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Numerical Solution of System of Fractional Differential Equations Using Polynomial Spline Functions

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Abstract

In this paper, we investigate the use of suitable spline functions of polynomial form to approximate the solution of system of fractional differential equations. The description of the proposed approximation method is first introduced. The error analysis and stability of the method are theoretically investigated. Numerical example is given to illustrate the applicability, accuracy and stability of the proposed method.

Keywords: Fractional differential equation; Spline functions; Taylor expansion; stability.

1. INTRODUCTION

In the last few years there has been increasing interest in the use of various types of spline function in the numerical treatment of ordinary differential equations [10,11,12,18] and delay differential equations [3,15,16,17]. Recently, fractional order differential equations have found interesting applications in the area of mathematical biology [1] and [2] due to the relation of such equations with memory that is inherit in corresponding biological systems. Also, a mathematical model of numerous

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engineering and physical phenomena involves either ordinary or partial differential equations of fractional order. The analysis of fractional differential equation (existence, uniqueness, and stability) of the form

$$y^{\alpha} = f(x, y(x)), \quad y(0) = y_0 \tag{1}$$

is studied by Kia Dithelm and Neville J. Ford [4]. A number of approximate solutions of the initial value problem (1) have been proposed in the literature, where the Adams-Bashforth-Moulton method is introduced in [6, 7]. An alternative is the backward differentiation formula, presented in [5] where the idea of this method is based on discretizing the differential operator in the fractional differential equation (1) by certain finite difference. The main result of [5] was that under suitable assumption we can expect an convergence behavior. In [15] an improvement of the performance of the method presented in [5] is achieved by applying extrapolation principles. Kia Dithelm et. al. in [8,9] are considered A fast algorithm for the numerical solution of initial value problems of the form (1) in the sense of caputo identify and discuss potential problems in the development of generally applicable schemes. More recently, Lagrange multiplier method and the homotopy perturbation method are used to solve numerically multi-order fractional differential equation see [14].

2. DESCRIPTION OF THE PROPOSED SPLINE APPROXIMATION METHOD

Consider the fractional ordinary differential equation of the form

$$\begin{aligned} y^{\alpha}(x) &= f_1(x, y(x), z(x)), \qquad a \le x \le b \\ z^{\alpha}(x) &= f_2(x, y(x), z(x)), \\ y(a) &= y_0, \quad z(a) = z_0, \quad \alpha \in (0,1). \end{aligned} \tag{2}$$

where $f_1,\ f_2$ are known functions, y and z are unknowns functions need to be found for $x > a$.
Let $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ be continuous and satisfies the Lipsechitz condition

$$|f_1^{(\alpha)}(x, y_1, z_1) - f_1^{(\alpha)}(x, y_2, z_2)| \le L_1\{|y_1 - y_2| + |z_1 - z_2|\} \tag{3}$$

$$|f_2^{(\alpha)}(x, y_1, z_1) - f_2^{(\alpha)}(x, y_2, z_2)| \le L_2\{|y_1 - y_2| + |z_1 - z_2|\} \tag{4}$$

with Lipsechitz constants L_1 and L_2 for all $(x, y_1, z_1),\ (x, y_2, z_2) \in [a, b] \times \mathbb{R} \times \mathbb{R}$.
These conditions assure the existence of unique solution of equation (2).
Let Δ be a uniform partition to the interval $[a, b]$ defined by the nodes

$$\Delta : a = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = b,\ x_k = x_0 + kh,\ h = \frac{b-a}{n} < 1$$

and $k = 0, 1, \ldots, n-1$.
Define the new form of system of fractional spline function $S(x)$ and $\tilde{S}(x)$ of polynomial form approximating the exact solution y and z by:

$$S_k(x) = S_{k-1}(x_k) + \sum_{i=0}^{r} M_k^{(i\alpha)} \frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)}, \tag{5}$$

$$\widetilde{S}_k(x) = \widetilde{S}_{k-1}(x_k) + \sum_{i=0}^{r} \widetilde{M}_k^{(i\alpha)} \frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)}, \tag{6}$$

where $M_k^{(\alpha)} = f_1^{(\alpha)}(x_k, S_{k-1}(x_k), \widetilde{S}_{k-1}(x_k)),\ \widetilde{M_k^{(i\alpha)}} = f_2^{(\alpha)}(x_k, S_{k-1}(x_k), \widetilde{S}_{k-1}(x_k))$ with $S_{-1}(x_0) = y_0,\ \widetilde{S}_{-1}(x_0) = z_0$, for $x \in [x_k, x_{k+1}],\ k = 0, 1, \ldots, n-1$.
Such $S_k(x)$ and $\widetilde{S}_k(x)$ exist and are unique.

