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Mathematical Study of a Model Arising from Stratigraphy

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Abstract

In this paper, we are interested in a mathematical problem arising from the modelling of maximal erosion rates in geological stratigraphy. The problem is nonlinear with a diffusion coefficient that is a nonlinear function of u and $\partial_t u$. Moreover, the problem degenerates in order to take implicitly into account a constraint on $\partial_t u$. We present a result of existence and uniqueness of solution to the continuous problem under a condition on a parameter denoted by τ.

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1 Introduction and Mathematical Model

This work deals with the study of a mathematical model arising from the modelling of geological basin formation. It takes into account sedimentation, transport and accumulation, erosion phenomena, and others. The original mathematical aspect of this model is the imposition of a constraint on the time-derivative of the unknown u.

Let us consider a sedimentary basin, we denote by Ω its basis, a fixed bounded domain of $\mathbb{R}^N$ $\big(N = 1, 2\big)$ with a Lipschitz boundary Γ. For any positive T, we set $Q =]0, T[\times\Omega$. Denote by u the sediments thickness. It naturally satisfies the mass balance equation

$$\partial_t u + \operatorname{div}(\vec{q}) = f(u) \text{ in } Q. \tag{1.1}$$

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According to Darcy–Barenblatt's law (see, e.g., [6]), the flux $\vec{q}$ is given by the relation

$$\vec{q} = -\lambda K(u)\nabla(u + \tau\partial_t u), \tag{1.2}$$

where λ is a parameter to be defined later and τ a positive time-scaled parameter.

In a sedimentary basin formation process, sediments must first be produced in situ by weathering effects prior to be transported by surfacing erosion. Thus, a maximum erosion rate $-\partial_t u \leq E$ in Q, where E is non-negative and takes into account the composition, the structure and the age of the sediments, has to be introduced [7]. In this paper, the authors introduce a flux limiter λ, $0 \leq \lambda \leq 1$, that satisfies

$$\partial_t u + E \geq 0,\ (1-\lambda)(\partial_t u + E) = 0, \text{ a.e. in } Q. \tag{1.3}$$

Following [2], one remarks that as soon as $f + E \geq 0$ for all $u \in L^2(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;H_0^1(\Omega))$, (1.1)–(1.3) is equivalent to the following formulation:

$$\partial_t u - \operatorname{div}[\lambda K(u)\nabla(u+\tau\partial_t u)] = f(u) \text{ in } Q,\ \lambda \in H(\partial_t u + E) \cap L^\infty(Q). \tag{1.4}$$

Here, homogeneous Dirichlet boundary conditions on u and $\partial_t u$ are considered:

$$u(0,\cdot) = u_0 \in H_0^1(\Omega), \quad E \in L^\infty(0,T;H^1(\Omega)), \quad f \in L^\infty(0,T;L^2(\Omega))$$

and H denotes the maximal monotone graph of the Heaviside function. Indeed, if $f + E \geq 0$ is assumed, using the admissible test function $(\partial_t u + E)^-$ (where $x^- = -\min(0,x)$ for $x \in \mathbb{R}$) in (1.4), we get that $\partial_t u + E \geq 0$ a.e. in Q since $\lambda \mathbf{1}_{\{\partial_t u + E < 0\}} = 0$ a.e., and therefore (1.3) and $\lambda \in H(\partial_t u + E)$ are equivalent assertions. Moreover, using that $\partial_t u + E \geq 0$, one has

$$\lambda\nabla(u+\tau\partial_t u) = \lambda\nabla[u - \tau E + \tau(\partial_t u + E)] = \lambda\nabla(u - \tau E) + \tau\nabla(\partial_t u + E).$$

Thus, the problem (1.4) is equivalent to the following one:

$$\begin{cases} \lambda \in H(\partial_t u + E) \cap L^\infty(Q), \\ \partial_t u - \operatorname{div}\Big\{\lambda\, K(u)\big[\nabla u - \tau\nabla E\big]\Big\} - \tau\operatorname{div}\Big\{K(u)\big[\nabla\partial_t u + \nabla E\big]\Big\} = f(u). \end{cases} \tag{1.5}$$

