

## Existence of Nonoscillatory Solutions to Second Order Neutral Type Difference Equations with Mixed Arguments

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### Abstract

In this paper, we present several sufficient conditions for the existence of nonoscillatory solutions to the following second order neutral type difference equation

$$\Delta(r_n \Delta(x_n + a_n x_{n-l} + b_n x_{n+m})) + p_n x_{n-k} - q_n x_{n+d} = 0, \quad n \geq n_0$$

via Banach contraction principle. Examples are provided to illustrate the main results. The results obtained in this paper extend and complement to some of the existing results.

**AMS Subject Classifications:** 39A10.

**Keywords:** Neutral type difference equation, fixed point, nonoscillatory solutions.

## 1 Introduction

This paper deals with the existence of nonoscillatory solutions of second order neutral type difference equation of the form

$$\Delta(r_n \Delta(x_n + a_n x_{n-l} + b_n x_{n+m})) + p_n x_{n-k} - q_n x_{n+d} = 0, \quad n \geq n_0 \quad (1.1)$$

where  $n_0$  is a nonnegative integer, subject to the following conditions:

( $H_1$ )  $\{a_n\}$  and  $\{b_n\}$  are real sequences, and  $\{p_n\}$  and  $\{q_n\}$  are non-negative real sequences for all  $n \geq n_0$ ;

( $H_2$ )  $\{r_n\}$  is a positive real sequence for all  $n \geq n_0$ ;

( $H_3$ )  $l$  and  $m$  are positive integers and  $k$  and  $r$  are non-negative integers.

Let  $\theta = \max\{l, k\}$ . By a solution of equation (1.1), we mean a real sequence  $\{x_n\}$  defined for all  $n \geq n_0 - \theta$ , and satisfying the equation (1.1) for all  $n \geq n_0$ . A nontrivial solution of equation (1.1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and oscillatory otherwise.

In recent years, many authors interested in studying the oscillatory and nonoscillatory behavior of various classes of difference equations, see for examples [1, 2] and the references cited therein.

In [3], the authors discussed the existence of nonoscillatory solutions of the following second order difference equation of the form

$$\Delta(|\Delta x_{n-1}|^{\alpha-1} \Delta x_{n-1}) + q_n |x_n|^{\beta-1} x_n = 0, \quad (1.2)$$

where  $\beta > \alpha > 0$ .

In [4], the author investigated the existence of nonoscillatory solutions for the equation

$$\Delta^2(x_n + p x_{n-m}) + p_n x_{n-k} - q_n x_{n-l} = 0, \quad n \geq n_0 \quad (1.3)$$

where  $p$  is a real number with  $p \neq \pm 1$ .

In [5], the authors discussed the existence of nonoscillatory solutions to the second order neutral delay dynamic equation of the form

$$(x(t) + p(t)x(\tau_0(t)))^{\Delta\Delta} + q_1(t)x(\tau_1(t)) - q_2(t)x(\tau_2(t)) = e(t) \quad (1.4)$$

on a time scale  $\mathbb{T}$ .

Recently in [6], the authors established sufficient conditions for the existence of nonoscillatory solutions for the equation

$$\Delta(r_n \Delta(x_n + c x_{n-m})) + p_n f(x_{n-k}) - q_n g(x_{n-l}) = 0, \quad n \geq n_0 \quad (1.5)$$

where  $c$  is a real number with  $c \neq -1$ .

In [7], the authors studied the existence of nonoscillatory solutions for the equation

$$\Delta(a_n \Delta(x_n + cx_{n-k})) + p_n x_{n-l} - q_n x_{n-m} = 0, \quad n \geq n_0 \tag{1.6}$$

where  $c$  is a real number with  $c \neq \pm 1$ .

On the other hand, there has been great interest in studying the oscillatory behavior of second order neutral type difference equations with mixed arguments, see for example [8–13], and the references cited therein.

In view of the above observations, in this paper we obtain sufficient conditions for the existence of nonoscillatory solutions for the difference equation (1.1). The equations (1.3), (1.5), and (1.6) are special cases of equation (1.1) and therefore the results presented in this paper generalize and complement to some of the results obtained in [4,6,7].

## 2 Existence of Nonoscillatory Solutions

In this section, we obtain sufficient conditions for the existence of nonoscillatory solutions of equations (1.1) using Banach contraction principle. We begin with the following theorem.

