International Journal of Difference Equations
ISSN 0973-6069, Volume 8, Number 1, pp. 111–124 (2013)
http://campus.mst.edu/ijde

Posteriori Analysis of a Finite Element Discretization for a Penalized Naghdi Shell

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Abstract

We consider a penalized Naghdi model in Cartesian coordinates for linearly elastic shells with little regularity. A posteriori analysis of the discrete problem leads to the construction of error indicators, which satisfy optimal estimates. We describe a mesh adaptivity strategy relying on these indicators and we present a numerical experiment that confirms its efficiency.

AMS Subject Classifications: 74K25, 74S05.
Keywords: Naghdi's shell model, finite element approximation, a posteriori error estimates, residuals.

1 Introduction

Naghdi's model is a linear elastic shell. The formulation of the model used here was introduced in [4, 6]. A posteriori analysis is now an important tool for improving the efficiency of the discretization. We refer to [7, 8] for the first works concerning a plate model and to [2] for shell models. The first aim of a posteriori analysis is mesh adaptivity. Indeed, a much smaller number of degrees of freedom are needed to obtain a given accuracy when the final mesh is adapted to the solution and the construction of such mesh relies on error indicators, which only depend on the discrete solution, and hence can be computed in an explicit way and often in a non expensive way. A posteriori estimate proves that these indicators provide a good representation of the local error – see [11] for a detailed presentation of all this. Here we perform a posteriori analysis of the discretization, relying on a penalized version studied in [5] and prove upper and lower bounds for the error as a function of residual type indicators. Finally, we describe

Received June 29, 2012; Accepted January 27, 2013
Communicated by Sandra Pinelas

the strategy that is used for the adaptivity mesh. Numerical experiments are in good agreement with the analysis.

The article is organized as follows. We first briefly recall the geometry of the midsurface and Naghdi shell formulation given in [4, 6]. This formulation involves the infinitesimal rotation vector, a vector unknown that is tangent to the midsurface. In Section 3, we recall the penalized version of Naghdi's model intended to approximate the above mentioned tangency. Section 4 is devoted to the a posteriori analysis of the finite element discretization. In Section 5 we present the adaptivity strategy and numerical experiments.

2 Presentation of the Model

Greek indices and exponents take their values in the set $\{1,2\}$, while Latin indices and exponents belong to the set $\{1,2,3\}$. Let (e_1,e_2,e_3) be the canonical orthonormal basis of $\mathbb{R}^3$. We denote by $u\cdot v$ the inner product of $\mathbb{R}^3$, $|u|=\sqrt{u\cdot u}$ the associated Euclidean norm and $u\wedge v$ the vector product of u and v. Let ω be a bounded connected domain of $\mathbb{R}^2$. We consider a shell of midsurface $S=\varphi(\overline{\omega})$, where $\varphi\in W^{2,\infty}(\omega,\mathbb{R}^3)$ is a one-to-one mapping such that the two vectors $a_\alpha(x)=\partial_\alpha\varphi(x)$ are linearly independent at each point x of $\overline{\omega}$. We let $a_3=\dfrac{a_1\wedge a_2}{|a_1\wedge a_2|}$ be the unit normal vector on the midsurface at point $\varphi(x)$. The vectors $a_i(x)$ define the local covariant basis at point $\varphi(x)$. The contravariant basis $a^i(x)$ is defined by $a_i\cdot a^j=\delta_i^j$, where δ_i^j is de Kronecker symbol. We let $a(x)=|a_1(x)\wedge a_2(x)|^2$, so that $\sqrt{a(x)}$ is the area element of the midsurface in the chart φ. The first and the second fundamental forms of the surface are given in covariant components by $a_{\alpha\beta}(x)=a_\alpha(x)\cdot a_\beta(x)$ and $b_{\alpha\beta}(x)=a_3(x)\cdot\partial_\beta a_\alpha(x)$. The contravariant components of the first fundamental form $a^{\alpha\beta}(x)=a^\alpha(x)\cdot a^\beta(x)$. The length element l on the boundary $\partial\omega$ is given by $\sqrt{a^{\alpha\beta}\tau_\alpha\tau_\beta}$, (τ_1,τ_2) being the covariant coordinates of a unit vector tangent to $\partial\omega$. Let $a^{\alpha\beta\rho\sigma}\in L^\infty(\omega)$ be the elasticity tensor. We consider here the case of a homogeneous, isotropic material with Young modulus $E>0$ and Poisson ratio $\nu,\ 0\le\nu\le\dfrac{1}{2}$, where these components are given by

$$a^{\alpha\beta\rho\sigma}=\frac{E}{2(1+\nu)}(a^{\alpha\rho}a^{\beta\sigma}+a^{\alpha\sigma}a^{\beta\rho})+\frac{E\nu}{1-\nu^2}a^{\alpha\beta}a^{\rho\sigma}. \tag{2.1}$$

