^1\|_{H^1(\Omega)}$, and using (3.9) and $b_0=1$, we obtain

$$\|u(t_1)-u^1\|_1\leq c_u b_0^{-1}\delta^2.$$

Then point (a) is verified for $j=1$. Suppose now we have proved (a) for all $k=1,\dots,j$, and prove it also for $k=j+1$. Collecting (2.1) and (3.13), we have

$$\begin{aligned}
\left(e^{k+1},v\right)&+\alpha_0\left(-\triangle e^{k+1},v\right)\\
&=(1-b_1)(e^k,v)+\sum_{j=1}^{k-1}(b_j-b_{j+1})\left(e^{k-j},v\right)+b_k(e^0,v)+(r^{k+1},v)\\
&\quad+\alpha_0\left(\sigma(u(x,t_{k+1}))|\nabla\varphi(x,t_{k+1})|^2,v\right)-\alpha_0\left(\sigma(u^{k+1})|\nabla\varphi^{k+1}|^2,v\right).
\end{aligned}\tag{4.7}$$

Taking $v = e^{k+1}$ in (4.7), then

$$\|e^{k+1}\|_2^2 + \alpha_0\|\nabla e^{k+1}\|_2^2$$
$$\leq (1-b_1)\|e^k\|_2\|e^{k+1}\|_2 + \sum_{j=1}^{k-1}(b_j - b_{j+1})\|e^{k-j}\|_2\|e^{k+1}\|_2 + b_k\|e^0\|_2\|e^{k+1}\|_2$$
$$+\|r^{k+1}\|_2\|e^{k+1}\|_2 + \alpha_0\left(\sigma\left(u(x,t_{k+1})\right)|\nabla\varphi(x,t_{k+1})|^2 - \sigma(u^{k+1})|\nabla\varphi^{k+1}|^2, e^{k+1}\right).$$

By Lemma 4.3,

$$\alpha_0\left(\sigma(u(x,t_{k+1}))|\nabla\varphi(x,t_{k+1})|^2 - \sigma(u^{k+1})|\nabla\varphi^{k+1}|^2, e^{k+1}\right)$$
$$\leq \alpha_0(\varepsilon + cc_\varepsilon)\|e^{k+1}\|_{H^1(\Omega)}^2 + \frac{\alpha_0}{2}\|\nabla e^{k+1}\|_2^2.$$

Using the induction assumption and the fact that $\dfrac{b_k^{-1}}{b_{k+1}^{-1}} < 1$ for a positive integer k, we have

$$\|e^{k+1}\|_2^2 + \alpha_0\|\nabla e^{k+1}\|_2^2$$
$$\leq \left\{(1-b_1)b_{k-1}^{-1} + \sum_{j=1}^{k-1}(b_j - b_{j+1})b_{k-j-1}^{-1}\right\}c_u\delta^2\|e^{k+1}\|_2$$
$$+ \alpha_0(\varepsilon + cc_\varepsilon)\|e^{k+1}\|_{H^1(\Omega)}^2 + \frac{\alpha_0}{2}\|\nabla e^{k+1}\|_2^2$$
$$\leq \left\{(1-b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k\right\}c_u b_{k-1}^{-1}\delta^2\|e^{k+1}\|_{H^1(\Omega)}$$
$$+ \alpha_0(\varepsilon + cc_\varepsilon)\|e^{k+1}\|_{H^1(\Omega)}^2 + \frac{\alpha_0}{2}\|\nabla e^{k+1}\|_2^2.$$

We then have

$$\|e^{k+1}\|_2^2 + \alpha_0\|\nabla e^{k+1}\|_2^2 \leq c_u b_{k-1}^{-1}\delta^2\|e^{k+1}\|_{H^1(\Omega)} + \alpha_0(\varepsilon + cc_\varepsilon)\|e^{k+1}\|_{H^1(\Omega)}^2,$$

since $(1-b_1) + \sum_{j=1}^{k-1}(b_j - b_{j+1}) + b_k = 1$. Then,

$$\|e^{k+1}\|_{H^1(\Omega)}^2 \leq c_u b_{k-1}^{-1}\delta^2\|e^{k+1}\|_{H^1(\Omega)} + \alpha_0(\varepsilon + cc_\varepsilon)\|e^{k+1}\|_{H^1(\Omega)}^2.$$

