International Journal of Difference Equations (IJDE).
ISSN 0973-6069, Volume 16, Number 1, (2021). 11-17

http://www.ripublication.com/ijde.htm

# Laplace Transform of Discriminant for Functions of Second Variable and its Generalized Form

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**Abstract**

We consider the Laplace transform of discriminant for functions of second variable and its generalized form. The generalized form is considered to be extended to the case of noninteger order. This method can be applied well to heat equations or other partial differential equations.

**AMS Subject Classification:** 44A05, 35K05, 35A22

**Key Words and Phrases:** discriminant, integral transform, noninteger order, PDEs

## 1. INTRODUCTION

Interpretation of partial differential equations(PDEs) using integral transforms have been tried in various ways. Since almost all the existing integral transforms[1-11] can be transformed by the Laplace transform in essence, it is known that the transform is the easiest to apply to the solution of PDEs. The discriminant for functions of second variable is given by

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2,$$

where

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}.$$

Since the discriminant $D$ is integer-order, its Laplace transform representation is not difficult. The proposed method can be usefully applied when the initial value or boundary value is uncertain. This is because arbitrary constants included in the solution of PDEs can be expressed in some detail as initial values. Let us extend the representation to the case of noninteger order. We recall that the Laplace transform of the derivatives $f^{(q)}$ of any integer is

$$\pounds(f^{(q)}) = s^q \pounds(f) - s^{q-1} f(0) - s^{q-2} f'(0) - \cdots - f^{(q-1)}(0),$$

where $q$ is an integer. Converting this equation to a simple form, we have

$$\pounds(\frac{d^q f}{dx^q}) = s^q \pounds(f) - \sum_{k=0}^{q-1} s^{q-1-k} \frac{d^k f}{dx^k}(0), \ \ q = 1, 2, 3, \cdots \tag{1}$$

Even if the positions of $k$ and $q-1-k$ are swapped in this equation, the equation does not matter. Therefore, we have

$$\pounds(\frac{d^q f}{dx^q}) = s^q \pounds(f) - \sum_{k=0}^{q-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0), \ \ q = 0, \pm 1, \pm 2, \cdots \tag{2}$$

because

$$\pounds(\frac{d^q f}{dx^q}) = s^q \pounds(f), \ \ q = 0, -1, -2, \cdots$$

Of course, (2) can be extended to noninteger $q$ by

$$\pounds(\frac{d^q f}{dx^q}) = s^q \pounds(f) - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0), \ \ all\ q, \tag{3}$$

where $n$ is the integer such that $n - 1 < q \le n$. If $q < 0$, the proof is made using the Riemann-Liouville integral

$$\frac{d^q f}{d(x-a)^q} = \frac{1}{\Gamma(-q)} \int_a^x [x-y]^{-q-1} f(y)\, dy, \ q < 0,$$

and if $q > 0$, it is made using

$$\frac{d^q f}{dx^q} = \frac{d^n}{dx^n} \frac{d^{q-n} f}{dx^{q-n}}.$$

The details are shown in [12].

This article has covered the expressions (1) and (3) of the discriminant $D$ for functions of second variable.

## 2. LAPLACE TRANSFORM OF DISCRIMINANT FOR FUNCTIONS OF SECOND VARIABLE AND ITS GENERALIZED FORM

**Lemma 1.** *In (2), consider the proof when $q$ is a natural number, i.e.,*

$$\pounds(\frac{d^q f}{dx^q}) = s^q \pounds(f) - \sum_{k=0}^{q-1} s^k \ \frac{d^{q-1-k} f}{dx^{q-1-k}}(0), \ \ q = 1, 2, 3 \cdots$$

Proof. First let us consider the case of $q = 1$.

$$\pounds(\frac{df}{dx}) = s\pounds(f) - f(0),$$

and by the direct calculation

$$\pounds(\frac{df}{dx}) = \int_0^\infty e^{-sx} \frac{df}{dx} \ dx$$

$$= e^{-sx} f(x)]_0^\infty + s \int_0^\infty e^{-sx} f(x) \ dx = s\pounds(f) - f(0).$$

