

## Small Functions and Uniqueness of Difference Differential Polynomials of L-functions

Nintu Mandal<sup>1\*</sup>, and Nirmal Kumar Datta

*Department of Mathematics, Chandernagore College, Chandernagore,  
Hooghly-712136, West Bengal, India.*

*Department of Physics, Suri Vidyasagar College, Suri, Birbhum-731101,  
West Bengal, India.*

### Abstract

In this paper, we study the value distributions of L-functions in the extended Selberg class. We prove two theorems which shows how difference differential polynomials of L-functions and difference differential polynomials of meromorphic functions uniquely determined concerning weighted sharing of small or rational functions. Our results improve and generalize some recent results due to W. J. Hao, J. F. Chen [3], W. Q. Zhu, J. F. Chen [16] and N. Mandal, N. K. Datta [10].

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### 1. INTRODUCTION

L-functions are the most important objects in the modern number theory. The Riemann hypothesis and its extension to the general classes of L-functions is the most important unsolved problem in pure mathematics.

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\*Corresponding Author

An L-function  $L$  in the Selberg class is defined by the Dirichlet series  $L(z) = \sum_{n=1}^{\infty} a(n)/n^z$  which satisfies the following

- (1)  $a(n) \ll n^\epsilon$ , for every  $\epsilon > 0$ ;
- (2) There exists a nonnegative integer  $k$  such that  $(z - 1)^k L(z)$  becomes an entire function of finite order;
- (3) Every L-function satisfies the functional equation

$$\lambda_L(z) = \omega \overline{\lambda_L(1 - \bar{z})},$$

where

$$\lambda_L(z) = L(z) Q^z \prod_{i=1}^k \Gamma(\eta_i z + \nu_i)$$

with positive real numbers  $Q$ ,  $\eta_i$  and complex numbers  $\nu_i$ ,  $\omega$  with  $Re \nu_i \geq 0$  and  $|\omega| = 1$ .

- (4)  $L(z)$  satisfies  $L(z) = \prod_p L_p(z)$ , where  $L_p(z) = \exp(\sum_{k=1}^{\infty} b(p^k)/p^{kz})$  with coefficients  $b(p^k)$  satisfying  $b(p^k) \ll p^{k\theta}$  for some  $\theta < 1/2$  and  $p$  denotes prime number.

Clearly the Riemann zeta function is an L-function in the Selberg class. If  $L$  satisfies (1) - (3) then we say that  $L$  is an L-function in the extended Selberg class. In this paper by an L-function we mean an L-function in the extended Selberg class with  $a(1) = 1$ .

In this paper, we study uniqueness problems with the help of Nevanlinna value distribution theory using the standard notations and definitions of the value distribution theory [4].

## 2. PRELIMINARIES

Let  $\xi$  be a nonconstant meromorphic function. We denote by  $S(r, \xi)$  any function satisfying  $S(r, \xi) = o(T(r, \xi))$  as  $r \rightarrow \infty$ , outside a possible exceptional set of finite linear measure. A meromorphic function  $\rho$  is said to be a small function of  $\xi$  if  $T(r, \rho) = S(r, \xi)$ . The hyper order  $\rho_2(\xi)$  of  $\xi$  is defined by  $\rho_2(\xi) = \limsup_{r \rightarrow \infty} \frac{\log \log T(r, \xi)}{\log r}$ . Let  $\alpha \in \mathbf{C} \cup \{\infty\}$  and  $n$  be a positive integer. We denote by  $E_n(\alpha; \xi) (\overline{E}_n(\alpha; \xi))$  the set of all zeros of  $\xi - \alpha$  with multiplicities not exceeding  $n$ , where zeros are counted according to their multiplicities (ignoring multiplicities).

In 2001 Lahiri [5, 6] introduced weighted sharing as follows

**Definition 2.1.** [5, 6]. Let  $\xi$  and  $\psi$  be meromorphic functions defined in the complex plane and  $n$  be an integer ( $\geq 0$ ) or infinity. For  $\alpha \in \mathbf{C} \cup \{\infty\}$  we denote by  $E_n(\alpha; \xi)$

the set of all zeros of  $\xi - \alpha$  where a zero of multiplicity  $k$  is counted  $k$  times if  $k \leq n$  and  $n + 1$  times if  $k > n$ . If  $E_n(\alpha; \xi) = E_n(\alpha; \psi)$ , we say that  $\xi, \psi$  share the value  $\alpha$  with weight  $n$ .