In this context, the covariant components of the change of metric tensor read

$$\gamma_{\alpha\beta}(u)=\frac{1}{2}(\partial_\alpha u\cdot a_\beta+\partial_\beta u\cdot a_\alpha), \tag{2.2}$$

the covariant components of the change of curvature tensor read

$$\chi_{\alpha\beta}(u,r)=\frac{1}{2}(\partial_\alpha u\cdot\partial_\beta a_3+\partial_\beta u\cdot\partial_\alpha a_3+\partial_\alpha r\cdot a_\beta+\partial_\beta r\cdot a_\alpha), \tag{2.3}$$

and the components of the change of transverse shear tensor read

$$\delta_{\alpha 3}(u,r) = \frac{1}{2}(\partial_\alpha u \cdot a_3 + r \cdot a_\alpha). \tag{2.4}$$

We assume that the boundary $\partial\omega$ of the chart domain is divided into two parts: γ_0 on which the shell is clamped and $\gamma_1 = \partial\omega\backslash\gamma_0$ on which the shell is subjected to applied traction and moment. Let us now consider the function space $\mathbb{V}(\omega)$ introduced in [4, 6]:

$$\mathbb{V}(\omega) = \Big\{ V = (v,s) \in \big[H^1(\omega,\mathbb{R}^3)\big]^2,\ s\cdot a_3 = 0 \text{ in } \omega,\ v = s = 0 \text{ on } \gamma_0 \Big\}. \tag{2.5}$$

This space is endowed with the natural Hilbert norm

$$\|V\|_{\mathbb{V}} = \Big(\|v\|^2_{H^1(\omega;\mathbb{R}^3)} + \|s\|^2_{H^1(\omega;\mathbb{R}^3)} \Big)^{\frac{1}{2}}. \tag{2.6}$$

We now recall the variational formulation of the problem corresponding to the linear Naghdi model with data $(f,N,M) \in L^2(\omega;\mathbb{R}^3)\times L^2(\gamma_1;\mathbb{R}^3)\times L^2(\gamma_1;\mathbb{R}^3)$: find $U = (u,r) \in \mathbb{V}(\omega)$ such that

$$\forall V \in \mathbb{V}, \qquad a(U,V) = L(V), \tag{2.7}$$

where the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(U,V) = \int_\omega \Big\{ e a^{\alpha\beta\rho\sigma} \Big[\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) + \frac{e^2}{12}\chi_{\alpha\beta}(U)\chi_{\rho\sigma}(V) \Big] + 2e\frac{E}{1+\nu} a^{\alpha\beta}\delta_{\alpha 3}(U)\delta_{\beta 3}(V) \Big\} \sqrt{a}\, dx, \tag{2.8}$$

and the linear form $L(\cdot)$ is given by

$$L(V) = \int_\omega f\cdot v\sqrt{a}\, dx + \int_{\gamma_1} (N\cdot v + M\cdot s)\, l\, d\tau. \tag{2.9}$$

The data $f,\ N,\ M$ represent a given resultant force density, an applied traction density and an applied moment density, respectively. In the above formulas, the thickness of the shell, denoted by e, is assumed to be constant and positive. We refer to [4, 6] for the proof of the following results:

- The form L is continuous on $\mathbb{V}(\omega)$ and its norm satisfies

$$\|L\| \le c\Big(\|f\|_{L^2(\omega;\mathbb{R}^3)} + \|N\|_{L^2(\gamma_1;\mathbb{R}^3)} + \|M\|_{L^2(\gamma_1;\mathbb{R}^3)} \Big). \tag{2.10}$$

- There exists a constant $c^* > 0$ such that

$$\forall V \in \mathbb{V}(\omega), \qquad a(V,V) \ge c^* \|V\|^2_{\mathbb{V}(\omega)}. \tag{2.11}$$

- Problem (2.7) admits a unique solution U in $\mathbb{V}(\omega)$.

3 Penalized Version

We consider the penalized Naghdi problem introduced in [5], in which the unknowns are the displacement u and the rotation r, elements of the space $H^1(\omega;\mathbb{R}^3)$ without any orthogonality constraint on r. The relaxed function space is defined by

$$\mathbb{X} = \left\{ V = (v,s) \in H^1(\omega,\mathbb{R}^3)^2,\ v = s = 0 \text{ on } \gamma_0 \right\}, \tag{3.1}$$

equipped with the norm defined in (2.6), which is now denoted by $\|\cdot\|_{\mathbb{X}(\omega)}$.

Theorem 3.1. *Let $p \in \mathbb{R}$ be such that $0 < p \leq 1$. Let $f \in L^2(\omega,\mathbb{R}^3)$, $N \in L^2(\gamma_1,\mathbb{R}^3)$ and $M \in L^2(\gamma_1,\mathbb{R}^3)$. Then there exists a unique solution to the following problem: find $U_p = (u_p, r_p) \in \mathbb{X}$ such that*

$$\forall V \in \mathbb{X}, \qquad a(U_p, V) + \frac{1}{p} b(r_p \cdot a_3; s \cdot a_3) = L(V), \tag{3.2}$$

where

$$b(\lambda;\mu) = \int_\omega \partial_\alpha \lambda \partial_\alpha \mu \, dx. \tag{3.3}$$