Existence and uniqueness of solution to such a problem is still an open question. A modified model, where H is replaced by a Lipschitz continuous function a, for example the Yosida approximation of H, is analyzed by Antontsev et al. in [2] with $K \equiv 1$, $f \equiv 0$ and a constant E. The existence result and numerical scheme based on the discontinuous Galerkin methods (DgFem) is considered in [5] for the non-homogeneous problem with space-time function E. The existence and the uniqueness of a solution to problem (1.5) is given in [3] with $\lambda = a(\partial_t u + E)$ and null source term, where E is assumed to be constant. Note that the above remark, concerning the equivalence between (1.4) and (1.5), doesn't hold anymore with a. Indeed, following to ideas of the book by Antontsev et al. [1], it has been proved in [8], under some hypothesis on a, that if u is a weak solution to the 1D problem

$$\partial_t u - \partial_x \Big\{ K(u) a(t, \partial_t u) \big[\partial_x u + \tau \partial_{xt} u \big] \Big\} = f \quad \text{in } Q =]0, T[\times \Omega, \tag{1.6}$$

with $\Omega =]0, 1[$ and the boundary and initial conditions

$$\partial_t u_{|t=0} = 0, \quad u(0, x) = u_0(x), \ x \in \Omega, \tag{1.7}$$

where $u_0(x) = 0, \ x \in]0, \rho_0[, \ 0 < \rho_0 < 1$, then there exists a positive $\delta > 0$ and $\rho(t) \in (0, \rho_0)$ such that, if $f(t, x) = 0$ in $]0, \delta[\times]0, \rho_0[$, then u satisfies the finite speed of propagation properties $u(t, x) = 0$ in $x \in]0, \rho(t)[, \ 0 \le t \le \delta$. However, this behavior is unknown to the following pseudo-parabolic problem:

$$\partial_t u - \operatorname{div}\Big\{ a(\partial_t u + E)\, K(u) \big[\nabla u - \tau \nabla E \big] \Big\} - \tau \operatorname{div}\Big\{ K(u) \big[\nabla \partial_t u + \nabla E \big] \Big\} = f \text{ in } Q. \tag{1.8}$$

On one hand, both (1.6) and (1.8) are approximations to the same problem when a converges towards the graph of Heaviside. On the other hand, they reveal distinct natures.

In this study, we consider the nonlinear pseudo-parabolic problem (1.8) with an initial height given by $u(0, \cdot) = u_0$ in Ω, where $u_0 \in H_0^1(\Omega)$, and homogeneous Dirichlet condition for u and $\partial_t u$. The main interest of this approach is that it will be possible to use the theorems of Meyers and Nečas, in order to obtain a more regular solution $\big($i.e., $u \in W^{1,\infty}\big(0, T; W_0^{1,p}(\Omega)\big)$, with $p > 2$ as soon as $u_0 \in W_0^{1,p}(\Omega)\big)$ and thus a uniqueness result. Furthermore, this approach is compatible with a wider class of functions a $\big($in particular, θ-Hölder functions of order $\theta \ge 1/2\big)$ as approximations of the Heaviside graph while, in the first approach, one is limited to Lipschitz continuous function a.

Let us set the assumptions made on the data:

$$\textbf{(H)} : \begin{cases} \tau > 0;\, E \in L^\infty(0, T; H^1(\Omega)), E \ge 0;\, f \in L^\infty(0, T; L^2(\Omega)),\, f + E \ge 0 \text{ in } Q; \\ a \in C^{0,\theta}(\mathbb{R}), \text{ with } \theta \ge 1/2, 0 \le a \le 1, \ \forall x \in] - \infty, 0], \ a(x) = 0; \\ K \text{ and } f \text{ are Lipschitz functions, with } 0 < k_m \le K \le k_M. \end{cases}$$

We now give the definition of solution.

Definition 1.1. Under assumption **(H)**, a solution to problem (1.8) is any function u in $W^{1,2}\big(0, T; H_0^1(\Omega)\big)$, such that for any v in $H_0^1(\Omega)$ and for a.a. t in $]0, T[$,

$$\begin{aligned} \int_\Omega \partial_t u v \, dx + \int_\Omega K(u) a(\partial_t u + E) \big(\nabla u - \tau \nabla E \big) \cdot \nabla v \, dx \\ + \tau \int_\Omega K(u) \big(\nabla \partial_t u + \nabla E \big) \cdot \nabla v \, dx = \int_\Omega f(u) v \, dx \end{aligned} \tag{1.9}$$

with the initial condition $u(0, \cdot) = u_0$ a.e. in Ω.

Since a vanishes on $\mathbb{R}^-$, the constraint $\partial_t u + E \ge 0$ in Q is implicitly satisfied.

2 Existence and Uniqueness Result

In this section, we establish the existence and uniqueness of a solution to problem (1.8). We proceed as follow: we first propose an existence and uniqueness lemma for the solution to an additional stationary problem. Next, we prove the existence of a solution to a semi-discretized problem and by passing to the limits with the time-discretization parameter to 0^+, we prove that the problem (1.9) has a solution via some *a priori* estimates. Finally, we use the theorems of Meyers and Nečas, in order to obtain a more regular solution and thus the uniqueness result.