We write  $\xi, \psi$  share  $(\alpha, n)$  to mean that  $\xi, \psi$  share the value  $\alpha$  with weight  $n$ .

In 2017, Liu, Li and Yi [9] proved the following uniqueness theorems of L-functions.

**Theorem 2.1.** [9]. Let  $j \geq 1$  and  $k \geq 1$  be integers such that  $j > 3k + 6$ . Also let  $L$  be an L-function and  $\xi$  be a nonconstant meromorphic function. If  $\{\xi^j\}^{(k)}$  and  $\{L^j\}^{(k)}$  share  $(1, \infty)$  then  $\xi \equiv \alpha L$  for some nonconstant  $\alpha$  satisfying  $\alpha^j = 1$ .

**Theorem 2.2.** [9]. Let  $j \geq 1$  and  $k \geq 1$  be integers such that  $j > 3k + 6$ . Also let  $L$  be an L-function and  $\xi$  be a nonconstant meromorphic function. If  $\{\xi^j\}^{(k)}(z) - (z)$  and  $\{L^j\}^{(k)}(z) - (z)$  share  $(0, \infty)$  then  $\xi \equiv \alpha L$  for some nonconstant  $\alpha$  satisfying  $\alpha^j = 1$ .

Considering differential polynomials in 2018, W. J. Hao and J. F. Chen [3] obtain the following uniqueness results on L-function:

**Theorem 2.3.** [3]. Let  $\xi$  be a nonconstant meromorphic function and  $L$  be an L-function such that  $[\xi^n(\xi - 1)^m]^{(\tau)}$  and  $[L^n(L - 1)^m]^{(\tau)}$  share  $(1, \infty)$ , where  $n, m, \tau \in \mathbb{Z}^+$ . If  $n > m + 3\tau + 6$  and  $\tau \geq 2$ , then,  $\xi \equiv L$  or,  $\xi^n(\xi - 1)^m \equiv L^n(L - 1)^m$ .

**Theorem 2.4.** [3]. Let  $\xi$  be a nonconstant meromorphic function and  $L$  be an L-function such that  $[\xi^n(\xi - 1)^m]^{(\tau)}$  and  $[L^n(L - 1)^m]^{(\tau)}$  share  $(1, 0)$ , where  $n, m, \tau \in \mathbb{Z}^+$ . If  $n > 4m + 7\tau + 11$  and  $\tau \geq 2$ , then,  $\xi \equiv L$  or,  $\xi^n(\xi - 1)^m \equiv L^n(L - 1)^m$ .

Using truncated sharing in 2019 W. Q. Zhu and J. F. Chen proved the following uniqueness theorem.

**Theorem 2.5.** [16]. Let  $L$  be an L-function and  $\xi$  be a transcendental meromorphic function defined in the complex plane  $\mathbf{C}$ . Also let  $n, k(\geq 2), l(\geq 2)$  be positive integers such that  $n \geq 7k + 17$ . If  $\overline{E}_l(1, (\xi^n(\xi - 1))^k) = \overline{E}_l(1, (L^n(L - 1))^k)$  then  $\xi \equiv L$ .

**Definition 2.2.** [8]. Let  $\xi$  be a meromorphic function defined in the complex plane. Let  $n$  be a positive integer and  $\alpha \in \mathbf{C} \cup \{\infty\}$ . By  $N(r, \alpha; \xi | \leq n)$  we denote the counting function of the  $\alpha$  points of  $\xi$  with multiplicity  $\leq n$  and by  $\overline{N}(r, \alpha; \xi | \leq n)$  the reduced counting function. Also by  $N(r, \alpha; \xi | \geq n)$  we denote the counting function of the  $\alpha$  points of  $\xi$  with multiplicity  $\geq n$  and by  $\overline{N}(r, \alpha; \xi | \geq n)$  the reduced counting function. We define

$$N_n(r, \alpha; \xi) = \overline{N}(r, \alpha; \xi) + \overline{N}(r, \alpha; \xi | \geq 2) + \dots + \overline{N}(r, \alpha; \xi | \geq n).$$

**Definition 2.3.** [10]. Let  $\xi, \chi$  be meromorphic functions defined in the complex plane and  $\psi$  be a rational function or a small function of  $\xi$  and  $\chi$ . Then we denote by  $E_m(\psi; \xi)$ ,  $\overline{E}_m(\psi; \xi)$  and  $E_m(\psi; \xi)$  the sets  $E_m(0; \xi - \psi)$ ,  $\overline{E}_m(0; \xi - \psi)$  and  $E_m(0; \xi - \psi)$  respectively.