Proof. See [5]. □

Remark 3.2. We note that the quantity $a(U,V)$ can be written in another form which seems more appropriate for implementation, since it uncouples the two components v and s of the test function $V = (v,s)$. Indeed, we introduce the contravariant components of the stress resultant,

$$n^{\rho\sigma}(u) = e a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(u), \tag{3.4}$$

of the stress couple,

$$m^{\rho\sigma}(U) = \frac{e^3}{12} a^{\alpha\beta\rho\sigma} \chi_{\alpha\beta}(U), \tag{3.5}$$

and of the transverse shear force,

$$t^\beta(U) = e \frac{E}{1+\nu} a^{\alpha\beta} \delta_{\alpha 3}(U). \tag{3.6}$$

We also have:

$$\chi_{\rho\sigma}(V) = \theta_{\rho\sigma}(v) + \gamma_{\rho\sigma}(s) \quad \text{with} \quad \theta_{\rho\sigma}(v) = \frac{1}{2}(\partial_\rho v \cdot \partial_\sigma a_3 + \partial_\sigma v \cdot \partial_\rho a_3). \tag{3.7}$$

Thus, the bilinear form $a(U_p, V)$ can be rewritten as

$$a(U_p, V) = \int_\omega \left(n^{\rho\sigma}(u_p) \gamma_{\rho\sigma}(v) + m^{\rho\sigma}(U_p) \theta_{\rho\sigma}(v) + t^\beta(U_p) \partial_\beta v \cdot a_3 \right) \sqrt{a}\, dx$$
$$+ \int_\omega \left(m^{\rho\sigma}(U_p) \gamma_{\rho\sigma}(s) + t^\beta(U_p) s \cdot a_\beta \right) \sqrt{a}\, dx. \tag{3.8}$$

To go further, using this new form together with the symmetry properties $n^{\rho\sigma}(u_p) = n^{\sigma\rho}(u_p)$ and $m^{\rho\sigma}(U_p) = m^{\sigma\rho}(U_p)$, and by integration by parts in problem (3.2), we obtain the strong formulation of the penalized problem:

$$\begin{cases} -\partial_\rho((n^{\rho\sigma}(u_p)a_\sigma + m^{\rho\sigma}(U_p)\partial_\sigma a_3 + t^\rho(U_p)a_3)\sqrt{a}) = f\sqrt{a} & \text{in } \omega, \\ -\partial_\rho(m^{\rho\sigma}(U_p)a_\sigma\sqrt{a}) + t^\beta(U_p)a_\beta\sqrt{a} - \dfrac{1}{p}\partial_{\rho\rho}(r_p \cdot a_3)a_3 = 0 & \text{in } \omega, \\ u_p = r_p = 0 & \text{on } \gamma_0, \\ \nu_\rho(n^{\rho\sigma}(u_p)a_\sigma + m^{\rho\sigma}(U_p)\partial_\sigma a_3 + t^\rho(U_p)a_3)\sqrt{a} = Nl & \text{on } \gamma_1, \\ \nu_\rho(m^{\rho\sigma}(U_p)a_\sigma\sqrt{a} + \dfrac{1}{p}\partial_\rho(r_p \cdot a_3)a_3) = Ml & \text{on } \gamma_1. \end{cases} \tag{3.9}$$

The discretization that we intend to study is constructed by the Galerkin method from problem (3.2). We refer to [5, §5.1] for more details.

4 A Posteriori Analysis of the Discrete Problem

Let $(\mathcal{T}_h)_h$ be a regular affine family of triangulations which covers the domain ω and $\mathcal{P}_k(K)$ denote the space of restrictions to K, element of $\mathcal{T}_h$, of polynomials with total degree $\leq k$. The discrete space is given by

$$\mathbb{X}_h = \left\{V_h = (v_h, s_h) \in C^0(\omega; \mathbb{R}^3)^2, V_h|K \in \mathcal{P}_1(K),\ v_h = s_h = 0 \text{ on } \gamma_0\right\}. \tag{4.1}$$

The discrete problem consists to find $U_{p,h} = (u_{p,h}, r_{p,h}) \in \mathbb{X}_h$ such that

$$\forall V_h \in \mathbb{X}_h, \quad a(U_{p,h}, V_h) + \frac{1}{p}\, b(r_{p,h} \cdot a_3, s_h \cdot a_3) = L(V_h). \tag{4.2}$$

This problem has a unique solution [5, Th. 3.1]. The a posteriori analysis of problem (4.2) relies on the residual equation

$$\begin{aligned} a(U_p - U_{p,h}, V) &+ \frac{1}{p}b((r_p \cdot a_3 - r_{p,h} \cdot a_3); s \cdot a_3) \\ &= L(V - V_h) - a(U_{p,h}, V - V_h) - \frac{1}{p}b\left(r_{p,h} \cdot a_3; (s \cdot a_3 - s_h \cdot a_3)\right), \end{aligned} \tag{4.3}$$

valid for all $V \in \mathbb{X}(\omega)$ and for all $V_h \in \mathbb{X}_h$. The construction of error indicators from these equations requires approximations of the data and of the coefficients [2, 11].