Therefore,

$$(1 - \alpha_0(\varepsilon + cc_\varepsilon))\|e^{k+1}\|_{H^1(\Omega)}^2 \leq c_u b_{k-1}^{-1}\delta^2\|e^{k+1}\|_{H^1(\Omega)}.$$

For a suitable choice of ε and dividing both sides by $\|e^{k+1}\|_{H^1(\Omega)}$, we get

$$\|e^{k+1}\|_{H^1(\Omega)} \leq c_u b_{k-1}^{-1}\delta^2.$$

One can show easily that

$$k^{-\alpha} b_{k-1}^{-1} \leq \frac{1}{1-\alpha}, \quad k = 1, \ldots, N.$$

Hence we have, for all k such that $k\delta \leq T$, that

$$\begin{aligned}\|u(t_k) - u^k\|_{H^1(\Omega)} \leq c_u b_{k-1}^{-1} \delta^2 &= c_u k^{-\alpha} b_{k-1}^{-1} k^{\alpha} \delta^2 \\ &\leq c_u \frac{1}{1-\alpha} (k\delta)^{\alpha} \delta^{2-\alpha} \\ &\leq \frac{c_u}{1-\alpha} T^{\alpha} \delta^{2-\alpha}.\end{aligned}$$

(b) We are now interested in the case $\alpha \to 1$. We will derive again the following estimation by induction:

$$\|u(t_j) - u^j\|_1 \leq c_u j \delta^2, \quad j = 1, 2, \ldots, N. \tag{4.8}$$

The above inequality is obvious for $j = 1$. Suppose now that (4.8) holds for all $j = 1, 2, \ldots, k$ and we need to prove that it holds also for $j = k+1$. Similarly to the previous case, by combining (2.1) and (3.13) and taking $v = e^{k+1}$ as a test function, we derive

$$\begin{aligned}
&\|e^{k+1}\|_2^2 + \alpha_0 \|\nabla e^{k+1}\|_2^2 \\
&\quad \leq (1-b_1)\|e^k\|_2 \|e^{k+1}\|_2 + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \|e^{k-j}\|_2 \|e^{k+1}\|_2 \\
&\quad + b_k \|e^0\|_2 \|e^{k+1}\|_2 + \|r^{k+1}\|_2 \|e^{k+1}\|_2 + \alpha_0(\varepsilon + c c_\varepsilon) \|e^{k+1}\|_{H^1(\Omega)} \\
&\quad \leq \left\{ (1-b_1)(c_u k \delta^2) + \sum_{j=1}^{k-1} (b_j - b_{j+1})(c_u (k-j)\delta^2) + c_u \delta^2 \right\} \|e^{k+1}\|_2 \\
&\quad + \alpha_0(\varepsilon + c c_\varepsilon) \|e^{k+1}\|_{H^1(\Omega)} \\
&\quad \leq \left\{ (1-b_1)\frac{k}{k+1} + \sum_{j=1}^{k-1} (b_j - b_{j+1}) \frac{k-j}{k+1} + \frac{1}{k+1} \right\} c_u (k+1)\delta^2 \|e^{k+1}\|_2 \\
&\quad + \alpha_0(\varepsilon + c c_\varepsilon) \|e^{k+1}\|_{H^1(\Omega)} \\
&\quad \leq \left\{ (1-b_1) + \sum_{j=1}^{k-1} (b_j - b_{j+1}) - (1-b_1)\frac{1}{k+1} \right. \\
&\quad \left. - \sum_{j=1}^{k-1} (b_j - b_{j+1}) \frac{j+1}{k+1} + \frac{1}{k+1} \right\} c_u (k+1)\delta^2 \|e^{k+1}\|_2 \\
&\quad + \alpha_0(\varepsilon + c c_\varepsilon) \|e^{k+1}\|_{H^1(\Omega)}.
\end{aligned}$$

Note that

$$(1-b_1)\frac{1}{k+1}+\sum_{j=1}^{k-1}(b_j-b_{j+1})\frac{j+1}{k+1}+b_k \geq \frac{1}{k+1}\{(1-b_1)+\sum_{j=1}^{k-1}(b_j-b_{j+1})+b_k\}=\frac{1}{k+1}.$$