We write  $\xi, \chi$  share  $(\psi, n)$  to mean that  $\xi - \psi, \chi - \psi$  share the value 0 with weight  $n$ . Clearly if  $\xi, \chi$  share  $(\psi, n)$  then  $\xi, \chi$  share  $(\psi, m)$  for all integers  $m, 0 \leq m < n$ . Also we note that  $\xi, \chi$  share  $\psi$  IM or CM if and only if  $\xi, \chi$  share  $(\psi, 0)$  or  $(\psi, \infty)$  respectively.

Considering truncated sharing of small functions in 2020 Mandal and Datta [10] proved the following theorem.

**Theorem 2.6.** [10]. Let  $L$  be a nonconstant  $L$ -function and  $\rho$  be a small function of  $L$  such that  $\rho \not\equiv 0, \infty$ . If  $\overline{E}_4(\rho; L) = \overline{E}_4(\rho; (L^m)^{(k)})$ ,  $E_2(\rho; L) = E_2(\rho; (L^m)^{(k)})$  and

$$2N_{2+k}(r, 0; L^m) \leq (\sigma + o(1))T(r, L),$$

where  $m \geq 1, k \geq 1$  are integers and  $0 < \sigma < 1$ , then  $L \equiv (L^m)^{(k)}$ .

Now the following questions comes naturally.

**Question 2.1.** If we take meromorphic function in place of transcendental meromorphic function in theorem 2.5 then what happens?

**Question 2.2.** Can we take difference differential polynomials in place of differential polynomials in theorem 2.3, 2.4, 2.5 and 2.6?

**Definition 2.4.** [8]. Let  $\xi$  and  $\psi$  be two meromorphic functions defined in the complex plane. Then we denote by  $N(r, \psi; \xi | \leq m)$ ,  $\overline{N}(r, \psi; \xi | \leq m)$ ,  $N(r, \psi; \xi | \geq m)$ ,  $\overline{N}(r, \psi; \xi | \geq m)$ ,  $N_m(r, \psi; \xi)$  etc. the counting functions  $N(r, 0; \xi - \psi | \leq m)$ ,  $\overline{N}(r, 0; \xi - \psi | \leq m)$ ,  $N(r, 0; \xi - \psi | \geq m)$ ,  $\overline{N}(r, 0; \xi - \psi | \geq m)$ ,  $N_m(r, 0; \xi - \psi)$  etc. respectively.

**Definition 2.5.** [5]. Let two nonconstant meromorphic functions  $\xi$  and  $\psi$  share a value  $(\alpha, 0)$ . We denote by  $\overline{N}_*(r, \alpha; \xi, \psi)$  the counting function of the  $\alpha$ -points of  $\xi$  and  $\psi$  with different multiplicities, where each  $\alpha$ -point is counted only once.

Clearly  $\overline{N}_*(r, \alpha; \xi, \psi) \equiv \overline{N}_*(r, \alpha; \psi, \xi)$ .

**Definition 2.6.** Let two nonconstant meromorphic functions  $\xi$  and  $\psi$  share a value  $\alpha$  IM. We denote by  $\overline{N}(r, \alpha; \xi | > \psi)$  the counting function of the  $\alpha$ -points of  $\xi$  and  $\psi$  with multiplicities with respect to  $\xi$  is greater than the multiplicities with respect to  $\psi$ , where each  $\alpha$ -point is counted once only.

**Definition 2.7.** Let two nonconstant meromorphic functions  $\xi$  and  $\psi$  share a value  $\alpha$  IM. We denote by  $\overline{N}_E(r, \alpha; \xi, \psi | > m)$  the counting function of the  $\alpha$ -points of  $\xi$  and  $\psi$  with multiplicities greater than  $m$  and the multiplicities with respect to  $\xi$  is equal to the multiplicities with respect to  $\psi$ , where each  $\alpha$ -point is counted once only.