4.1 Approximation of the Data

Let E_h^1 denote the set of edges of elements of $\mathcal{T}_h$ which are contained in $\overline{\gamma}_1$. We consider an approximation f_h of f in $\mathbb{Z}_h$ and approximations N_h and M_h of N and M in $\mathbb{Z}_h^1$, where the spaces $\mathbb{Z}_h$ and $\mathbb{Z}_h^1$ are defined by

$$\begin{aligned} \mathbb{Z}_h &= \left\{g_h \in L^2(\omega)^3;\ \forall K \in \mathcal{T}_h,\ g_{h|K} \in \mathcal{P}_0(K)^3\right\}, \\ \mathbb{Z}_h^1 &= \left\{\mathcal{E}_h \in L^2(\gamma_1)^3;\ \forall e \in E_h^1,\ \mathcal{E}_{h|e} \in \mathcal{P}_0(e)^3\right\}. \end{aligned} \tag{4.4}$$

4.2 Approximation of the Coefficients

We denote by $a_h^{\alpha\beta}$, $a_h^{\alpha\beta\rho\sigma}$, $(\sqrt{a})_h$ and l_h, the approximations of $a^{\alpha\beta}$, $a^{\alpha\beta\rho\sigma}$, $\sqrt{a}$ and l, respectively, in the space $\mathbb{M}_h$ which is given by

$$\mathbb{M}_h = \left\{\chi_h \in H^1(\omega);\ \forall K \in \mathcal{T}_h,\ \chi_{h|K} \in P_1(K)\right\}. \tag{4.5}$$

Similarly, we consider approximations a_k^h of the vectors a_k and d_α^h of the $\partial_\alpha a_3$ in the space $\mathbb{M}_h$. We also agree to denote by $\gamma_{\alpha\beta}^h(\cdot)$, $\chi_{\alpha\beta}^h(\cdot)$ and $\delta_{\alpha 3}^h(\cdot)$ the approximations of the tensors introduced in (2.2) to (2.4). For instance, $\gamma_{\alpha\beta}^h(\cdot)$ is given by

$$\gamma_{\alpha\beta}^h(u) = \frac{1}{2}\left(\partial_\alpha u \cdot a_\beta^h + \partial_\beta u \cdot a_\alpha^h\right). \tag{4.6}$$

This leads to the definition of the approximate linear form

$$L_h(V) = \int_\omega f_h \cdot v(\sqrt{a})_h \, dx + \int_{\gamma_1} (N_h \cdot v + M_h \cdot s) l_h \, d\tau, \tag{4.7}$$

and also of approximate bilinear forms

$$\begin{aligned} a_h(U,V) = \int_\omega \Bigg\{ & e a_h^{\alpha\beta\rho\sigma} \left[\gamma_{\alpha\beta}^h(u)\, \gamma_{\rho\sigma}^h(v) + \frac{e^2}{12}\chi_{\alpha\beta}^h(U)\, \chi_{\rho\sigma}^h(V)\right] \\ & + 2e\frac{E}{1+\nu} a_h^{\alpha\beta} \delta_{\alpha 3}^h(U)\, \delta_{\beta 3}^h(V)\Bigg\} (\sqrt{a})_h \, dx, \end{aligned} \tag{4.8}$$

$$b_h(r \cdot a_3, s \cdot a_3) = \int_\omega \left(\partial_\alpha r \cdot a_3^h + r \cdot d_\alpha^h\right) \partial_\alpha s \cdot a_3 \, dx. \tag{4.9}$$

It is easy to check that $\forall V \in \mathbb{X}(\omega)$ and $\forall V_h \in \mathbb{X}_h$ one has

$$\begin{aligned} L(V - V_h) - a\left(U_{p,h}, V - V_h\right) - \frac{1}{p} b\left(r_{p,h} \cdot a_3; (s \cdot a_3 - s_h \cdot a_3)\right) \\ = (L - L_h)(V - V_h) + L_h(V - V_h) - (a - a_h)(U_{p,h}, V - V_h) \\ - a_h(U_{p,h}, V - V_h) - \frac{1}{p}(b - b_h)(r_{p,h} \cdot a_3; (s \cdot a_3 - s_h \cdot a_3)) \\ - \frac{1}{p} b_h\left(r_{p,h} \cdot a_3; (s \cdot a_3 - s_h \cdot a_3)\right). \end{aligned} \tag{4.10}$$

To go further, we recall some standard notations:

(i) E_K denotes the set of edges of K which are not contained in $\overline{\gamma_0}$ and E_K^1 the set of elements of E_K which are contained in $\overline{\gamma_1}$;

(ii) for each $e \in E_K$, $\nu = (\nu_1, \nu_2)$ is a unit vector normal to e, with the further assumption that, when e belongs to E_K^1, ν is outward to ω;

(iii) for each $e \in E_K,\ h_e$ stands for the length of e;

(iv) for each $e \in E_K \backslash E_K^1$, $[\cdot]_e$ denotes the jump through e;

(v) ω_K is the union of triangles of $\mathcal{T}_h$ that share an edge with K;

(vi) Δ_K is the union of triangles of $\mathcal{T}_h$ that intersect K.