Lemma 2.1. Suppose $b \in C^{0,\theta}(\mathbb{R})$, a bounded function with $\theta \geq \dfrac{1}{2}$, κ in $H^1(\Omega)$, f in $L^2(\Omega)$, and E in $H^1(\Omega)$. Then the variational problem: find $w \in H_0^1(\Omega)$, such that for all v in $H_0^1(\Omega)$

$$\begin{aligned}\int_\Omega wv\,dx + \int_\Omega K(\kappa)b(w+E)\nabla(\kappa - \tau E)\cdot\nabla v\,dx \\ + \tau\int_\Omega K(\kappa)\nabla(w+E)\cdot\nabla v\,dx = \int_\Omega f(\kappa)v\,dx\end{aligned} \tag{2.1}$$

has a unique solution.

Proof. The existence of a solution is based on the Schauder fixed point theorem. The uniqueness result is based on the L^1 method by using the Lipschitz approximation of the sign function: $t \mapsto p_\mu(t) = \min\left[1, \ln\left(\max(1, \dfrac{e\,t}{\mu})\right)\right]$ (see [4]). □

In the sequel of the section, we consider a uniform partition of $(0,T)$ into N subintervals $[t_k, t_{k+1}[$, $k = 0,\ldots,N-1$ and we denote by $\Delta t = t_{k+1} - t_k$ the time step. Assume that $E^{k+1} \in H^1(\Omega)$ with $E^{k+1} \geq 0$, $f^{k+1} \in L^2(\Omega)$ with $f^{k+1} + E^{k+1} \geq 0$. Denote by $\|\cdot\|_0$ the L^2 norm and by $\|\cdot\|_1$ the H^1 norm. The semi-discretized problem consists to find $u^{k+1} \in H_0^1(\Omega)$ for $k = 0,\ldots,N-1$, such that for all v in $H_0^1(\Omega)$

$$\begin{cases} \displaystyle\int_\Omega \frac{u^{k+1}}{\Delta t} v\,dx + \int_\Omega K\left(u^{k+1}\right) a\left(\frac{u^{k+1}-u^k}{\Delta t} + E^{k+1}\right)\nabla\left[u^{k+1} - \tau E^{k+1}\right]\cdot\nabla v\,dx \\ \displaystyle + \tau\int_\Omega K(u^{k+1})\nabla\frac{u^{k+1}}{\Delta t}\cdot\nabla v\,dx = \int_\Omega\left[f^{k+1}(u^{k+1}) + \frac{u^k}{\Delta t}\right]v\,dx \\ \displaystyle \qquad\qquad + \tau\int_\Omega K(u^{k+1})\nabla\left[\frac{u^k}{\Delta t} - E^{k+1}\right]\cdot\nabla v\,dx, \\ u^0 = u_0 \text{ in } \Omega. \end{cases} \tag{2.2}$$

Note that it is sufficient to prove the result for the first iteration. Let $u = u^1$, $E = E^1$, $f = f^1$, and $w := \dfrac{u - u_0}{\Delta t}$. The problem becomes: find $w \in H_0^1(\Omega)$ such that for all v

in $H_0^1(\Omega)$

$$\int_\Omega wv\,dx + \Delta t\int_\Omega K(u)\left[a(w+E)+\frac{\tau}{\Delta t}\right]\nabla w\cdot\nabla v\,dx$$
$$+\int_\Omega K(u)a(w+E)\nabla(u_0-\tau E)\cdot\nabla v\,dx = \int_\Omega f(u)v\,dx - \tau\int_\Omega K(u)\nabla E\cdot\nabla v\,dx. \tag{2.3}$$

Lemma 2.2. Problem (2.3) admits a unique solution. Moreover, the constraint is implicitly satisfied in the discrete sense: $\dfrac{u-u_0}{\Delta t}+E\geq 0$.

Proof. The existence of w is based on the fixed point theorem of Schauder–Tikhonov. Let $\psi: H_0^1(\Omega)\to H_0^1(\Omega),\ S\mapsto w=\psi(S)$, where w is the unique solution of the following linear problem: find $w\in H_0^1(\Omega)$ such that for all $v\in H_0^1(\Omega)$:

$$\begin{aligned}\int_\Omega wv\,dx + \Delta t\int_\Omega K(\Delta t\,S+u_0)\left[a(S+E)+\frac{\tau}{\Delta t}\right]\nabla w\cdot\nabla v\,dx\\ =\int_\Omega f(\Delta t\,S+u_0)\,v\,dx - \int_\Omega K(\Delta t\,S+u_0)\,a(S+E)\nabla(u_0-\tau E)\nabla v\,dx\\ -\tau\int_\Omega \nabla E\nabla v dx.\end{aligned} \tag{2.4}$$