**Definition 2.8.** [7]. We denote by  $N_{\otimes}(r, 0; \xi^{(k)})$  ( $\overline{N}_{\otimes}(r, 0; \xi^{(k)})$ ) the counting function (reduced counting function) of those zeros of  $\xi^{(k)}$  which are not the zeros of  $\xi(\xi - 1)$ .

### 3. MAIN RESULTS

Using weighted sharing we try to solve Questions 2.1 and 2.2 and prove the following theorems.

**Theorem 3.1.** Let  $L$  be a nonconstant L-function and  $\xi$  be a meromorphic function. Let  $\tau, n, \eta, \mu_j (j = 1, 2, \dots, \eta), \lambda = \sum_{j=1}^{\eta} \mu_j$  be positive integers such that  $n > \lambda + (\eta + 1)(3\tau + 4)$  and  $\omega_j \in \mathbf{C} - \{0\}$  ( $j = 1, 2, \dots, \eta$ ) be distinct constants. Also let  $\rho_2(L) < 1, \rho_2(\xi) < 1, [L^n(z) \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)}$  and  $[\xi^n(z) \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$  share  $(\rho(z), l)$  and  $\xi, L$  share  $(\infty, 0)$ , where  $2 \leq l < \infty$  and  $\rho(z)$  is a small function of  $\xi$  and  $L$ , then one of the following holds

$$(i) [L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$$

$$(ii) [L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv \rho(z)^2.$$

**Theorem 3.2.** Let  $L$  be a nonconstant L-function and  $\xi$  be a meromorphic function. Let  $\tau, n, \eta, \mu_j (j = 1, 2, \dots, \eta), \lambda = \sum_{j=1}^{\eta} \mu_j$  be positive integers such that  $n > \lambda + (\eta + 1)(3\tau + 4)$  and  $\omega_j \in \mathbf{C} - \{0\}$  ( $j = 1, 2, \dots, \eta$ ) be distinct constants. Also let  $\rho_2(L) < 1, \rho_2(\xi) < 1, [L^n(z) \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)}$  and  $[\xi^n(z) \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$  share  $(Q(z), l)$  and  $\xi, L$  share  $(\infty, 0)$ , where  $2 \leq l < \infty$  and  $Q(z)$  is a rational function, then one of the following holds

$$(i) [L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$$

$$(ii) [L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv Q(z)^2.$$

#### 4. LEMMAS

In this section we present some necessary lemmas.

Henceforth we denote by  $\Omega$  the function defined by

$$\Omega = \left( \frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi - 1} \right) - \left( \frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi - 1} \right)$$

**Lemma 4.1.** [11]. *Let  $L$  be an  $L$ -function with degree  $q$ . Then*

$$T(r, L) = \frac{q}{\pi} r \log r + O(r).$$

**Lemma 4.2.** [10]. *Let  $L$  be an  $L$ -function. Then  $N(r, \infty; L) = S(r, L) = O(\log r)$ .*

**Lemma 4.3.** *Let  $\xi$  be a nonconstant meromorphic function and  $L$  be an  $L$ -function. If  $\xi$  and  $L$  share  $(\infty, 0)$  then  $\overline{N}(r, \infty; \xi) = S(r, L) = O(\log r)$ .*

*Proof.* Since  $\xi$  and  $L$  share  $(\infty, 0)$  therefore by lemma 4.2 we have  $\overline{N}(r, \infty; \xi) = \overline{N}(r, \infty; L) = S(r, L) = O(\log r)$ . This completes the proof.  $\square$