We recall that from [3, Theorem IX.3.11 and Corollary IX.3.12] there exists a Clément type operator $\mathcal{R}_h$ which maps $H^1_{\gamma_0}(\omega)$ into $\mathbb{M}_h^{\gamma_0} = \mathbb{M}_h \cap H^1_{\gamma_0}$ and satisfies, for all functions $\chi \in H^1_{\gamma_0}(\omega)$, each $K \in \mathcal{T}_h$ and each e of K which is not contained in γ_0,

$$\begin{gathered} \|\chi - \mathcal{R}_h\chi\|_{L^2(K)} + h_K\,|\chi - \mathcal{R}_h\chi|_{H^1(K)} \le c\,h_K\,\|\chi\|_{H^1(\Delta_K)}\,, \\ \|\chi - \mathcal{R}_h\chi\|_{L^2(e)} \le c\,h^{\frac{1}{2}}\,\|\chi\|_{H^1(\Delta_K)}\,. \end{gathered} \tag{4.11}$$

The idea is to take V_h equal to $(\mathcal{R}_h v, \mathcal{R}_h s)$ and χ_h equal to $\mathcal{R}_h\chi$ in (4.3). From [2, Lemma 3.3, Lemma 3.4 and Lemma 3.5], we define the quantities linked to the local approximation error on the data: for each $K \in \mathcal{T}_h$,

$$\varepsilon_K^{(d)} = h_K\,\|f - f_h\|_{L^2(K)^3} + \sum_{e\in E_K^1} h_e^{\frac{1}{2}}\left(\|N - N_h\|_{L^2(e)^3} + \|M - M_h\|_{L^2(e)^3}\right), \tag{4.12}$$

and also from the global approximation error on the coefficients

$$\begin{aligned} \varepsilon_h^{(c)} = \Bigg(& h\left\|\sqrt{a} - (\sqrt{a})_h\right\|_{L^\infty(\omega)} + h^{\frac{1}{2}}\,\|l - l_h\|_{L^\infty(\gamma_1)} \\ & + \sup_{1\le\alpha,\beta,\rho,\sigma\le 2}\left\|a^{\alpha\beta\rho\sigma} - a_h^{\alpha\beta\rho\sigma}\right\|_{L^\infty(\omega)} + \sup_{1\le\alpha,\beta\le 2}\left\|a^{\alpha\beta} - a_h^{\alpha\beta}\right\|_{L^\infty(\omega)} \\ & + \sup_{1\le k\le 3}\left\|a_k - a_k^h\right\|_{L^\infty(\omega)^3} + \sup_{1\le\alpha\le 2}\left\|\partial_\alpha a_3 - d_\alpha^h\right\|_{L^\infty(\omega)^3}\Bigg)\,\|L\|\,. \end{aligned} \tag{4.13}$$

We are now in a position to prove the a posteriori error estimate. In order to state it, we define the error indicators. We use Remark 3.2 to write $a(U,V)$ and observe that a similar form holds for $a_h(U,V)$, with the obvious notation for the quantities $n_h^{\rho\sigma}(\cdot),\ m_h^{\rho\sigma}(\cdot)$, $t_h^\beta(\cdot)$ and $\theta^h_{\rho\sigma}(\cdot)$ (in comparison with (3.4) to (3.7), all coefficients are replaced by their approximations). For each $K \in \mathcal{T}_h$, the error indicator η_K is defined by

$$\eta_K = \eta_{K1} + \eta_{K2} \tag{4.14}$$

with

$$\begin{aligned} \eta_{K1} = {} & h_K\left\|f_h(\sqrt{a})_h + \partial_\rho\left((n_h^{\rho\sigma}(u_{p,h})a_\sigma^h + m_h^{\rho\sigma}(U_{p,h})d_\sigma^h + t_h^\rho(U_{p,h})\,a_3^h)(\sqrt{a})_h\right)\right\|_{L^2(K)^3} \\ & + \sum_{e\in E_K\backslash E_K^1} h_e^{\frac{1}{2}}\left\|\left[\nu_\rho\left(n_h^{\rho\sigma}(u_{p,h})a_\sigma^h + m_h^{\rho\sigma}(U_{p,h})d_\sigma^h + t_h^\rho(U_{p,h})a_3^h\right)(\sqrt{a})_h\right]_e\right\|_{L^2(e)^3} \\ & + \sum_{e\in E_K^1} h_e^{\frac{1}{2}}\left\|N_h l_h - \nu_\rho\left(n_h^{\rho\sigma}(u_{p,h})a_\sigma^h + m_h^{\rho\sigma}(U_{p,h})d_\sigma^h + t_h^\rho(U_{p,h})a_3^h\right)(\sqrt{a})_h\right\|_{L^2(e)^3} \end{aligned} \tag{4.15}$$

and

$$
\begin{aligned}
\eta_{K2} = \; & h_K \Big\| \partial_\rho \left(m_h^{\rho\sigma}(U_{p,h}) a_\sigma^h (\sqrt{a})_h \right) - t_h^\beta (U_{p,h}) a_\beta^h (\sqrt{a})_h \\
& + \frac{1}{p} \left(\partial_\rho (\partial_\rho (r_{p,h} \cdot a_3) a_3^h) - \partial_\rho (r_{p,h} \cdot a_3) d_\rho^h \right) \Big\|_{L^2(K)^3} \\
& + \sum_{e \in E_K \setminus E_K^1} h_e^{\frac{1}{2}} \left\| \left[\nu_\rho m_h^{\rho\sigma}(U_{p,h}) a_\sigma^h (\sqrt{a})_h + \frac{1}{p} \nu_\rho \partial_\rho (r_{p,h} \cdot a_3) a_3^h \right]_e \right\|_{L^2(e)^3} \\
& + \sum_{e \in E_K^1} h_e^{\frac{1}{2}} \left\| M_h l_h - \nu_\rho m_h^{\rho\sigma}(U_{p,h}) a_\sigma^h (\sqrt{a})_h - \frac{1}{p} \nu_\rho \partial_\rho (r_{p,h} \cdot a_3) a_3^h \right\|_{L^2(e)^3} .
\end{aligned}
\tag{4.16}
$$

Note that these indicators are of residual type and easy to compute since they only involve polynomial functions.