The existence of a solution to this linear problem is guaranteed by the Lax–Milgram lemma since $K(\Delta t\,S+u_0)(\Delta t\,a(S+E)+\tau)\geq\tau k_m>0$. Using $v=w$ in the weak formulation, we have

$$\begin{aligned}\|w\|_0^2 + \int_\Omega K(\Delta t\,S+u_0)[\Delta t\,a(S+E)+\tau]|\nabla w|^2dx\\ =\int_\Omega f(\Delta t\,S+u_0)wdx - \int_\Omega K(\Delta t\,S+u_0)a(S+E)\nabla(u_0-\tau E)\nabla wdx\\ -\tau\int_\Omega K(\Delta t\,S+u_0)\nabla E\nabla wdx.\end{aligned}$$

Using the assumption, Cauchy–Schwarz and Young's inequalities, we obtain

$$\frac{1}{2}\|w\|_0^2+\frac{\tau}{2}k_m\|\nabla w\|_0^2\leq\frac{1}{2}\|f\|_0^2+\frac{k_M^2}{k_m}\left(4\tau\|\nabla E\|_0^2+\frac{1}{\tau}\|\nabla u_0\|_0^2\right).$$

We conclude that

$$\|w\|_1\leq C=C(\tau,k_m,k_M,\|f\|_0,\|\nabla u_0\|_0,\|\nabla E\|_0).$$

The set $K=\{v\in H_0^1(\Omega);\ \|v\|_1\leq C\}$ is not empty, bounded, convex and strongly closed. Thus, weakly closed in $H_0^1(\Omega)$. Moreover, $\psi(K)\subset K$. Consider $S_n\rightharpoonup S$ in

$H_0^1(\Omega)$. Up to a subsequence, $S_{n'} \to S$ in $L^2(\Omega)$ and a.e. in Ω, since the embedding of $H_0^1(\Omega) \subset L^2(\Omega)$ is compact. Then, since K and a are continuous and bounded,

$$K(\Delta t\, S_{n'}+u_0)a(S_{n'}+E)\nabla v \to K(\Delta t\, S+u_0)a(S+E)\nabla v \quad \text{a.e. in } \Omega \text{ and in } \left(L^2(\Omega)\right)^d.$$

Let $w_{S_{n'}} := \psi(S_{n'})$. The boundedness of $(w_{S_{n'}})$ allows to extract a sub-sequence $(w_{S_{n''}})$ such that

$$w_{S_{n''}} \rightharpoonup \xi \text{ in } H_0^1(\Omega).$$

By passing to the limits $(n'' \to \infty)$, we conclude that ξ satisfies: $\xi \in H_0^1(\Omega)$ and for all $v \in H_0^1(\Omega)$

$$\begin{aligned}
&\int_\Omega \xi v\, dx + \Delta t \int_\Omega K\left(\Delta t\, S + u_0\right)\left(a(S+E) + \frac{\tau}{\Delta t}\right)\nabla\xi\cdot\nabla v\, dx \\
&\qquad = \int_\Omega f\left(\Delta t\, S + u_0\right) v\, dx - \int_\Omega K\left(\Delta t\, S + u_0\right) a(S+E)\nabla(u_0 - \tau E)\cdot\nabla v\, dx \\
&\qquad\quad - \tau \int_\Omega K\left(\Delta t\, S + u_0\right)\nabla E\cdot\nabla v\, dx.
\end{aligned} \tag{2.5}$$

Thus, $\xi = \psi(S)$ by the uniqueness of solution, and any sequence (w_{S_n}) converges weakly to w in $H_0^1(\Omega)$. Then, ψ is sequentially continuous with respect to the weak convergence. The existence of a fixed point w then results from the Schauder fixed point theorem in the separable Hilbert framework. The constraint follows as before by taking $v = \left(\dfrac{u-u_0}{\Delta t} + E\right)^-$ in (2.3). □

By induction, we obtain the following result.

Lemma 2.3. u^{k+1} in $H_0^1(\Omega)$ exists, solution to the problem (2.2). As expected, the discrete version of the constraint is satisfied: $\dfrac{u^{k+1}-u^k}{\Delta t} + E^{k+1} \geq 0$ a.e. in Ω.

Let us now prove the following existence result for (1.8).

Proposition 2.4. There exists at least one u in $H^1(0,T;H_0^1(\Omega))$ such that, for a.a. $t \in (0,T)$ and for all $v \in H_0^1(\Omega)$,

$$\begin{aligned}
&\int_\Omega \partial_t u v\, dx + \int_\Omega a(\partial_t u + E)K(u)\nabla(u - \tau E)\cdot\nabla v\, dx \\
&\qquad\qquad + \tau\int_\Omega K(u)\nabla(\partial_t u + E)\cdot\nabla v dx = \int_\Omega f(u) v\, dx.
\end{aligned} \tag{2.6}$$

Moreover, the constraint $\partial_t u + E \geq 0$ is implicitly satisfied.