**Theorem 2.1** (Banach’s contraction mapping principle). *A contraction mapping on a complete metric space has a unique fixed point.*

**Theorem 2.2.** *Assume that  $0 \leq a_n \leq a < 1$ , and  $0 \leq b_n \leq b \leq 1 - a$  for all  $n \geq n_0$ . If*

$$\sum_{n=n_0}^{\infty} R_n p_n < \infty, \quad \text{and} \quad \sum_{n=n_0}^{\infty} R_n q_n < \infty, \tag{2.1}$$

where  $R_n = \sum_{s=n_0}^{n-1} \frac{1}{r_s}$ , then equation (1.1) has a bounded nonoscillatory solution.

*Proof.* From condition (2.1), one can choose an integer  $N > n_0$  sufficiently large such that

$$\sum_{s=n}^{\infty} R_s p_s \leq \frac{M_2 - \alpha}{M_2} \quad \text{for all } n \geq N, \tag{2.2}$$

and

$$\sum_{s=n}^{\infty} R_s q_s \leq \frac{\alpha - (a + b)M_2 - M_1}{M_2} \quad \text{for all } n \geq N, \tag{2.3}$$

where  $M_1$  and  $M_2$  are positive constants such that

$$(a + b)M_2 + M_1 < M_2, \quad \text{and} \quad \alpha \in ((a + b)M_2 + M_1, M_2).$$

Let  $B$  be the set of all bounded real sequences  $\{x_n\}$  defined for all  $n \geq n_0$  with supremum norm  $\|x\| = \sup_{n \geq n_0} |x_n|$ . Then clearly  $B$  is a Banach space. Set

$$S = \{x = \{x_n\} \in B : M_1 \leq x_n \leq M_2, n \geq n_0\}.$$

It is clear that  $S$  is a bounded, closed and convex subset of  $B$ . Define an operator  $T : S \rightarrow B$  by

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} + R_{n-1} \sum_{s=n-1}^{\infty} (p_s x_{s-k} - q_s x_{s+d}) \\ \quad + \sum_{s=N}^{n-2} R_s (p_s x_{s-k} - q_s x_{s+d}), n \geq N, \\ (Tx)_N, n_0 \leq n \leq N. \end{cases}$$

Clearly  $(Tx)_n$  is a real sequence and it is not difficult to show that  $T$  is a continuous mapping on  $S$ . For every  $x = \{x_n\} \in S$  and  $n \geq N$ , we have

$$(Tx)_n \leq \alpha + M_2 \sum_{s=N}^{\infty} R_s p_s \leq M_2,$$

and

$$(Tx)_n \geq \alpha - aM_2 - bM_2 - M_2 \sum_{s=n}^{\infty} R_s q_s \geq M_1,$$

where we have used (2.2) and (2.3). Thus  $TS \subset S$ .

Next, we show that  $T$  is a contraction mapping on  $S$ . Let  $x, y \in S$  and  $n \geq N$ . Then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq a_n |x_{n-l} - y_{n-l}| + b_n |x_{n+m} - y_{n+m}| \\ &\quad + R_{n-1} \sum_{s=n-1}^{\infty} p_s |x_{s-k} - y_{s-k}| + R_{n-1} \sum_{s=n-1}^{\infty} q_s |x_{s+d} - y_{s+d}| \\ &\quad + \sum_{s=N}^{n-2} R_s p_s |x_{s-k} - y_{s-k}| + \sum_{s=N}^{n-2} R_s q_s |x_{s+d} - y_{s+d}| \\ &\leq \|x - y\| (a + b + L_1 \sum_{s=N}^{\infty} R_s (p_s + q_s)) \\ &\leq \left( a + b + \frac{M_2 - \alpha}{M_2} + \frac{\alpha - (a + b)M_2 - M_1}{M_2} \right) \|x - y\| \end{aligned}$$

or

$$\|T_x - T_y\| \leq \lambda_1 \|x - y\|$$

where  $\lambda_1 = (1 - \frac{M_1}{M_2})$ . This implies that  $T$  is a contraction mapping on  $S$  since  $\lambda_1 < 1$ . By Theorem 2.1,  $T$  has a unique fixed point which is a positive and bounded solution of equation (1.1). The proof is now completed.  $\square$

**Theorem 2.3.** *Assume that  $0 \leq a_n \leq a < 1$ , and  $a - 1 < b \leq b_n \leq 0$  for all  $n \geq n_0$ . If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.*