3. ERROR ESTIMATION AND CONVERGENCE ANALYSIS

To estimate the error of the approximate solution, we write the exact solution $y(x)$ and $z(x)$ in the following Taylor form [19]:

$$y(x) = \sum_{i=0}^{r} y_k^{(i\alpha)} \frac{(x-x_k)^{(i\alpha)}}{\Gamma(i\alpha+1)} + y^{((r+1)\alpha)}(\zeta_k) \frac{(x-x_k)^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)} \tag{7}$$

$$z(x) = \sum_{i=0}^{r} z_k^{(i\alpha)} \frac{(x-x_k)^{(i\alpha)}}{\Gamma(i\alpha+1)} + z^{((r+1)\alpha)}(\zeta_k) \frac{(x-x_k)^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)}, \tag{8}$$

where $\zeta_k \in (x_k, x_{k+1}),\ y_k = y(x_k)$ and $z_k = z(x_k)$.
Moreover, we denote to the estimated error of $y(x)$ and $z(x_k)$ at any point $x \in [a, b]$ by:

$$e_\alpha(x) = |y(x) - S_k(x)|,\ \tilde{e}_\alpha(x) = |\tilde{z}(x) - \tilde{S}_k(x)|$$

and at x_k denote to the error

$$e_{k,\alpha}(x) = |y_k - S_{k-1,\alpha}(x_k)|,\ \tilde{e}_{k,\alpha}(x) = |z_k - \tilde{S}_{k-1,\alpha}(x_k)| \tag{9}$$

Define the modulus of continuity of $\omega(y^{((r+1)\alpha)}(\zeta_k, h))$ and $\omega(z^{((r+1)\alpha)}(\zeta_k, h))$ as follows:

$$\omega(y^{((r+1)\alpha)}(\zeta_k, h)) = \max_{\zeta_k \in [a,b]} (|y^{((r+1)\alpha)}(\zeta_k + h) - y^{((r+1)\alpha)}(\zeta_k)|)$$

and

$$\omega(z^{((r+1)\alpha)}(\zeta_k, h)) = \max_{\zeta_k \in [a,b]} (|z^{((r+1)\alpha)}(\zeta_k + h) - z^{((r+1)\alpha)}(\zeta_k)|)$$

Next lemma gives an upper bound to the error.

Lemma 1 *Let $e_\alpha(x)$ and $\tilde{e}_\alpha(x)$ are defined as in (9) then there exist constant d_1 and d_2 independent of h such that the following inequality:*

$$\begin{aligned} e_\alpha(x) &\leq (1+d_1h)e_k + d_1 h\tilde{e}_k + \omega(h)\frac{h^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)} \\ \tilde{e}_\alpha(x) &\leq (1+d_2h)\tilde{e}_k + d_2 h e_k + \omega(h)\frac{h^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)}, \end{aligned}$$

holds for all $x \in [a,b]$ where, $d_1 = \sum_{i=0}^{r}\frac{L_1}{\Gamma((i+1)\alpha+1)}$ and $d_2 = \sum_{i=0}^{r}\frac{L_2}{\Gamma((i+1)\alpha+1)}$.