The proof is based on *a priori* estimations on the sequence (u^k).
Let us denote by $\mathbf{1}_k$ be the characteristic function of the interval $[k\Delta t, (k+1)\Delta t[$ and define the functions

$$u_{\Delta t}(t) = \sum_{k=0}^{N-1} u^k\ \mathbf{1}_k(t)$$

and, for any t in $[k\Delta t, (k+1)\Delta t[$,

$$\tilde{u}_{\Delta t}(t) = (t - k\Delta t)\frac{u^{k+1} - u^k}{\Delta t} + u^k. \tag{2.7}$$

We note that the derivative $\partial_t \tilde{u}_{\Delta t}$ is given by

$$\partial_t \tilde{u}_{\Delta t}(t) = \sum_{k=0}^{N-1} \frac{u^{k+1} - u^k}{\Delta t}\ \mathbf{1}_k(t).$$

Thus, for a.a. t in $(0, T)$ and for all v in $H_0^1(\Omega)$, one has that

$$\begin{aligned} &\int_\Omega \partial_t \tilde{u}_{\Delta t} v\, dx + \int_\Omega K(u_{\Delta t}) a(\partial_t \tilde{u}_{\Delta t} + E_{\Delta t}) \nabla(u_{\Delta t} - \tau E_{\Delta t}) \cdot \nabla v\, dx \\ &\qquad + \tau \int_\Omega K(u_{\Delta t}) \nabla(\partial_t \tilde{u}_{\Delta t} + E_{\Delta t}) \cdot \nabla v\, dx = \int_\Omega f_{\Delta t}(u_{\Delta t}) v\, dx, \end{aligned} \tag{2.8}$$

where $E_{\Delta t}$ and $f_{\Delta t}$ are given by

$$E_{\Delta t} = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} E(x,s)ds, \quad f_{\Delta t} = \frac{1}{\Delta t}\int_{k\Delta t}^{(k+1)\Delta t} f(u)ds.$$

Lemma 2.5. For small Δt,

- the sequences $(u_{\Delta t})$ and $(\tilde{u}_{\Delta t})$ are bounded in $L^\infty(0, T; H_0^1(\Omega))$;
- the sequence $(\partial_t \tilde{u}_{\Delta t})$ is bounded in $L^\infty(0, T; H_0^1(\Omega))$.

Proof. Let us consider $v = \dfrac{u^{k+1} - u^k}{\Delta t}$ in (2.2), in order to get

$$\begin{aligned} &\left\| \frac{u^{k+1} - u^k}{\Delta t} \right\|_0^2 + \tau k_m \left\| \nabla\left(\frac{u^{k+1} - u^k}{\Delta t} \right) \right\|_0^2 \\ &\qquad \le k_M \left\| \nabla\left(u^{k+1} - \tau E^{k+1} \right) \right\|_0 \left\| \nabla\left(\frac{u^{k+1} - u^k}{\Delta t} \right) \right\|_0 \\ &\qquad + \tau k_M \left\| \nabla E^{k+1} \right\|_0 \left\| \nabla\left(\frac{u^{k+1} - u^k}{\Delta t} \right) \right\|_0 + \left\| f^{k+1} \right\|_0 \left\| \frac{u^{k+1} - u^k}{\Delta t} \right\|_0. \end{aligned}$$

Thus, for any positive ϵ_1, ϵ_2, one has that

$$\begin{aligned} &\left\| \frac{u^{k+1} - u^k}{\Delta t} \right\|_0^2 + \tau k_m \left\| \nabla\left(\frac{u^{k+1} - u^k}{\Delta t} \right) \right\|_0^2 \le k_M \frac{\epsilon_1 + \epsilon_2}{2} \left\| \nabla\left(\frac{u^{k+1} - u^k}{\Delta t} \right) \right\|_0^2 \\ &\qquad + \frac{k_M}{2\epsilon_1} \left\| \nabla\left(u^{k+1} - \tau E^{k+1} \right) \right\|_0^2 + k_M \frac{\tau^2}{2\epsilon_2} \left\| \nabla E^{k+1} \right\|_0^2 \\ &\qquad + \frac{1}{2} \left\| f^{k+1} \right\|_0^2 + \frac{1}{2} \left\| \frac{u^{k+1} - u^k}{\Delta t} \right\|_0^2. \end{aligned}$$