*Proof.* In view of condition (2.1), we can choose an integer  $N > n_0$  sufficiently large satisfying  $N \geq n_0 + \theta$  such that

$$\sum_{s=n}^{\infty} R_s p_s \leq \frac{(1+a)M_4 - \alpha}{M_4} \quad \text{for all } n \geq N, \tag{2.4}$$

and

$$\sum_{s=n}^{\infty} R_s q_s \leq \frac{\alpha - aM_4 - M_3}{M_4} \quad \text{for all } n \geq N, \tag{2.5}$$

where  $M_3$  and  $M_4$  are positive constants such that

$$M_3 + aM_4 < (1+b)M_4, \text{ and } \alpha \in (M_3 + aM_4, (1+b)M_4).$$

Let  $B$  be the Banach space as defined in Theorem 2.2. Set

$$S = \{x = \{x\} \in B : M_3 \leq x_n \leq M_4, n \geq n_0\}.$$

It is clear that  $S$  is a bounded, closed and convex subset of  $S$ . Define an operator  $T : S \rightarrow B$  by

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} + R_{n-1} \sum_{s=n-1}^{\infty} (p_s x_{s-k} - q_s x_{s+d}) \\ + \sum_{s=N}^{n-2} R_s (p_s x_{s-k} - q_s x_{s+d}), n \geq N, \\ (Tx)_N, n_0 \leq n \leq N. \end{cases}$$

Clearly  $T$  is a continuous mapping on  $S$ . For  $n \geq N$  and  $x \in S$ , we have from (2.4) and (2.5), respectively that

$$(Tx)_n \leq \alpha - bM_4 + M_4 \sum_{s=N}^{\infty} R_s p_s \leq M_4,$$

and

$$(Tx)_n \geq \alpha - aM_4 - M_4 \sum_{s=N}^{\infty} R_s q_s \geq M_3.$$

This proves that  $TS \subset S$ . Let  $x, y \in S$  and  $n \geq N$ . Then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq \|x - y\| \left( a - b + \sum_{s=n}^{\infty} R_s(p_s + q_s) \right) \\ &\leq \lambda_2 \|x - y\|, \end{aligned}$$

where  $\lambda_2 = (1 - \frac{M_3}{M_4})$ . This implies that  $\|Tx - Ty\| \leq \lambda_2 \|x - y\|$ . Since  $\lambda_2 < 1$ ,  $T$  is a contraction mapping on  $S$ . Hence by Theorem 2.1,  $T$  has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof.  $\square$

**Theorem 2.4.** *Assume that  $1 < a \leq a_n \leq c < \infty$ , and  $0 \leq b_n \leq b < a - 1$  for all  $n \geq n_0$ . If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.*

*Proof.* In view of condition (2.1), one can choose an integer  $N > n_0$  so that  $N + l \geq n_0 + k$  sufficiently large such that

$$\sum_{s=n}^{\infty} R_s p_s \leq \frac{aM_6 - \alpha}{M_6} \text{ for all } n \geq N, \quad (2.6)$$

and

$$\sum_{s=n}^{\infty} R_s p_s \leq \frac{\alpha - cM_5 - (1+b)M_6}{M_6} \text{ for all } n \geq N, \quad (2.7)$$

where  $M_5$  and  $M_6$  are positive constants such that

$$cM_5 + (1+b)M_6 < aM_6, \text{ and } \alpha \in (cM_5 + (1+b)M_6, aM_6).$$

Let  $B$  be the Banach space as defined in Theorem 2.2. Set

$$S = \{x = \{x\} \in B : M_5 \leq x_n \leq M_6, n \geq n_0\}.$$

Obviously  $S$  is a bounded, closed and convex subset of  $B$ . Define an operator  $T : S \rightarrow B$  as follows:

$$(Tx)_n = \begin{cases} \frac{1}{a_{n+l}} \left\{ \alpha - x_{n+l} - b_n x_{n+l+m} + R_{n+l-1} \sum_{s=n-1+l}^{\infty} (p_s x_{s-k} - q_s x_{s+d}) \right. \\ \left. + \sum_{s=N}^{n-2+l} R_s (p_s x_{s-k} - q_s x_{s+d}) \right\}, & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly,  $T$  is a continuous mapping on  $S$ . For  $n \geq N$  and  $x \in S$ , we have from (2.6), and (2.7), respectively that

$$(Tx)_n \leq \frac{1}{a_{n+l}} \left( \alpha + M_6 \sum_{s=n}^{\infty} R_s p_s \right) \leq \frac{1}{a} \left( \alpha + M_6 \sum_{s=n}^{\infty} R_s p_s \right) \leq M_6,$$

and

$$\begin{aligned} (Tx)_n &\geq \frac{1}{a_{n+l}} \left( \alpha - (1+b)M_6 - M_6 \sum_{s=n}^{\infty} R_s q_s \right) \\ &\geq \frac{1}{c} \left( \alpha - (1+b)M_6 - M_6 \sum_{s=n}^{\infty} R_s q_s \right) \geq M_5. \end{aligned}$$