4.3 The Main Results

Theorem 4.1. *For any data* (f, N, M) *in* $L^2(\omega; \mathbb{R}^3) \times L^2(\gamma_1; \mathbb{R}^3) \times L^2(\gamma_1; \mathbb{R}^3)$*, the following a posteriori error estimate between the solution* U_p *of problem* (3.2) *and the solution* $U_{p,h}$ *of problem* (4.2) *holds:*

$$
\|U_p - U_{p,h}\|_{\mathbb{X}(\omega)} \leq c \left(\left(\sum_{K \in \mathcal{T}_h} \left(\eta_K^2 + \varepsilon_K^{(d)2} \right) \right)^{\frac{1}{2}} + \varepsilon_h^{(c)} \right) . \tag{4.17}
$$

Proof. We give an abridged proof. From the ellipticity property (2.11), using the residual equation (4.3) by replacing its second member by (4.10), we then use the triangle inequality that, combined with [2, Lemma 3.3, Lemma 3.4 and Lemma 3.5], leads to the bound of the following quantity:

$$
L_h(V - V_h) - a_h(U_{p,h}, V - V_h) - \frac{1}{p} b_h(r_{p,h} \cdot a_3; (s \cdot a_3 - s_h \cdot a_3))
$$

with $V_h = (\mathcal{R}_h v, \mathcal{R}_h s)$. We can write

$$
L_h(V - V_h) = L_h((v - \mathcal{R}_h v), 0) + L_h(0, (s - \mathcal{R}_h s)),
$$

$$
a_h(U_{p,h}, V - V_h) = a_h(U_{p,h}, (v - \mathcal{R}_h v, 0)) + a_h(U_{p,h}, (0, s - \mathcal{R}_h s),
$$

and

$$
\begin{aligned}
\frac{1}{p} b_h \left(U_{p,h}, V - V_h \right) = \; & \frac{1}{p} b_h(r_{p,h} \cdot a_3; (v - \mathcal{R}_h v, 0)) \\
& + \frac{1}{p} b_h \left(r_{p,h} \cdot a_3; (0, s \cdot a_3 - \mathcal{R}_h s \cdot a_3) \right) .
\end{aligned}
$$

Note that $b_h(r_{p,h}\cdot a_3; (v-\mathcal{R}_h v, 0)) = 0$. It remains to bound the following two quantities:

$$A_1 = \sup_{v\in H^1_{\gamma_0}(\omega;\mathbb{R}^3)} \frac{L_h(v-\mathcal{R}_h v,0) - a_h(U_{p,h},(v-\mathcal{R}_h v,0))}{\|v\|_{H^1(\omega;\mathbb{R}^3)}}$$

and

$$A_2 = \sup_{s\in H^1_{\gamma_0}(\omega;\mathbb{R}^3)} \frac{1}{\|s\|_{H^1(\omega;\mathbb{R}^3)}} \Bigg\{ L_h(0,s-\mathcal{R}_h s) - a_h(U_{p,h},(0,s-\mathcal{R}_h s)) - \frac{1}{p} b_h(r_{p,h}\cdot a_3;(0,(s-\mathcal{R}_h s)\cdot a_3)) \Bigg\}.$$

Setting $w = v - \mathcal{R}_h v$ and using the symmetry properties of the $n_h^{\rho\sigma}(\cdot)$ and $m_h^{\rho\sigma}(\cdot)$, we have

$$L_h(v-\mathcal{R}_h v,0) - a_h(U_{p,h},(v-\mathcal{R}_h v,0)) = \int_\omega f_h\cdot w(\sqrt{a})_h\,dx + \int_{\gamma_1} N_h\cdot w l_h d\tau - \int_\omega n_h^{\rho\sigma}(u_{p,h})\partial_\rho w\cdot a_\sigma^h + m_h^{\rho\sigma}(U_{p,h})\partial_\rho w\cdot d_\sigma^h + t_h^\beta(U_{p,h})\partial_\beta w\cdot a_3^h)(\sqrt{a})_h\,dx.$$