**Lemma 4.4.** [15]. *Let  $\xi(z) = \frac{\alpha_0 + \alpha_1 z + \dots + \alpha_n z^n}{\beta_0 + \beta_1 z + \dots + \beta_m z^m}$  be a nonconstant rational function defined in the complex plane  $\mathbf{C}$ , where  $\alpha_0, \alpha_1, \dots, \alpha_n (\neq 0)$  and  $\beta_0, \beta_1, \dots, \beta_m (\neq 0)$  are complex constants. Then*

$$T(r, \xi) = \max\{m, n\} \log r + O(1).$$

**Lemma 4.5.** [12]. *Let  $\xi$  be a transcendental meromorphic function of hyper order  $\rho_2(\xi) < 1$ . Then for any  $\alpha \in \mathbf{C} - 0$*

$$T(r, \xi(z + \alpha)) = T(r, \xi(z)) + S(r, \xi(z))$$

$$N(r, \infty; \xi(z + \alpha)) = N(r, \infty; \xi(z)) + S(r, \xi(z))$$

$$N(r, 0; \xi(z + \alpha)) = N(r, 0; \xi(z)) + S(r, \xi(z))$$

**Lemma 4.6.** [1]. *Let  $\Phi$  and  $\Psi$  be two nonconstant meromorphic functions sharing  $(1, l)$  and  $(\infty, 0)$  where  $2 \leq l < \infty$  and  $\Omega \not\equiv 0$ . Then*

$$\begin{aligned} T(r, \Phi) &\leq N_2(r, 0; \Phi) + N_2(r, 0; \Psi) + \overline{N}(r, \infty; \Phi) + \overline{N}(r, \infty; \Psi) + \overline{N}_*(r, \infty; \Phi, \Psi) \\ &\quad - m(r, 1; \Psi) - N_E(r, 1; \Phi, \Psi | > 3) - \overline{N}(r, 1; \Psi | > \Phi) + S(r, \Phi) + S(r, \Psi) \end{aligned}$$

$$\begin{aligned} T(r, \Psi) &\leq N_2(r, 0; \Phi) + N_2(r, 0; \Psi) + \overline{N}(r, \infty; \Phi) + \overline{N}(r, \infty; \Psi) + \overline{N}_*(r, \infty; \Phi, \Psi) \\ &\quad - m(r, 1; \Phi) - N_E(r, 1; \Psi, \Phi | > 3) - \overline{N}(r, 1; \Phi | > \Psi) + S(r, \Phi) + S(r, \Psi) \end{aligned}$$

**Lemma 4.7.** [14]. *Let  $\Phi$  be a nonconstant meromorphic function and  $k, p$  be two positive integers. Then*

$$T(r, \Phi^{(k)}) \leq T(r, \Phi) + k\bar{N}(r, \infty; \Phi) + S(r, \Phi)$$

$$N_p(r, 0; \Phi^{(k)}) \leq T(r, \Phi^{(k)}) - T(r, \Phi) + N_{p+k}(r, 0; \Phi) + S(r, \Phi)$$

$$N_p(r, 0; \Phi^{(k)}) \leq N_{p+k}(r, 0; \Phi) + k\bar{N}(r, \infty; \Phi) + S(r, \Phi)$$

$$N(r, 0; \Phi^{(k)}) \leq N(r, 0; \Phi) + k\bar{N}(r, \infty; \Phi) + S(r, \Phi)$$

**Lemma 4.8.** [2]. *Let  $\xi$  be a transcendental meromorphic function of hyper order  $\rho_2(\xi) < 1$  and  $\phi(z) = \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}$ , where  $n, \eta, \mu_j (j = 1, 2, \dots, \eta), \lambda = \sum_{j=1}^{\eta} \mu_j$  are positive integers and  $\omega_j \in \mathbf{C} - \{0\} (j = 1, 2, \dots, \eta)$  be distinct constants. Then*

$$(n - \lambda)T(r, \xi) + S(r, \xi) \leq T(r, \xi^n \phi) \leq (n + \lambda)T(r, \xi) + S(r, \xi)$$

**Lemma 4.9.** [13]. *Let  $\xi$  be a nonconstant meromorphic function and let  $l$  be a positive integer. If  $\rho$  be a small function of  $\xi$  then*

$$T(r, \xi) \leq \bar{N}(r, \infty; \xi) + N(r, 0; \xi) + N(r, \rho, \xi^{(l)}) - N(r, 0, (\frac{\xi^{(l)}}{\rho})') + S(r, \xi).$$