Thus  $TS \subset S$ . Next we show that  $T$  is a contraction mapping on  $S$ . If  $x, y \in S$  and  $n \geq N$ , then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq \frac{1}{a} \|x - y\| \left( 1 + b + \sum_{s=N}^{\infty} R_s (p_s + q_s) \right) \\ &\leq \lambda_3 \|x - y\| \end{aligned}$$

where  $\lambda_3 = (1 - \frac{cM_5}{M_6})$ . This implies that  $\|Tx - Ty\| \leq \lambda_3 \|x - y\|$ . Since  $\lambda_3 < 1$ ,  $T$  is a contraction mapping on  $S$ . Therefore by Theorem 2.1,  $T$  has a unique fixed point which is a positive and bounded solution of equation (1.1). The completes the proof.  $\square$

**Theorem 2.5.** Assume that  $1 < a \leq a_n \leq c < \infty$ , and  $1 - a < b \leq b_n \leq 0$  for all  $n \geq n_0$ . If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.

*Proof.* In view of condition (2.1), one can choose an integer  $N > n_0$  sufficiently large satisfying  $N + l \geq n_0 + k$  such that

$$\sum_{s=n}^{\infty} R_s p_s \leq \frac{(a+b)M_8 - \alpha}{M_8} \text{ for all } n \geq N, \tag{2.8}$$

and

$$\sum_{s=n}^{\infty} R_s q_s \leq \frac{\alpha - cM_7 - M_8}{M_8} \text{ for all } n \geq N, \tag{2.9}$$

where  $M_7$  and  $M_8$  are positive constants such that

$$cM_7 + M_8 < (a+b)M_8, \text{ and } \alpha \in (cM_7 + M_8, (a+b)M_8).$$

Let  $B$  be the Banach space as defined in Theorem 2.2. Define

$$S = \{x = \{x\} \in B : M_7 \leq x_n \leq M_8, n \geq n_0\}.$$

Clearly  $S$  is a bounded, closed and convex subset of  $B$ . Define a mapping  $T : S \rightarrow B$  as follows:

$$(Tx)_n = \begin{cases} \frac{1}{a_{n+l}} \left\{ \alpha - x_{n+l} - b_{n+l}x_{n+l+m} + R_{n+l-1} \sum_{s=n+l-1}^{\infty} (p_s x_{s-k} - q_s x_{s+d}) \right. \\ \left. + \sum_{s=N}^{n+l-2} R_s (p_s x_{s-k} - q_s x_{s+d}) \right\}, & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

It is clear that  $T$  is a continuous mapping on  $S$ . For  $n \geq N$  and  $x \in S$ , we have from (2.8), and (2.9), respectively that

$$(Tx)_n \leq \frac{1}{a} \left( \alpha - bM_8 + M_8 \sum_{s=N}^{\infty} R_s p_s \right) \leq M_8,$$

and

$$(Tx)_n \geq \frac{1}{c} \left( \alpha - M_8 - M_8 \sum_{s=n}^{\infty} R_s q_s \right) \geq M_7.$$

This implies that  $TS \subset S$ . If  $x, y \in S$  and  $n \geq N$ , then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq \frac{1}{a} \|x - y\| \left( 1 - b + \sum_{s=n}^{\infty} R_s (p_s + q_s) \right) \\ &= \lambda_4 \|x - y\| \end{aligned}$$

where  $\lambda_4 = (1 - \frac{cM_7}{M_8})$ . This implies that  $\|Tx - Ty\| \leq \lambda_4 \|x - y\|$ . Since  $\lambda_4 < 1$ ,  $T$  is a contraction mapping on  $S$ . Therefore by Theorem 2.1,  $T$  has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof.  $\square$

**Theorem 2.6.** Assume that  $-1 < a \leq a_n \leq 0$ , and  $0 \leq b_n \leq b \leq 1 + a$  for all  $n \geq n_0$ . If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.