Proof.
Using the Lipschitz condition, Taylor expansion, definition of error estimation and (9) we get, by dropping α:

$$\begin{aligned} e(x) &= |y(x) - S_k(x)| \\ &= \Big|(y_k - S_{k-1}(x_k)) + \sum_{i=1}^{r} y_k^{(i\alpha)}\frac{(x-x_k)^{i\alpha}}{\Gamma(i\alpha+1)} \\ &- \sum_{i=0}^{r} M_k^{(i\alpha)}\frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} + y^{(r+1)\alpha}(\zeta_k)\frac{(x-x_k)^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)}\Big| \\ &= \Big|(y_k - S_{k-1}(x_k)) + \sum_{i=0}^{r-1} y_k^{((i+1)\alpha)}\frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} - \sum_{i=0}^{r-1} M_k^{(i\alpha)}\frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \\ &+ y^{((r+1)\alpha)}(\zeta_k)\frac{(x-x_k)^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)} - M^{r\alpha}\frac{(x-x_k)^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)}\Big| \\ &\leq e_k + \sum_{i=0}^{r-1}\Big|y_k^{((i+1)\alpha)} - M_k^{i\alpha}\Big|\frac{\Big|(x-x_k)^{(i+1)\alpha}\Big|}{\Gamma((i+1)\alpha+1)} \\ &+ \Big|y^{(r+1)\alpha}(\zeta_k) - M^{r\alpha}\Big|\frac{\Big|(x-x_k)^{(r+1)\alpha}\Big|}{\Gamma((r+1)\alpha+1)} \end{aligned} \tag{10}$$

where

$$\begin{aligned} \Big|y_k^{((i+1)\alpha)} - M_k^{(i\alpha)}\Big| &= \Big|f^{(i\alpha)}(x_k, y_k, z_k) - f^{(i\alpha)}(x_k, S_{k-1}, \tilde{S}_{k-1}(x_k))\Big| \\ &\leq L_1\Big\{|y_k - S_{k-1}(x_k)| + |z_k - \tilde{S}_{k-1}(x_k)|\Big\} \leq L_1(e_k + \tilde{e}_k) \end{aligned}$$

Similarly,

$$\begin{aligned} |y^{((r+1)\alpha)}(\zeta_k) - M_k^{(r\alpha)}| &\leq |y^{((r+1)\alpha)}(\zeta_k) - y_k^{((r+1)\alpha)}| \\ &+ |y_k^{((r+1)\alpha)} - M_k^{(r\alpha)}| \leq \omega(h) + L_1(e_k + \tilde{e}_k) \end{aligned}$$

where the constant $L_1 > 0$ is the Lipsechitz constant independent of h, $\omega(h)$ is the modulus of continuity of $\omega(y^{((r+1)\alpha)}(\zeta_k, h))$ and $|x - x_k| < |h| < 1$. The inequality (12) is then reduced to

$$\begin{aligned} e_\alpha(x) &\leq e_k + \sum_{i=0}^{r-1} L_1(e_k + \tilde{e}_k)\frac{|(x-x_k)^{(i+1)\alpha}|}{\Gamma((i+1)\alpha+1)} + (\omega(h) + L_1(e_k+\tilde{e}_k))\frac{|(x-x_k)^{(r+1)\alpha}|}{\Gamma((r+1)\alpha+1)} \\ &\leq e_k + \sum_{i=0}^{r-1} \frac{L_1(e_k+\tilde{e}_k)h^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} + (\omega(h) + L_1(e_k+\tilde{e}_k))\frac{h^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)} \\ &= e_k + \sum_{i=0}^{r} \frac{L_1 e_k h^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} + \sum_{i=0}^{r} \frac{L_1 \tilde{e}_k h^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} + \omega(h)\frac{h^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)} \\ &\leq e_k + \sum_{i=0}^{r} \frac{L_1 e_k h}{\Gamma((i+1)\alpha+1)} + \sum_{i=0}^{r} \frac{L_1 \tilde{e}_k h}{\Gamma((i+1)\alpha+1)} + \omega(h)\frac{h^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)} \\ &= e_k + d_1 e_k h + d_1 h \tilde{e}_k + \omega(h)\frac{h^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)} \\ &= (1 + d_1 h)e_k + d_1 h \tilde{e}_k + \omega(h)\frac{h^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)}; \end{aligned}$$

where $d_1 = \sum_{i=0}^{r} \frac{L_1}{\Gamma((i+1)\alpha+1)}$ is constant independent of h.

In the same manner we can prove that

$$\tilde{e}_\alpha(x) \leq (1 + d_2 h)\tilde{e}_k + d_2 h e_k + \omega(h)\frac{h^{(r+1)\alpha}}{\Gamma((r+1)\alpha+1)}$$

where $d_2 = \sum_{i=0}^{r} \frac{L_2}{\Gamma((i+1)\alpha+1)}$ is constant independent of h. The lemma is proved.