Then,

$$\begin{aligned}(1-\alpha_0(\varepsilon+cc_\varepsilon))\|e^{k+1}\|^2_{H^1(\Omega)} &\leq \left\{(1-b_1)+\sum_{j=1}^{k-1}(b_j-b_{j+1})+b_k\right\}c_u(k+1)\delta^2\|e^{k+1}\|_{H^1(\Omega)}\\ &= c_u(k+1)\delta^2\|e^{k+1}\|_{H^1(\Omega)}\end{aligned}$$

and it follows, for an ε well chosen such that $1-\alpha_0(\varepsilon+cc_\varepsilon)>0$ and after dividing by $\|e^{k+1}\|_{H^1(\Omega)}$, that $\|e^{k+1}\|_{H^1(\Omega)}\leq c_u(k+1)\delta^2$. The estimate (b) is proved.

The proof is complete. □

References

[1] S. Abbas, M. Benchohra, and G. M. N'Guérékata, Topics in fractional differential equations, Springer, New York, 2012.

[2] S. N. Antontsev and M. Chipot, The thermistor problem; existence, smoothness, uniqueness, blowup, SIAM J. Math. Anal., 25 (1994), pp. 1128–1156.

[3] K. Atkinson and W. Han, Theoretical numerical analysis. A functional analysis framework, Springer, New York, 2001.

[4] G. Cimatti, A bound for the temperature in the thermistor's problem, IMA J. Appl. Math., 40 (1988), pp. 15–22.

[5] G. Cimatti, Existence of weak solutions for the nonstationary problem of the Joule heating of a conductor, Ann. Mat. Pura. Appl. (4), 162 (1992), pp. 33–42.

[6] A. C. Fowler, I. Frigaard, and S. D. Howison, Temperature surges in current-limiting circuit devices, SIAM J. Appl. Math., 52 (1992), pp. 998–1011.

[7] R. Gorenflo, F. Mainardi, D. Moretti, P. Paradisi, Time fractional diffusion: a discrete random walk approach, Nonlinear Dynam. 29 (2002), pp. 129–143.

[8] S. D. Howison, J. F. Rodrigues, and M. Shillor, Stationary solutions to the thermistor problem, J. Math. Anal. Appl., 174 (1993), pp. 573–588.

[9] V. Hrynkiv, S. Lenhart, and V. protopopescu, Optimal control of a connective boundary condition in a thermistor problem, SIAM J. Control. Optim. Vol. 47, No. 1, pp. 20–39.

[10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, 204, Elsevier Science B. V. Amsterdam, 2006.

[11] K. Kwok, Complete guide to semiconductor devices, McGraw-Hill, New York, 1995.

[12] H.-C. Lee and T. shiklin, Analysis of optimal control problem for the two-dimernsional thermistor system, SIAM J. Control Optim., 44 (2005), pp. 268–282.

[13] Y. Lin, C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, Journal of Computational Physics, 225 (2007), pp. 1533–1552.

[14] E. D. Maclen, Thermistors, Electrochemical Publications, Glascow, 1979.

[15] I. Podlubny, Fractional differential equations, Academic Press, San Diego, 1999.

[16] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional integrals and derivatives, theory and applications, Gordon and Breach, Yverdon, 1993.

[17] P. Shi, M. Shillor, and X. Xu, Existence of a solution to the Stefan problem with Joule's heating, J. Differential Equations, 105 (1993), pp. 239–263.

[18] M. R. Sidi Ammi, Application of the L^∞-energy method to the nonlocal thermistor problem, Arabian Journal for Science and Engineering: Section A Sciences, 35, pp. 1-12.

[19] M. R. Sidi Ammi and D. F. M. Torres, Existence and uniqueness of a positive solution to generalized nonlocal thermistor problems with fractional-order derivatives, Differ. Equ. Appl. 4 (2012), No. 2, pp. 267–276.

[20] M. R. Sidi Ammi and D. F. M. Torres, Optimal control of nonlocal thermistor equations, International Journal of Control, Vol. 85, No. 11, November 2012, pp. 1789–1801.

[21] M. E. Taylor, Partial Differential Equations III. Nonlinear Equations, Springer, New York, 1996.

[22] E. Zeidler, Nonlinear functional analysis and its applications IV, Springer, New York, 1988.