*Proof.* From condition (2.1), one can choose an integer  $N > n_0$  sufficiently large satisfying  $N \geq n_0 + \theta$  such that

$$\sum_{s=n}^{\infty} R_s p_s \leq \frac{(1+a)M_{10} - \alpha}{M_{10}} \text{ for all } n \geq N, \quad (2.10)$$



and

$$\sum_{s=n}^{\infty} R_s q_s \leq \frac{\alpha - bM_{10} - M_9}{M_{10}} \text{ for all } n \geq N, \quad (2.11)$$

where  $M_9$  and  $M_{10}$  are positive constants such that

$$M_9 + bM_{10} < (a + 1)M_{10}, \text{ and } \alpha \in (M_9 + bM_{10}, (1 + a)M_{10}).$$

Let  $B$  be the Banach space as defined in Theorem 2.2. Let

$$S = \{x = \{x_n\} \in B : M_9 \leq x_n \leq M_{10}, n \geq n_0\}.$$

Clearly  $S$  is a bounded, closed and convex subset of  $B$ . Define an operator  $T : S \rightarrow B$  as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} + R_{n-1} \sum_{s=n-1}^{\infty} (p_s x_{s-k} - q_s x_{s+d}) \\ + \sum_{s=N}^{n-2} R_s (p_s x_{s-k} - q_s x_{s+d}), & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly,  $T$  is a continuous mapping on  $S$ . For  $n \geq N$  and  $x \in S$ , from (2.10) and (2.11) it follows that

$$(Tx)_n \leq \alpha - aM_{10} + M_{10} \sum_{s=N}^{\infty} R_s p_s \leq M_{10},$$

and

$$(Tx)_n \geq \alpha - bM_{10} - M_{10} \sum_{s=N}^{\infty} R_s q_s \geq M_9.$$

Hence,  $TS \subset S$ . Next, we show that  $T$  is a contraction mapping on  $S$ . For  $x, y \in S$ , and  $n \geq N$ , then

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq \|x - y\| \left( -a + b + \sum_{s=N}^{\infty} R_s (p_s + q_s) \right) \\ &= \lambda_5 \|x - y\| \end{aligned}$$

where  $\lambda_5 = (1 - \frac{M_9}{M_{10}})$ . This implies that  $\|Tx - Ty\| \leq \lambda_5 \|x - y\|$ . Since  $\lambda_5 < 1$ ,  $T$  is a contraction mapping on  $S$ . Hence by Theorem 2.1,  $T$  has a unique fixed point which is a positive and bounded solution of equation (1.1). The proof is now completed.  $\square$

**Theorem 2.7.** Assume that  $-1 < a \leq a_n \leq 0$ , and  $-1 - a < b \leq b_n \leq 0$  for all  $n \geq n_0$ . If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.

*Proof.* In view of condition (2.1), one can choose an integer  $N > n_0$  sufficiently large satisfying  $N \geq n_0 + \theta$  such that

$$\sum_{s=n}^{\infty} R_s p_s \leq \frac{(1+a+b)M_{12} - \alpha}{M_{12}} \text{ for all } n \geq N, \quad (2.12)$$

and

$$\sum_{s=n}^{\infty} R_s q_s \leq \frac{\alpha - M_{11}}{M_{12}} \text{ for all } n \geq N, \quad (2.13)$$

where  $M_{11}$  and  $M_{12}$  are positive constants such that

$$M_{11} < (1+a+b)M_{12}, \text{ and } \alpha \in (M_{11}, (1+a+b)M_{12}).$$

Let  $B$  be the Banach space as defined in Theorem 2.2. Set

$$S = \{x = \{x\} \in B : M_{11} \leq x_n \leq M_{12}, n \geq n_0\}.$$

It is clear that  $S$  is a bounded, closed and convex subset of  $B$ . Define an operator  $T : S \rightarrow B$  as follows:

$$(Tx)_n = \begin{cases} \alpha - a_n x_{n-l} - b_n x_{n+m} + R_{n-1} \sum_{s=n-1}^{\infty} (p_s x_{s-k} - q_s x_{s+d}) \\ + \sum_{s=N}^{n-2} R_s (p_s x_{s-k} - q_s x_{s+d}), & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly,  $T$  is continuous on  $S$ . For  $n \geq N$  and  $x \in S$ , from (2.12) and (2.13), it follows that

$$(Tx)_n \leq \alpha - aM_{12} - bM_{12} + M_{12} \sum_{s=N}^{\infty} R_s p_s \leq M_{12},$$