By cutting the integrals on ω into the sum of integrals on the K in $\mathcal{T}_h$ and integrating by parts on each K, we derive

$$\begin{aligned} L_h\,(v-\mathcal{R}_h v,0) - a_h(U_{p,h},(v-\mathcal{R}_h v,0)) &= \int_\omega f_h\cdot w(\sqrt{a})_h dx + \int_{\gamma_1} N_h\cdot w\; l_h d\tau \\ &+ \sum_{K\in\mathcal{T}_h}\Bigg(\int_K \partial_\rho((n_h^{\rho\sigma}(u_{p,h})a_\sigma^h + m_h^{\rho\sigma}(U_{p,h})d_\sigma^h + t_h^\beta(U_{p,h})a_3^h)(\sqrt{a})_h)\cdot w\,dx \\ &- \int_{\partial K} \nu_\rho(n_h^{\rho\sigma}(u_{p,h})a_\sigma^h + m_h^{\rho\sigma}(U_{p,h})d_\sigma^h + t_h^\beta(U_{p,h})a_3^h)(\sqrt{a})_h\cdot w\,d\tau\Bigg). \end{aligned} \tag{4.18}$$

Using the Cauchy–Schwarz inequality combined with (4.11) leads to

$$A_1 \le c\left(\sum_{K\in\mathcal{T}_h}\eta_{K1}^2\right)^{\frac{1}{2}}.$$

Setting $t = s - \mathcal{R}_h s$, and using the same arguments as previously, we arrive to

$$A_2 \le c\left(\sum_{K\in\mathcal{T}_h}\eta_{K2}^2\right)^{\frac{1}{2}}.$$

This concludes the proof. □

Theorem 4.2. *For any data* $(f, N, M) \in L^2(\omega;\mathbb{R}^3) \times L^2(\gamma_1;\mathbb{R}^3) \times L^2(\gamma_1;\mathbb{R}^3)$, *the following bounds hold for all indicators defined in* (4.15)–(4.16)*:*

$$\eta_{Ki} \leq c\left(\|U_p - U_{p,h}\|_{\mathbb{X}(\omega_K)} + \left(\sum_{K\subset\omega_K} \left(\varepsilon_K^{(d)}\right)^2 \right)^{\frac{1}{2}} + \varepsilon_h^{(c)} \right), \qquad i = 1, 2. \tag{4.19}$$

Proof. This is an abridged proof of the estimate for η_{K1}. We write η_{K1} in compact form as

$$\eta_{K1} = h_K \|\mathbf{F}_h\|_{L^2(K)^3} + \sum_{e\in E_K\setminus E_K^1} h_e^{\frac{1}{2}} \|[\mathbf{G}_h]_e\|_{L^2(e)^3} + \sum_{e\in E_K^1} h_e^{\frac{1}{2}} \|\mathbf{N}_h l_h - \mathbf{G}_h\|_{L^2(e)^3}.$$

We take $\mathcal{R}_h v = 0$ in (4.18) and $w = v$ equal to

$$w = \begin{cases} \mathbf{F}_h\psi_K & \text{on } K, \\ 0 & \text{on } \omega\backslash K, \end{cases}$$

where ψ_K denotes the bubble function on K. Thus, all the terms on the right-hand side of (4.18) vanish but the integral on K. We observe that function w (thus $\mathbf{F}_h$) on K is a polynomial of degree ≤ 3. From the appropriate inverse inequality [3, Proposition VII.4.1] together with (4.10) and (4.3) combined with [2, Lemma 3.3, Lemma 3.4 and Lemma 3.5], leads to

$$h_K \|\mathbf{F}_h\|_{L^2(K)^3} \leq c\left(\|U_p - U_{p,h}\|_{\mathbb{X}(K)} + \varepsilon_K^{(d)} + \varepsilon_K^{(c)} \right). \tag{4.20}$$

Similarly, to bound η_{K1} for any edge e shared by two elements K and K', we take w in (4.18) equal to

$$w = \begin{cases} P_{e,k}([\mathbf{G}_h]_e\,\psi_e) & \text{on } k \in \left\{K, K'\right\}, \\ 0 & \text{on } \omega\backslash\left\{K\cup K'\right\}, \end{cases}$$

where ψ_e is the bubble function on e and $P_{e,k}$ is the lifting operator introduced in [3, Lemma XI 2.7] from polynomials on e vanishing on ∂e into polynomials on K vanishing on $\partial K\backslash e$, constructed by an affine transformation from a fixed lifting operator on the reference triangle. Finally, for each e in E_K^1, we take the function w in (4.18) equal to

$$w = \begin{cases} P_{e,k}((\mathbf{N}_h l_h - \mathbf{G}_h)\psi_e) & \text{on } K, \\ 0 & \text{on } \omega\backslash K, \end{cases}$$

and this gives the bound for η_{K1}. □

5 The Adaptivity Strategy and Numerical Experiment

Now we present the adaptivity strategy and some numerical experiments.

5.1 Adaptivity Strategy

(i) Construct a first mesh $\mathcal{T}_h^0$. Set $i = 0$.

(ii) Solve the numerical problem on $\mathcal{T}_h^i$. Let u_h^i denote the solution.

(iii) Compute $\eta_{K_i}(u_h^i)$ on $K_i \in \mathcal{T}_h^i$.

(iv) If the global estimator is small enough, then stop.

(v) Otherwise, compute the new step of mesh $h_{K_{i+1}}$: $h_{K_{i+1}} = \dfrac{1}{2} h_{K_i}$ if $\eta_{K_i}(u_h^i) \geq TOL$, otherwise $h_{K_{i+1}} = l h_{K_i},\ l \geq 1$. Here

$$TOL = \frac{1}{n_{t_i}} \sum_{K_i \in \mathcal{T}_h^i} \eta_{K_i}(u_h^i),$$

where n_{t_i} is the number of triangles of $\mathcal{T}_h^i$.