4. STABILITY ANALYSIS OF THE PROPOSED METHOD

For analyzing the stability properties of the given method, we make a small change of the starting values and study the changes in the numerical solution produced by the method. Now, we define the spline approximating function $W(x)$ and $\tilde{W}(x)$ as:

$$W_k(x) = W_{k-1}(x_k) + \sum_{i=0}^{r} N_k^{(i\alpha)} \frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \tag{11}$$

$$\tilde{W}_k(x) = \tilde{W}_{k-1}(x_k) + \sum_{i=0}^{r} \tilde{N}_k^{(i\alpha)} \frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \tag{12}$$

Where $N_k^{(\alpha)} = f_1^{(\alpha)}(x_k, W_{k-1}(x_k), \tilde{W}_{k-1}(x_k))$
and $\tilde{N}_k^{(\alpha)} = f_2^{(\alpha)}(x_k, W_{k-1}(x_k), \tilde{W}_{k-1}(x_k))$
with $W_{-1}(x_0) = y_0^*,\ \tilde{W}{-1}(x_0) = \tilde{y}_0^*$ for $x \in [x_k, x_{k+1}]$,$k = 0, 1, ..., n-1$.

and use the notation

$$e_\alpha^*(x) = |S_k(x) - W_k^{(\alpha)}(x)| \quad \text{and} \quad e_{\alpha,k}^* = |S_{k-1,\alpha} - W_{k-1,\alpha}(x_k)| \tag{13}$$

$$\tilde{e}_\alpha^*(x) = |\tilde{S}_k(x) - \tilde{W}_k^{(\alpha)}(x)| \quad \text{and} \quad \tilde{e}_{\alpha,k}^* = |\tilde{S}_{k-1,\alpha} - \tilde{W}_{k-1,\alpha}(x_k)| \tag{14}$$

Lemma 2 *Let $e_\alpha^*(x)$ and $\tilde{e}_\alpha^*(x)$ be defined as in (13), then the inequality*

$$e_\alpha^*(x) \le (1 + d_1 h)e_k^* + dh\tilde{e}_k^*, \quad \tilde{e}_\alpha^*(x) \le (1 + d_2 h)\tilde{e}_k^* + dhe_k^*$$

holds where $d_1 = \sum_{i=0}^{r} \frac{L_1}{\Gamma((i+1)\alpha+1)}$ and $d_2 = \sum_{i=0}^{r} \frac{L_2}{\Gamma((i+1)\alpha+1)}$ are constants independent of h.

Proof.
Using Lipsechitz condition and (5), (11), (13) and (14) we get, by dropping α:

$$\begin{aligned} e^*(x) &= |S_k(x) - W_k(x)| \\ &= \Big|(S_{k-1}(x_k) - W_{k-1}(x_k)) + \sum_{i=0}^{r} M_k^{(i\alpha)} \frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \\ &- \sum_{i=0}^{r} \tilde{M}_k^{(i\alpha)} \frac{(x-x_k)^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)}\Big| \\ &\le e_k^* + \sum_{i=0}^{r} \Big|M_k^{(i\alpha)} - \tilde{M}_k^{(i\alpha)}\Big| \frac{|(x-x_k)^{(i+1)\alpha}|}{\Gamma((i+1)\alpha+1)} \\ &\le e_k^* + \sum_{i=0}^{r} \Big|M_k^{(i\alpha)} - \tilde{M}_k^{(i\alpha)}\Big| \frac{h^{(i+1)\alpha}}{\Gamma((i+1)\alpha+1)} \end{aligned} \tag{15}$$

but

$$|M_k^{(i\alpha)} - N_k^{(i\alpha)}| \le L_1(e_k^* + \tilde{e}_k^*). \tag{16}$$

Thus from (15) and (16) we obtain:

$$e^*(x) \le e_k^* + d_1 e_k^* h + d_1 \tilde{e}_k^* h \le (1+dh)e_k^* + d_1 \tilde{e}_k^* h,$$

where, $d_1 = \sum_{i=0}^{r} \frac{L_1}{\Gamma((i+1)\alpha+1)}$ is constant independent of h.
In the same manner we can prove that

$$\tilde{e}^*(x) \le \tilde{e}_k^* + d_2 \tilde{e}_k^* h + d_2 e_k^* h \le (1+dh)\tilde{e}_k^* + d_2 e_k^* h,$$

where $d_2 = \sum_{i=0}^{r} \frac{L_2}{\Gamma((i+1)\alpha+1)}$ is constant independent of h. And the lemma is proved.