and

$$(Tx)_n \geq \alpha - M_{12} \sum_{s=N}^{\infty} R_s q_s \geq M_{11}.$$

This implies that  $TS \subset S$ . If  $x, y \in S$  and  $n \geq N$ , then we have

$$|(Tx)_n - (Ty)_n| \leq \|x - y\| \left( -a - b + \sum_{s=N}^{\infty} R_s (p_s + q_s) \right)$$

$$= \lambda_6 \|x - y\|$$

where  $\lambda_6 = (1 - \frac{M_{11}}{M_{12}})$ . Hence  $\|Tx - Ty\| \leq \lambda_6 \|x - y\|$ . Since  $\lambda_6 < 1$ , it follows that  $T$  is a contraction mapping on  $S$ . By Theorem 2.1,  $T$  has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof.  $\square$

**Theorem 2.8.** *Assume that  $-\infty < c \leq a_n \leq a < -1$ , and  $0 \leq b_n \leq b < -a - 1$  for all  $n \geq n_0$ . If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.*

*Proof.* In view of condition (2.1), one can choose an integer  $N > n_0$  sufficiently large satisfying  $N + l \geq n_0 + k$  such that

$$\sum_{s=n}^{\infty} R_s q_s \leq \frac{(1 - a - b)M_{14} - \alpha}{M_{14}} \text{ for all } n \geq N, \tag{2.14}$$

and

$$\sum_{s=n}^{\infty} R_s p_s \leq \frac{cM_{13} + \alpha}{M_{14}} \text{ for all } n \geq N, \tag{2.15}$$

where  $M_{13}$  and  $M_{14}$  are positive constants such that

$$-cM_{13} < (-1 - a - b)M_{14}, \text{ and } \alpha \in (-cM_{13}, (-1 - a - b)M_{14}).$$

Let  $B$  be the Banach space as defined in Theorem 2.2. Set

$$S = \{x = \{x\} \in B : M_{13} \leq x_n \leq M_{14}, n \geq n_0\}.$$

Clearly  $S$  is a bounded, closed and convex subset of  $B$ . Define a mapping  $T : S \rightarrow B$  by

$$(Tx)_n = \begin{cases} -\frac{1}{a_{n+l}} \left\{ \alpha + x_{n+l} + b_{n+l}x_{n+l+m} - R_{n+l-1} \sum_{s=n+l-1}^{\infty} (p_s x_{s-k} - q_s x_{s+d}) \right. \\ \left. + \sum_{s=N}^{n+l-2} R_s (p_s x_{s-k} - q_s x_{s+d}) \right\}, & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Clearly,  $T$  is continuous on  $S$ . For  $n \geq N$ , and  $x \in S$ , from (2.14) and (2.15), we have

$$(Tx)_n \leq -\frac{1}{a} \left( \alpha + M_{14} + bM_{14} + M_{14} \sum_{s=N}^{\infty} R_s p_s \right) \leq M_{14},$$

and

$$(Tx)_n \geq -\frac{1}{c} \left( \alpha - M_{14} \sum_{s=N}^{\infty} R_s q_s \right) \geq M_{13}.$$

Thus  $TS \subset S$ . If  $x, y \in S$ , and  $n \geq N$ , then we have

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq -\frac{1}{a} \|x - y\| \left( 1 + b + \sum_{s=N}^{\infty} R_s (p_s + q_s) \right) \\ &= \lambda_7 \|x - y\| \end{aligned}$$

where  $\lambda_7 = (1 - \frac{M_{13}}{M_{14}})$ . This implies that  $\|Tx - Ty\| \leq \lambda_7 \|x - y\|$ . Since  $\lambda_7 < 1$ ,  $T$  is a contraction mapping on  $S$ . By Theorem 2.1,  $T$  has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof.  $\square$

**Theorem 2.9.** Assume that  $-\infty < c \leq a_n \leq a < -1$ , and  $a + 1 < b \leq b_n \leq 0$  for all  $n \geq n_0$ . If condition (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.