With $\epsilon_1 = \epsilon_2 = \dfrac{\tau k_m}{2k_M}$, we have

$$\begin{aligned}\frac{1}{2}\left\|\frac{u^{k+1}-u^k}{\Delta t}\right\|_0^2 + \frac{\tau k_m}{2}\left\|\nabla\left(\frac{u^{k+1}-u^k}{\Delta t}\right)\right\|_0^2 &\leq \frac{k_M^2}{\tau k_m}\left\|\nabla\left(u^{k+1}-\tau E^{k+1}\right)\right\|_0^2 \\ &+ \frac{\tau k_M^2}{k_m}\left\|\nabla E^{k+1}\right\|_0^2 + \frac{1}{2}\left\|f^{k+1}\right\|_0^2 \\ &\leq \frac{2k_M^2}{\tau k_m}\left\|\nabla u^{k+1}\right\|_0^2 + \frac{3\tau k_M^2}{k_m}\left\|\nabla E^{k+1}\right\|_0^2 + \frac{1}{2}\left\|f^{k+1}\right\|_0^2\end{aligned}$$

and

$$\begin{aligned}\left\|\frac{u^{k+1}-u^k}{\Delta t}\right\|_0^2 + \tau k_m\left\|\nabla\left(\frac{u^{k+1}-u^k}{\Delta t}\right)\right\|_0^2 &\leq \frac{8\Delta t^2 k_M^2}{\tau k_m}\left\|\sum_{m=0}^{k}\nabla\left(\frac{u^{m+1}-u^m}{\Delta t}\right)\right\|_0^2 \\ &+ \frac{8k_M^2}{\tau k_m}\left\|\nabla u_0\right\|_0^2 + \frac{6\tau k_M^2}{k_m}\left\|\nabla E^{k+1}\right\|_0^2 + \left\|f^{k+1}\right\|_0^2 \\ &\leq \frac{8\Delta t^2 k_M^2}{\tau k_m}(k+1)\sum_{m=0}^{k}\left\|\nabla\left(\frac{u^{m+1}-u^m}{\Delta t}\right)\right\|_0^2 + M \\ &\leq \frac{8T\Delta t k_M^2}{\tau k_m}\sum_{m=0}^{k}\left\|\nabla\left(\frac{u^{m+1}-u^m}{\Delta t}\right)\right\|_0^2 + M,\end{aligned}$$

where M is given by

$$M = \frac{8k_M^2}{\tau k_m}\left\|\nabla u_0\right\|_0^2 + \frac{6\tau k_M^2}{k_m}\left\|\nabla E^{k+1}\right\|_0^2 + \left\|f^{k+1}\right\|_0^2.$$

Let $x_k = \left\|\nabla\left(\dfrac{u^{k+1}-u^k}{\Delta t}\right)\right\|_0^2$. Using hypothesis (H) for data E and f, there is a constant C independent of Δt such that

$$x_k \leq \frac{8T\Delta t k_M^2}{\tau^2 k_m^2}\sum_{m=0}^{k} x_m + C.$$

If $\Delta t \leq \dfrac{\tau^2 k_m^2}{16 T k_M^2}$, we get

$$x_k \leq \frac{16T\Delta t k_M^2}{\tau^2 k_m^2}\sum_{m=0}^{k-1} x_m + 2C.$$

On one hand, by using the discrete version of the lemma of Gronwall, one obtains

$$x_k \leq 2C\exp\left(\frac{16T k_M^2}{\tau^2 k_m^2}k\Delta t\right) \leq C\exp\left(\frac{16T^2 k_M^2}{\tau^2 k_m^2}\right).$$

Thus,

$$\left\|\nabla\left(\frac{u^{k+1}-u^k}{\Delta t}\right)\right\|_0^2 \leq C \text{ and } \|\partial_t\tilde{u}_{\Delta t}\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C. \tag{2.9}$$

On the other hand, one has that

$$\begin{aligned}\left\|\nabla u^k\right\|_0 &= \left\|\sum_{m=0}^{k-1}\nabla\left(\frac{u^{m+1}-u^m}{\Delta t}\right)\Delta t+\nabla u_0\right\|_0\\ &\leq \sum_{m=0}^{k-1}\Delta t\left\|\nabla\left(\frac{u^{m+1}-u^m}{\Delta t}\right)\right\|_0+\|\nabla u_0\|_0\\ &\leq \left(\sum_{m=0}^{k-1}\Delta t\right)^{1/2}\left(\sum_{m=0}^{k-1}\frac{\left\|\nabla\left(u^{m+1}-u^m\right)\right\|_0^2}{\Delta t}\right)^{1/2}+\|\nabla u_0\|_0\\ &\leq \sqrt{T}C+\|\nabla u_0\|_0.\end{aligned}$$

Thus, the following estimate holds:

$$\|u_{\Delta t}\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C. \tag{2.10}$$