5. NUMERICAL EXAMPLE

To demonstrate the applicability, accuracy and stability of the proposed method, numerical example is considered where the closed form solution is available for the example.

5.1. Example

Consider the system of fractional differential equation

$$D^\alpha y(x) = -y(x) + z(x) + \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}, \quad D^\alpha z(x) = y(x) - z(x) + \frac{2}{\Gamma(3-\alpha)} x^{2-\alpha}$$

The exact solution is $y = x^2$ and $z = x^2$.
The obtained numerical results are summarized in Tables 1 and 2 to illustrate the accuracy and the stability of the proposed spline method using spline function of polynomial form. The first column represents different values of α, the second column represents the values of x, the third column gives the approximate solutions at the corresponding points while the fourth column gives the absolute error between the exact solution and the obtained approximate numerical solution with the initial conditions $y(0) = 0$ and $z(0) = 0$. With small change in the initial conditions, $y^* = .00001$ and $z^* = .00001$, the approximate solution is computed as shown in the fifth column. To test the stability, the difference between the two approximate solutions is computed as shown in the six column.

Table 1. The accuracy and stability of the proposed spline method using spline function of polynomial form(using $h = 0.01$).

α	x	Appr. Solution for the problem	Absolute Error	Appr. solution for the perturbed problem	Absolute diff. between the two Appr. Solutions
0.1	0.01	$y = 0.000940111$	8.4×10^{-5}	0.000940127	1.53289×10^{-8}
	0.02	$y = 0.00292199$	2.5×10^{-3}	0.00292201	2.39914×10^{-8}
	0.03	$y = 0.00569601$	4.8×10^{-3}	0.00569604	3.13401×10^{-8}
	0.04	$y = 0.00916692$	7.6×10^{-3}	0.00916696	3.79869×10^{-8}
	0.05	$y = 0.0132783$	1.1×10^{-3}	0.0132783	4.41767×10^{-8}
0.2	0.01	$y = 0.000548301$	4.5×10^{-5}	0.000548311	1.00000×10^{-8}
	0.02	$y = 0.00182564$	1.4×10^{-3}	0.00182566	1.58953×10^{-8}
	0.03	$y = 0.00370481$	2.9×10^{-3}	0.00370483	2.16077×10^{-8}
	0.04	$y = 0.00613465$	4.5×10^{-3}	0.00613467	2.69391×10^{-8}
	0.05	$y = 0.00908433$	6.6×10^{-3}	0.00908436	3.20198×10^{-8}
0.3	0.01	$y = 0.000318381$	2.2×10^{-4}	0.000318387	5.82450×10^{-9}
	0.02	$y = 0.00113567$	7.4×10^{-4}	0.00113568	1.04516×10^{-8}
	0.03	$y = 0.00239922$	1.5×10^{-3}	0.00239923	1.47858×10^{-8}
	0.04	$y = 0.00408762$	2.5×10^{-3}	0.00408764	1.89618×10^{-8}
	0.05	$y = 0.00618821$	3.7×10^{-3}	0.00618823	2.30361×10^{-8}
0.4	0.01	$y = 0.000184096$	8.4×10^{-4}	0.000184099	3.55100×10^{-9}
	0.02	$y = 0.000703506$	3.1×10^{-4}	0.000703513	6.82362×10^{-9}
	0.03	$y = 0.00154724$	6.5×10^{-3}	0.00154725	1.00467×10^{-8}
	0.04	$y = 0.00271233$	1.1×10^{-3}	0.00271235	1.32538×10^{-8}
	0.05	$y = 0.00419792$	1.7×10^{-3}	0.00419794	1.55516×10^{-8}
0.5	0.01	$y = 0.000106018$	6.0×10^{-6}	0.00010602	2.15046×10^{-9}
	0.02	$y = 0.000434043$	3.4×10^{-5}	0.000434048	4.42555×10^{-9}
	0.03	$y = 0.000993812$	9.4×10^{-5}	0.000993818	6.78178×10^{-9}
	0.04	$y = 0.00179258$	1.9×10^{-4}	0.00179259	9.20362×10^{-9}
	0.05	$y = 0.00283642$	3.4×10^{-4}	0.00283643	1.16821×10^{-8}