*Proof.* In view of condition (2.1), one can choose an integer  $N > n_0$  sufficiently large satisfying  $N + l \geq n_0 + k$  such that

$$\sum_{s=n}^{\infty} R_s q_s \leq \frac{(-a-1)M_{16} - \alpha}{M_{16}} \text{ for all } n \geq N, \quad (2.16)$$

and

$$\sum_{s=n}^{\infty} R_s p_s \leq \frac{cM_{15} + bM_{16} + \alpha}{M_{16}} \text{ for all } n \geq N, \quad (2.17)$$

where  $M_{15}$  and  $M_{16}$  are positive constants such that

$$-cM_{15} - bM_{16} < (-a-1)M_{16}, \text{ and } \alpha \in (-cM_{15} - bM_{16}, (-a-1)M_{16}).$$

Let  $B$  be the Banach space as defined in Theorem 2.2. Set

$$S = \{x = \{x\} \in B : M_{15} \leq x_n \leq M_{16}, n \geq n_0\}.$$

It is clear that  $S$  is a bounded, closed and convex subset of  $B$ . Define an operator  $T : S \rightarrow B$  by

$$(Tx)_n = \begin{cases} -\frac{1}{a_{n+l}} \left\{ \alpha + x_{n+l} + b_{n+l} x_{n+l+m} - R_{n+l-1} \sum_{s=n+l-1}^{\infty} (p_s x_{n-k} - q_s x_{s+d}) \right. \\ \left. - \sum_{s=N}^{n+l-2} R_s (p_s x_{s-k} - q_s x_{s+d}) \right\}, & n \geq N, \\ (Tx)_N, & n_0 \leq n \leq N. \end{cases}$$

Obviously  $T$  is continuous on  $S$ . For  $n \geq N$  and  $x \in S$ , it follows from (2.16), and (2.17) that

$$(Tx)_n \leq -\frac{1}{a} \left( \alpha + M_{16} + M_{16} \sum_{s=N}^{\infty} R_s p_s \right) \leq M_{16},$$

and

$$(Tx)_n \geq -\frac{1}{c} \left( \alpha + bM_{16} - M_{16} \sum_{s=N}^{\infty} q_s \right) \geq M_{15}.$$

This implies that  $TS \subset S$ . If  $x, y \in S$  and  $n \geq N$ , then we have

$$\begin{aligned} |(Tx)_n - (Ty)_n| &\leq -\frac{1}{a} \|x - y\| \left( 1 - b + \sum_{s=N}^{\infty} R_s (p_s + q_s) \right) \\ &= \lambda_8 \|x - y\| \end{aligned}$$

where  $\lambda_8 = (1 - \frac{cM_{15}}{M_{16}})$ . This implies that  $\|Tx - Ty\| \leq \lambda_8 \|x - y\|$ . Since  $\lambda_8 < 1$ ,  $T$  is a contraction mapping on  $S$ . By Theorem 2.1,  $T$  has a unique fixed point which is a positive and bounded solution of equation (1.1). This completes the proof.  $\square$

### 3 Examples

In this section, we present some examples to illustrate the main results.

**Example 3.1.** Consider the neutral difference equation of the form

$$\begin{aligned} \Delta \left( n(n+1) \Delta \left( x_n + \frac{1}{4} x_{n-1} + \frac{1}{4} x_{n+1} \right) \right) + \frac{(n-1)}{2n(n+2)(n+3)} x_{n-1} \\ - \frac{(n+2)}{2n(n-1)(n+3)} x_{n+2} = 0, \quad n \geq 3. \end{aligned} \quad (3.1)$$

Here  $r_n = n(n-1)$ ,  $a_n = \frac{1}{4}$ ,  $b_n = \frac{1}{4}$ ,  $p_n = \frac{(n-1)}{2n(n+2)(n+3)}$ ,  $q_n = \frac{(n+2)}{2n(n-1)(n+3)}$ ,  $l = 1$ ,  $m = 1$ ,  $k = 1$  and  $d = 2$ . Simple calculation shows that  $R_n = \frac{n-2}{2n}$  and then one can easily verify other conditions of Theorem 2.2. Hence equation (3.1) has a bounded nonoscillatory solution. In fact  $\{x_n\} = \{1 + \frac{1}{n}\}$  is one such nonoscillatory solution of equation (3.1).

**Example 3.2.** Consider a neutral difference equation of the form

$$\Delta \left( 2^n \Delta \left( x_n + \frac{1}{4}x_{n-2} - \frac{1}{4}x_{n+1} \right) \right) + \frac{1}{4^n}x_{n-2} - \frac{2}{4^n} \left( \frac{2^n + 4}{2^{n+1} + 1} \right) x_{n+1} = 0, \quad n \geq 1. \quad (3.2)$$

Here  $r_n = 2^n$ ,  $a_n = \frac{1}{4}$ ,  $b_n = \frac{1}{4}$ ,  $p_n = \frac{1}{4^n}$ ,  $q_n = \frac{2}{4^n} \left( \frac{2^n + 4}{2^{n+1} + 1} \right)$ ,  $l = 2$ ,  $m = 1$ ,  $k = 2$  and  $d = 1$ . Simple calculation shows that  $R_n = \frac{2^n - 2}{2^n}$  and then one can easily verify other conditions of Theorem 2.3. Hence equation (3.2) has a bounded nonoscillatory solution. In fact  $\{x_n\} = \{1 + \frac{1}{2^n}\}$  is one such nonoscillatory solution of equation (3.2).