**Lemma 4.10.** *Let  $L$  be a nonconstant L-function and  $\xi$  be a meromorphic function. Let  $\tau, n, \eta, \mu_j (j = 1, 2, \dots, \eta), \lambda = \sum_{j=1}^{\eta} \mu_j$  be positive integers such that  $n > \lambda + (\eta + 1)(\tau + 1)$ . and  $\omega_j \in \mathbf{C} - \{0\} (j = 1, 2, \dots, \eta)$  be distinct constants. Also let  $\rho_2(L) < 1, \rho_2(\xi) < 1, [L^n(z) \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)}$  and  $[\xi^n(z) \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$  share  $(\rho(z), l)$  and  $\xi, L$  share  $(\infty, 0)$ , where  $2 \leq l < \infty$  and  $\rho(z)$  is a small function of  $L$ , then  $L$  and  $\xi$  are transcendental meromorphic functions.*

*Proof.* By lemma 4.1 we have  $T(r, L) = \frac{q}{\pi} r \log r + O(r)$ , where  $L$  be an L-function with degree  $q$ . Hence  $L$  is a transcendental meromorphic function. Let  $\phi(z) = \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}$  and  $\psi(z) = \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}$  Since  $[L^n(z) \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)}$  and  $[\xi^n(z) \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$  share  $(\rho(z), l)$  therefore by lemma 4.2, lemma 4.3, lemma

4.4, lemma 4.5, lemma 4.7 and lemma 4.9 we have

$$\begin{aligned}
T(r, L^n \psi) &\leq \bar{N}(r, \infty; L^n \psi) + N(r, 0; L^n \psi) + N(r, z; (L^n \psi)^{(\tau)}) \\
&\quad - N(r, 0; (\frac{(L^n \psi)^{(\tau)}}{\rho})') + S(r, L) \\
&\leq (\tau + 1)\bar{N}(r, 0; L^n \psi) + \bar{N}(r, 0; \frac{(L^n \psi)^{(\tau)}}{\rho} - 1) \\
&\quad - N_{\otimes}(r, 0; (\frac{(L^n \psi)^{(\tau)}}{\rho})') + S(r, L) \\
&\leq (\tau + 1)[T(r, L) + \eta T(r, L)] + \bar{N}(r, 0; \frac{(\xi^n \phi)^{(\tau)}}{\rho} - 1) + S(r, L) \\
&\leq (\tau + 1)(1 + \eta)T(r, L) + T(r, (\xi^n \phi)^{(\tau)}) + S(r, L) \tag{4.1}
\end{aligned}$$

By lemma 4.8 we have from (4.1)

$$(n - \lambda)T(r, L) \leq (\tau + 1)(1 + \eta)T(r, L) + T(r, (\xi^n \phi)^{(\tau)}) + S(r, L) \tag{4.2}$$

From (4.2) we have

$$(n - \lambda - (\tau + 1)(1 + \eta))T(r, L) \leq T(r, (\xi^n \phi)^{(\tau)}) + S(r, L) \tag{4.3}$$

From (4.3) it is clear that  $\xi$  is a transcendental meromorphic function since  $n > \lambda + (\tau + 1)(1 + \eta)$  and  $L$  is a transcendental meromorphic function. This completes the proof of the lemma.  $\square$

## 5. PROOF OF THE MAIN RESULTS

### Proof of Theorem 3.1

Let  $\phi(z) = \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}$ ,  $\psi(z) = \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}$ ,  $\Phi = \frac{(\xi^n \phi)^{(\tau)}}{\rho}$  and  $\Psi = \frac{(L^n \psi)^{(\tau)}}{\rho}$ . Then  $\Phi, \Psi$  share  $(1, l)$ ,  $2 \leq l < \infty$  and  $\Phi, \Psi$  share  $(\infty, 0)$  except for zeros and poles of  $\rho(z)$ .

By lemma 4.10  $\xi$  and  $L$  are transcendental meromorphic functions.

We have by lemma 4.7 and lemma 4.8

$$\begin{aligned}
N_2(r, 0; \Phi) &\leq N_2(r, 0; (\xi^n \phi)^{(\tau)}) + S(r, \xi) \\
&\leq T(r, (\xi^n \phi)^{(\tau)}) - T(r, \xi^n \phi) + N_{2+\tau}(r, 0; \xi^n \phi) + S(r, \xi) \\
&\leq T(r, \frac{(\xi^n \phi)^{(\tau)}}{\rho}) - (n - \lambda)T(r, \xi) + N_{2+\tau}(r, 0; \xi^n \phi) + S(r, \xi) \tag{5.1}
\end{aligned}$$