Table 2. The accuracy and stability of the proposed spline method using spline function of integral form (using h=0.01

α	x	Appr. Solution for the problem	Absolute Error	Appr. solution for the perturbed problem	Absolute diff. between the two Appr. Solutions
0.1	0.01	$z = 0.000940111$	8.4×10^{-5}	0.000940127	1.53289×10^{-8}
	0.02	$z = 0.00292199$	2.5×10^{-3}	0.00292201	2.39914×10^{-8}
	0.03	$z = 0.00569601$	4.8×10^{-3}	0.00569604	3.13401×10^{-8}
	0.04	$z = 0.00916692$	7.6×10^{-3}	0.00916696	3.79869×10^{-8}
	0.05	$z = 0.0132783$	1.1×10^{-3}	0.0132783	4.41767×10^{-8}
0.2	0.01	$z = 0.000548301$	4.5×10^{-5}	0.000548311	1.00000×10^{-8}
	0.02	$z = 0.00182564$	1.4×10^{-3}	0.00182566	1.58953×10^{-8}
	0.03	$z = 0.00370481$	2.9×10^{-3}	0.00370483	2.16077×10^{-8}
	0.04	$z = 0.00613465$	4.5×10^{-3}	0.00613467	2.69391×10^{-8}
	0.05	$z = 0.00908433$	6.6×10^{-3}	0.00908436	3.20198×10^{-8}
0.3	0.01	$z = 0.000318381$	2.2×10^{-4}	0.000318387	5.82450×10^{-9}
	0.02	$z = 0.00113567$	7.4×10^{-4}	0.00113568	1.04516×10^{-8}
	0.03	$z = 0.00239922$	1.5×10^{-3}	0.00239923	1.47858×10^{-8}
	0.04	$z = 0.00408762$	2.5×10^{-3}	0.00408764	1.89618×10^{-8}
	0.05	$z = 0.00618821$	3.7×10^{-3}	0.00618823	2.30361×10^{-8}
0.4	0.01	$z = 0.000184096$	8.4×10^{-4}	0.000184099	3.55100×10^{-9}
	0.02	$z = 0.000703506$	3.1×10^{-4}	0.000703513	6.82362×10^{-9}
	0.03	$z = 0.00154724$	6.5×10^{-3}	0.00154725	1.00467×10^{-8}
	0.04	$z = 0.00271233$	1.1×10^{-3}	0.00271235	1.32538×10^{-8}
	0.05	$z = 0.00419792$	1.7×10^{-3}	0.00419794	1.55516×10^{-8}
0.5	0.01	$z = 0.000106018$	6.0×10^{-6}	0.00010602	2.15046×10^{-9}
	0.02	$z = 0.000434043$	3.4×10^{-5}	0.000434048	4.42555×10^{-9}
	0.03	$z = 0.000993812$	9.4×10^{-5}	0.000993818	6.78178×10^{-9}
	0.04	$z = 0.00179258$	1.9×10^{-4}	0.00179259	9.20362×10^{-9}
	0.05	$z = 0.00283642$	3.4×10^{-4}	0.00283643	1.16821×10^{-8}

From the obtained results in Tables 1 and2, we can see that the proposed method using the spline function of polynomial form gives acceptable accuracy and the method is shown to be very efficient where its algorithm has recursive nature which makes it easy and simple to be programmed. Also, from column six, one can observe that the method is stable. We used Mathematica program for numerical results.

6. CONCLUSION

In this paper, we investigated the possibility of extending and generalizing the spline functions of integral form given in Ramadan M. A. et al. [15] in fractional form with some additional assumptions and definitions for approximating the solution of single and system fractional ordinary differential equations. The error analysis and stability are theoretically investigated. A numerical example is given to illustrate the applicability, accuracy and stability of the proposed method. The obtained numerical results reveal that the method is stable and gives high accuracy.

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