**Example 3.3.** Consider the neutral difference equation of the form

$$\Delta \left( 3^n \Delta \left( x_n + 2x_{n-2} - \frac{1}{3}x_{n+1} \right) \right) + \frac{1}{9^n}x_{n-2} - \frac{3}{9^n} \left( \frac{3^n + 9}{3^{n+1} + 1} \right) x_{n+1} = 0, \quad n \geq 1. \quad (3.3)$$

Here  $r_n = 3^n$ ,  $a_n = 2$ ,  $b_n = -\frac{1}{3}$ ,  $p_n = \frac{1}{9^n}$ ,  $q_n = \frac{3}{9^n} \left( \frac{3^n + 9}{3^{n+1} + 1} \right)$ ,  $l = 2$ ,  $m = 1$ ,  $k = 2$  and  $d = 1$ . Simple calculation shows that  $R_n = \frac{1}{2} \left( 1 - \frac{1}{3^{n-1}} \right)$  and then one can easily verify the other conditions of Theorem 2.5. Hence equation (3.3) has a bounded nonoscillatory solution. In fact  $\{x_n\} = \{1 + \frac{1}{3^n}\}$  is one such nonoscillatory solution of equation (3.3).

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## References

- [1] R.P.Agarwal, *Difference Equations and Inequalities, Theory, Methods and Applications*, Second Edition, Marcel Dekker, New York, 2000.
- [2] R.P.Agarwal, M.Bohner, S.R.Grace and D.O.Regan, *Discrete Oscillation Theory*, Hindawi Publ. Corp., New York, 2005.
- [3] S.Chen and C.Li, Nonoscillatory solutions of second order nonlinear difference equations, *Appl.Math.Comput.*, 205(2008), 478–481.
- [4] J.Cheng, Existence of nonoscillatory solutions of a second order linear neutral difference equation, *Appl.Math.Lett.*, 20(2007), 892–899.

- [5] T.Li, Z.Han, S.Sun and D.Yang, Existence of nonoscillatory solutions to second order neutral delay dynamic equations on time scales, *Adv.Differ.Equ.*, Volume 2009, Article ID 562329, 10 pages.
- [6] Y.Tian, Y.Cai and T.Li, Existence of nonoscillatory solutions to second order nonlinear neutral difference equations, *J.Nonlinear Sci.Appl.*, 8(2015), 884–892.
- [7] E.Thandapani and K.Mahalingam, Existence of nonoscillatory solution of second order difference equations of neutral type, *Indian J. Pure Appl. Math.*, 33(2002), 625–633.
- [8] E.Thandapani and N.Kavitha, Oscillation theorems for second order nonlinear neutral difference equations of mixed type, *J.Math.Comput.Sci.*, 1(2011), 89–102.
- [9] E.Thandapani, N.Kavitha and S.Pinelas, Comparison and oscillation theorem for second order nonlinear neutral difference equations of mixed type, *Dynam. Sys. Appl.*, 21(2012), 83–92.
- [10] E.Thandapani, N.Kavitha and S.Pinelas, Oscillation criteria for second order nonlinear neutral difference equations of mixed type, *Advances in Differ.Equ.*, 4(2012), 1–15.
- [11] E.Thandapani and V.Balasubramanian, Oscillation and asymptotic behavior of second order nonlinear neutral difference equations of mixed arguments, *Transylv.Math.Mech.*, 5(2013), 149–156.
- [12] E.Thandapani, K.Thangavelu and E.Chandrasekaran, Oscillatory behavior of second order neutral difference equations with positive and negative coefficients, *Elec.J.Differ.Eqns.*, 145(2009), 1–8.
- [13] S.Selvarangam, M.Madhan and E.Thandapani, Oscillation theorems for second order nonlinear neutral type difference equations with positive and negative coefficients, *Romanian J.Math.Comput.Sci.*, 7(2017), 1–10.