Hence from (5.1) we have

$$(n - \lambda)T(r, \xi) \leq T(r, \Phi) - N_2(r, 0; \Phi) + N_{2+\tau}(r, 0; \xi^n \phi) + S(r, \xi) \tag{5.2}$$



Similarly we have

$$(n - \lambda)T(r, L) \leq T(r, \Psi) - N_2(r, 0; \Psi) + N_{2+\tau}(r, 0; L^n\psi) + S(r, L) \tag{5.3}$$

Now we have to consider the following two cases

**Case 1** Let  $\Omega \not\equiv 0$ .

Using lemma 4.2, lemma 4.3 and lemma 4.6 we get from (5.2)

$$\begin{aligned} (n - \lambda)T(r, \xi) &\leq N_2(r, 0; \Psi) + \overline{N}(r, \infty; \Phi) + \overline{N}(r, \infty; \Psi) \\ &\quad + \overline{N}_*(r, \infty; \Phi, \Psi) - m(r, 1; \Psi) - N_E(r, 1; \Phi, \Psi | > 3) \\ &\quad - \overline{N}(r, 1; \Psi | > \Phi) + N_{2+\tau}(r, 0; \xi^n\phi) + S(r, \xi) + S(r, L) \\ &\leq N_2(r, 0; (L^n\psi)^{(\tau)}) + N_{2+\tau}(r, 0; \xi^n\phi) + S(r, \xi) + S(r, L) \\ &\leq (2 + \tau)(1 + \eta)T(r, L) + (\tau + 2)(\eta + 1)T(r, \xi) \\ &\quad + S(r, \xi) + S(r, L) \end{aligned} \tag{5.4}$$

Similarly we have by lemma 4.3 and lemma 4.6 from (5.3)

$$\begin{aligned} (n - \lambda)T(r, L) &\leq (2 + \tau)(1 + \eta)T(r, \xi) + (\tau + 2)(\eta + 1)T(r, L) \\ &\quad + S(r, \xi) + S(r, L) \end{aligned} \tag{5.5}$$

Using (5.4) and (5.5) we get

$$\begin{aligned} (n - \lambda)\{T(r, L) + T(r, \xi)\} &\leq (4 + 2\tau)(1 + \eta)\{T(r, L) + T(r, \xi)\} \\ &\quad + S(r, \xi) + S(r, L) \end{aligned} \tag{5.6}$$

Hence from (5.6) we arrive at a contradiction since  $n > \lambda + (4 + 2\tau)(1 + \eta)$ .

**Case 2** Let  $\Omega \equiv 0$ . Then  $(\frac{\Phi''}{\Phi'} - \frac{2\Phi'}{\Phi-1}) - (\frac{\Psi''}{\Psi'} - \frac{2\Psi'}{\Psi-1}) \equiv 0$ .

Integrating we have

$$\Phi - 1 \equiv \frac{\Psi - 1}{C - D(\Psi - 1)}, \tag{5.7}$$

where  $C(\neq 0)$  and  $D$  are constants.

Now we have to consider the following three cases

**Subcase 2.1** Let  $D = 0$ . Then from (5.7) we have

$$\Phi - 1 \equiv \frac{(\Psi - 1)}{C}, \tag{5.8}$$

If  $C \neq 1$ , then from (5.8)

$$\bar{N}(r, 0; \Phi) = \bar{N}(r, 1 - C; \Psi) \quad (5.9)$$

By lemma 4.2, lemma 4.7 and the second fundamental theorem we have from (5.3)