From (2.9) and (2.10), we deduce that

$$\|\tilde{u}_{\Delta t}\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C, \tag{2.11}$$

and the assertions in Lemma 2.5 are proved. □

Proof of Proposition 2.4. Because u in $H^1\left(0,T;H_0^1(\Omega)\right)$ exists, a sub-sequence, still denoted by $(\tilde{u}_{\Delta t})$, may be extracted such that $\tilde{u}_{\Delta t} \rightharpoonup u$ in $H^1(0,T;H_0^1(\Omega))$ and, for all $t \in (0,T)$,

$$\tilde{u}_{\Delta t}(t) \rightharpoonup u(t) \text{ in } H_0^1(\Omega).$$

In particular, we have $u(0,\cdot)=u_0$ a.e. in Ω. By construction, for almost all $t \in [0,T]$, $E_{\Delta t}(t)$ tends to $E(t)$ in $H^1(\Omega)$ and $f_{\Delta t}(t)$ tends to $f(t)$ in $L^2(\Omega)$. Moreover, we have

$$\begin{aligned}\forall\, t \in [k\Delta t,(k+1)\Delta t[,\ \|u_{\Delta t}(t)-\tilde{u}_{\Delta t}(t)\|_{H_0^1(\Omega))} &= \|\tilde{u}_{\Delta t}(k\Delta t)-\tilde{u}_{\Delta t}(t)\|_{H_0^1(\Omega))}\\ &\leq \int_{k\Delta t}^{(k+1)\Delta t}\|\partial_t\tilde{u}_{\Delta t}(s)\|_{H_0^1(\Omega))}ds,\\ &\leq C\Delta t.\end{aligned}$$

Then,

$$u_{\Delta t}(t) \rightharpoonup u(t) \text{ in } H_0^1(\Omega)$$

for all $t \in (0,T)$. Since the sequence $(\partial_t\tilde{u}_{\Delta t})$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$, a measurable subset Z of $(0,T)$ with $\mathcal{L}((0,T),Z)=0$ exists such that for all $t \in Z$ the sequence $(\partial_t \tilde{u}_{\Delta t})(t)$ stays in a fixed bounded subset of $H_0^1(\Omega)$. A sub-sequence denoted by $(\partial_t \tilde{u}_{\Delta t_t})$ may be extracted such that

$$\partial_t \tilde{u}_{\Delta t_t}(t) \rightharpoonup \xi(t) \text{ in } H_0^1(\Omega).$$

The embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact, thus

$$\partial_t \tilde{u}_{\Delta t_t}(t) \to \xi(t) \text{ in } L^2(\Omega) \text{ and a.e. in } \Omega, \tag{2.12}$$

for a sub-sequence denoted by the same way. Thus, since K and a are continuous and bounded, for all v in $H_0^1(\Omega)$, we have

$$K(u_{\Delta t})a(\partial_t \tilde{u}_{\Delta t_t}(t) + E_{\Delta t})\nabla v \to K(u)a(\xi(t)+E)\nabla v \text{ in } (L^2(\Omega))^d. \tag{2.13}$$

The derivative operators $\dfrac{\partial}{\partial x_i}$ are linear and continuous from $H_0^1(\Omega)$ into $L^2(\Omega)$, thus we get that

$$\begin{aligned} \forall\, t \in]0,T[\text{ a.e., } & \nabla(\tilde{u}_{\Delta t}(t) - \tau E_{\Delta t}) \rightharpoonup \nabla(u(t) - \tau E) \text{ in } (L^2(\Omega))^d, \\ \forall\, t \in Z, \ & \nabla(\partial_t \tilde{u}_{\Delta t_t}(t) + E_{\Delta t}) \rightharpoonup \nabla(\xi(t) + E) \text{ in } (L^2(\Omega))^d. \end{aligned}$$

Passing to the limits on Δt ($\Delta t \to 0^+$) in (2.8), for all v in $H_0^1(\Omega)$, $\xi(t)$ satisfies the equation

$$\int_\Omega \xi(t) v\, dx + \int_\Omega K(u)a(\xi(t)+E)\nabla(u(t) - \tau E)\cdot \nabla v\, dx \\ + \tau \int_\Omega K(u)\nabla(\xi(t)+E)\cdot\nabla v dx = \int_\Omega f(u) v\, dx.$$

Then, using Lemma 2.1, with $b = a$ and $\kappa = u(t)$ and hypothesis (H), the solution $\xi(t)$ is unique in $H_0^1(\Omega)$. Then, any sequence $(\partial_t \tilde{u}_k(t))_{\Delta t}$, and not the sub-sequences $(\partial_t \tilde{u}_{\Delta t_t})(t)$, $(\partial_t \tilde{u}_{\Delta t})(t)$, converges in $H_0^1(\Omega)$ toward $\xi(t)$. Thus, we have for all t in Z

$$\partial_t \tilde{u}_{\Delta t}(t) \rightharpoonup \xi(t) \text{ in } H_0^1(\Omega).$$