(vi) Generate the new mesh $\mathcal{T}_h^{i+1}$ and return to (ii).

5.2 Numerical Experiment

The numerical experiment that we present has been performed on the finite element code `FreeFem++`, see [10]. Three-dimensional visualization of the deformed shells uses `Medit`, a free mesh visualization software available at `http://www.ann.jussieu.fr/~frey/logiciels/medit.html`.

Hyperbolic paraboloid shell

The reference domain ω is the square

$$\omega = \left\{ (x, y); \mid x \mid + \mid y \mid \leq \sqrt{2} b \right\}, \tag{5.1}$$

as illustrated in [1, §1.3.3 and §2.4.2] and the cart φ is defined by

$$\varphi(x, y) = \left(x, y, \frac{c}{2b^2}(x^2 - y^2) \right)^T. \tag{5.2}$$

We choose $b = 50$ cm, c $= 10$ cm and $e = 0.8$ cm. We assume that the shell is clamped on $\gamma_0 = \partial\omega$ and that it is subjected to uniform pressure $q = -0.01$ kp/cm^2. The mechanical data are $E = 2.8\ 10^4$ kp/cm^2, $\nu = 0.4$. The reference value for this test is the normal displacement at the center $C(0,0)$ of the shell. Its value computed by various methods is of -0.0240000 cm; see [1]. We take $p = 10^3 \dfrac{E}{2(1+\nu)}$, see [5].

Results with mesh adaptation

Iteration	1	2	3
Degrees of freedom	3009	7188	13424
$U_3(C)$	-0.0241186	-0.0242561	-0.0240096

Results without mesh adaptation

Degrees of freedom	20157	23849
$U_3(C)$	-0.0239921	-0.0239905

Note that the resolution of the discrete problem with mesh adaptation gives the convergence to the solution after the third iteration with a number of degrees of freedom equal to 13424 but the resolution of the problem without mesh adaptation gives an approximate solution with an error of order 4.10^{-3} and a number of degrees of freedom was lower than calculated by mesh adaptation. Figure 5.1 presents the initial and the final adapted meshes according to the strategy described above. Figure 5.2 and 5.3 present the "over-deformed" shell.

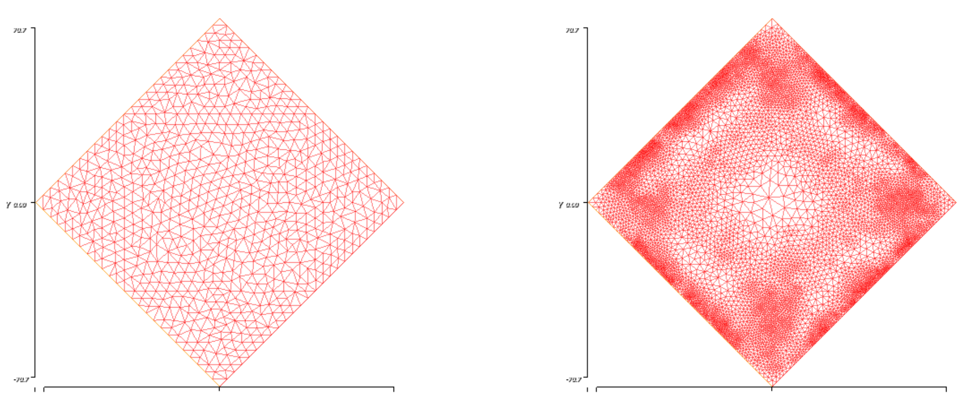

Figure 5.1: The initial and adapted meshes

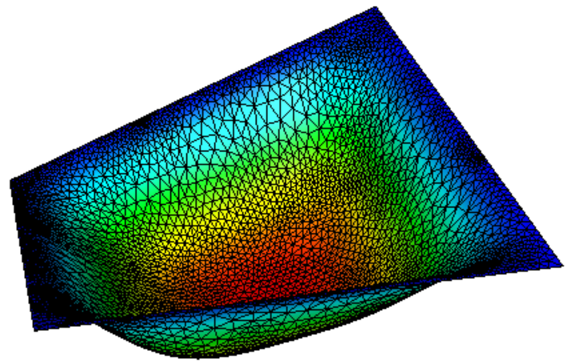

Figure 5.2: The "over-deformed" top side

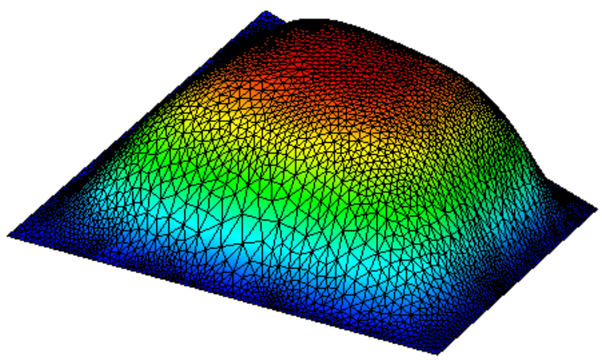

Figure 5.3: The "over-deformed" bottom side

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