$$\begin{aligned} (n - \lambda)T(r, L) &= T(r, \Psi) - N_2(r, 0; \Psi) + N_{\tau+2}(r, 0; L^n\psi) + S(r, L) \\ &\leq \bar{N}(r, 0; \Psi) + \bar{N}(r, 1 - C; \Psi) + \bar{N}(r, \infty; \Psi) \\ &\quad - N_2(r, 0; \Psi) + N_{\tau+2}(r, 0; L^n\psi) + S(r, L) \\ &\leq \bar{N}(r, 0; \Psi) + \bar{N}(r, 0; \Phi) - N_2(r, 0; \Psi) \\ &\quad + N_{\tau+2}(r, 0; L^n\psi) + S(r, L) \\ &\leq \bar{N}(r, 0; (\xi^n\phi)^{(\tau)}) + \bar{N}(r, 0; (L^n\psi)^{(\tau)}) + N_{\tau+2}(r, 0; L^n\psi) + S(r, L) \\ &\leq N_{\tau+1}(r, 0; \xi^n\phi) + N_{\tau+1}(r, 0; L^n\psi) + N_{\tau+2}(r, 0; L^n\psi) + S(r, L) \\ &\leq (\tau + 1)(\eta + 1)T(r, L) + (\tau + 1)(\eta + 1)T(r, \xi) \\ &\quad + (\tau + 2)(\eta + 1)T(r, L) + S(r, L) \\ &\leq (2\tau + 3)(\eta + 1)T(r, L) + (\tau + 1)(\eta + 1)T(r, \xi) \\ &\quad + S(r, L) + S(r, \xi) \end{aligned} \quad (5.10)$$

Similarly we have from (5.2)

$$\begin{aligned} (n - \lambda)T(r, \xi) &\leq (2\tau + 3)(\eta + 1)T(r, \xi) + (\tau + 1)(\eta + 1)T(r, L) \\ &\quad + S(r, L) + S(r, \xi) \end{aligned} \quad (5.11)$$

From (5.10) and (5.11) we have

$$\begin{aligned} (n - \lambda)(T(r, L) + T(r, \xi)) &\leq (3\tau + 4)(\eta + 1)(T(r, L) + T(r, \xi)) \\ &\quad + S(r, \xi) + S(r, L) \end{aligned} \quad (5.12)$$

From (5.12) we arrive at a contradiction since  $n > \lambda + (3\tau + 4)(\eta + 1)$ .

Hence  $C = 1$  and therefore we get from (5.8)

$$[L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)}$$

**Subcase 2.2** Let  $D \neq 0$  and  $C = -D$ .

If  $D = 1$ , then from (5.7) we have  $\Phi\Psi \equiv 1$ . Hence

$$[L(z)^n \prod_{j=1}^{\eta} L(z + \omega_j)^{\mu_j}]^{(\tau)} [\xi(z)^n \prod_{j=1}^{\eta} \xi(z + \omega_j)^{\mu_j}]^{(\tau)} \equiv \rho(z)^2.$$

If  $D \neq 1$ , then from (5.7) we have  $\frac{1}{\Phi} = \frac{-D\Psi}{(1-D)\Psi-1}$ .

Hence  $\bar{N}(r, 0; \Phi) = \bar{N}(r, \frac{1}{1-D}; \Psi)$ .

Now proceeding as in the Subcase 2.1 we arrive at a contradiction.

**Subcase 2.3** Let  $D \neq 0$  and  $C \neq -D$ .

If  $D = 1$ , then from (5.7) we have

$$\Phi \equiv \frac{-C}{\Psi - C - 1} \tag{5.13}$$

Since  $\xi, L$  share  $(\infty, 0)$  therefore by lemma 4.2 we have from (5.13)

$$\begin{aligned} \bar{N}(r, C + 1; \Psi) &= \bar{N}(r, \infty; \Phi) \\ &= \bar{N}(r, \infty; \xi) + O(\log r) \\ &= S(r, L) \end{aligned}$$

Now proceeding as in Subcase 2.1 we arrive at a contradiction.

If  $D \neq 1$ , then from (5.7) we have

$$\Phi - (1 - \frac{1}{D}) \equiv \frac{-C}{D^2(\Psi - \frac{C+D}{D})}$$

Therefore by lemma 4.3 we have

$$\begin{aligned} \bar{N}(r, \frac{C + D}{D}; \Psi) &= \bar{N}(r, \infty; \Phi) \\ &= \bar{N}(r, \infty; \xi) + O(\log r) \\ &= S(r, L) \end{aligned}$$

Hence proceeding as in Subcase 2.1 we arrive at a contradiction.

This completes the proof of the theorem.

**Proof of Theorem 3.2**

By lemma 4.1  $L$  is a transcendental meromorphic function. Hence by lemma 4.4  $Q$  is a small function of  $L$ . Therefore by lemma 4.10  $\xi$  is a transcendental meromorphic function.

So by theorem 3.1 we get the required result.

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