Therefore, the function $\xi\ :]0,T[\to H_0^1(\Omega)$ is weakly measurable (indeed, for any g in $H^{-1}(\Omega)$, $t \mapsto \langle g, \xi(t)\rangle$ is the limit of a sequence of measurable functions $t \mapsto \langle g, \partial_t \tilde{u}_{\Delta t}(t)\rangle$) and consequently ξ is a measurable function thanks to the theorem of Pettis [10], since $H_0^1(\Omega)$ is a separable set. Moreover, for any v in $L^2(0,T;H_0^1(\Omega))$,

$$(\partial_t \tilde{u}_{\Delta t}(t), v(t)) \rightharpoonup (\xi(t), v(t)) \text{ a.e. in }]0,T[.$$

As the sequence $(\partial_t \tilde{u}_{\Delta t})$ is bounded in $L^\infty(0,T;H_0^1(\Omega))$, there exists a constant C such that

$$|(\partial_t \tilde{u}_{\Delta t}(t), v(t))| \le C\|v(t)\|_1.$$

Thus, we conclude that $\partial_t \tilde{u}_{\Delta t} \rightharpoonup \xi$ in $L^2(0,T;H_0^1(\Omega))$. Therefore, we have $\xi = \partial_t u$, and passing to the limit ($\Delta t \to 0^+$), we have, for t a.e. in $(0,T)$, for all v in $L^2(0,T;H_0^1(\Omega))$

$$\int_\Omega \partial_t u v \, dx + \int_\Omega K(u) a(\partial_t u + E) \nabla (u - \tau E) \cdot \nabla v \, dx + \tau \int_\Omega K(u) \nabla(\partial_t u + E) \cdot \nabla v \, dx = \int_\Omega f(u) v \, dx, \tag{2.14}$$

i.e., a solution exists. □

Now, by applying the theorem of Meyers [9], we get the following lemma.

Lemma 2.6. Assuming $u^{k-1} \in W_0^{1,p_0}(\Omega)$, $f^k \in L^{p_0}(\Omega)$ with $p_0 > 2$ and considering the unique solution $u^k \in H_0^1(\Omega)$, there exists a real $\overline{p}(p_0) > 2$, depending on p_0 and $\dfrac{k_M(a_{\max} + \frac{\tau}{\Delta t})}{\frac{\tau}{\Delta t} k_m}$, and a positive constant $C(\overline{p}(p_0))$ such that

$$u^k \in W_0^{1,\overline{p}(p_0)}(\Omega) \quad \text{and} \quad \|\nabla u^k\|_{L^{\overline{p}(p_0)}(\Omega)^N} \leq C(\overline{p}(p_0)) \Big(\|u_0\|_{W_0^{1,p_0}(\Omega)}, \|f\|_{L^{p_0}(\Omega)} \Big).$$

The regularity $W_0^{1,p_0}(\Omega)$ can be obtained by using the theorem of Nečas.

Lemma 2.7. Let u^k be the solution given in Lemma 2.2. Then, $u^k \in W_0^{1,p_0}(\Omega)$, for any $k = 1, \cdots, N$, with $p_0 > 2$ and there exist a positive constant $C(p_0)$ such that

$$\|\nabla u^k\|_{L^{p_0}(\Omega)^N} \leq C(p_0) \Big(\|u_0\|_{W_0^{1,p_0}(\Omega)}, \|f\|_{L^{p_0}(\Omega)} \Big).$$

Thanks to previous lemmata, one gets the following result.

Theorem 2.8. Let u be the solution of (1.8). If $u_0 \in W_0^{1,p_0}(\Omega)$ and $f \in L^2\big(0,T;L^{p_0}(\Omega)\big)$ for a given $p_0 > 2$, then one has $u \in W^{1,\infty}\big(0,T;W_0^{1,p_0}(\Omega)\big)$.

One is able to adapt the proofs involving tri-linear terms and, based on Hölder type inequalities, to prove the following theorem.

Theorem 2.9. There exist a real $\tau^* \geq 0$ such that, for any $\tau > \tau^*$, the problem (1.8) has a unique solution in the space $W^{1,\infty}\big(0,T;W_0^{1,p_0}(\Omega)\big)$ with $p_0 > 2$. Moreover, the application: $u_0 \mapsto \partial_t u$ is a locally Lipschitz continuous function in the space $H_0^1(\Omega)$ to the space $L^\infty\big(0,T;H_0^1(\Omega)\big)$.

3 Conclusion

We proved existence and uniqueness of solution to a more realistic problem where the diffusion coefficient and source term depend of the unknown u. However, many open questions have yet to be treated, as to deal with the physic boundary conditions, i.e., non-homogenuous Neumann boundary conditions on the input part and unilateral boundary conditions on the output part.

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