Next, let us assume that $q = m$ is valid for an arbitrary natural number $m$. Thus,

$$\pounds(\frac{d^m f}{dx^m}) = s^m \pounds(f) - \sum_{k=0}^{m-1} s^k \ \frac{d^{m-1-k}}{dx^{m-1-k}} f(0)$$

holds. It is sufficient to show that it is

$$\pounds(\frac{d^{m+1} f}{dx^{m+1}}) = s^{m+1} \pounds(f) - \sum_{k=0}^{m} s^k \ \frac{d^{m-k}}{dx^{m-k}} f(0). \tag{4}$$

Since

$$\pounds(\frac{d^{m+1} f}{dx^{m+1}}) = \pounds[\frac{d}{dx}(\frac{d^m f}{dx^m})] = s\pounds(f^{(m)}) - f^{(m)}(0)$$

for $f^{(m)}$ is the $m$-th derivative of $f$, we get

$$\pounds(\frac{d^{m+1} f}{dx^{m+1}})$$

$$= s^{m+1} \pounds(f) - s \sum_{k=0}^{m-1} s^k \ \frac{d^{m-1-k}}{dx^{m-1-k}} f(0) - f^{(m)}(0)$$

$$= s^{m+1} \pounds(f) - \sum_{k=0}^{m} s^k \ \frac{d^{m-k}}{dx^{m-k}} f(0)$$

because

$$s \sum_{k=0}^{m-1} s^k \ \frac{d^{m-1-k}}{dx^{m-1-k}} f(0)$$

$$= \sum_{k=0}^{m} s^k \frac{d^{m-k}}{dx^{m-k}} f(0) - \frac{d^m f}{dx^m} f(0).$$

Verification of this equality can be easily verified by substitution. Hence, the theorem is valid in an arbitrary natural number $k$.

We would like to consider the Laplace transform of discriminant for functions of second variable.

**Theorem 2.** *The Laplace transform of discriminant of second partial derivative test can be expressed as*

$$\Big(s^2F^2 - sf(x,0)F - f_y(x,0)F, \quad -s^2F^2, \quad 2sf_x(x,0)F, \quad -(f_x(x,0))^2\Big)$$

$$\times \begin{pmatrix} \frac{\partial^2}{\partial x^2} \\ (\frac{\partial}{\partial x})^2 \\ \frac{\partial}{\partial x} \\ 1 \end{pmatrix},$$

*where*

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2.$$

*Alternately, $\pounds(D)$ can be expressed as*

$$\pounds(D) = (s^2F^2 - sf(x,0)F - f_y(x,0)F)\ \frac{\partial^2}{\partial x^2} - s^2F^2\ (\frac{\partial}{\partial x})^2$$

$$+2sf_x(x,0)F\ \frac{\partial}{\partial x} - (f_x(x,0))^2.$$

*The generalization of this result to non-integer order is as*

$$\pounds(D^q) = (s^{2q}F - \sum_{k=0}^{n-1} s^k \frac{d^{2q-1-k}f}{dx^{2q-1-k}}(0))(s^{2q}F - \sum_{k=0}^{n-1} s^k \frac{d^{2q-1-k}f}{dy^{2q-1-k}}(0))$$

$$-(s^qF - \sum_{k=0}^{n-1\ /2} s^k \frac{d^{q-1-k}f}{dx^{q-1-k}}(0))(s^qF - \sum_{k=0}^{n-1\ /2} s^k \frac{d^{q-1-k}f}{dy^{q-1-k}}(0)),$$

*where $n$ is the integer such that $n-1 < 2q \le n$.*

Proof. First, let us take Laplace transform for each terms with respect to $y$. Then

$$\pounds(f_{yy}) = s^2F - sf(x,0) - f_y(x,0)$$

$$\pounds(f_{xx}) = \int_0^\infty e^{-sy}\frac{\partial^2 f}{\partial x^2}\,dy = \frac{\partial^2}{\partial x^2}\pounds(f) = \frac{\partial^2}{\partial x^2}F$$
$$\pounds(f_{xy}) = s\pounds(f_x) - f_x(x,0) = s\frac{\partial}{\partial x}F - f_x(x,0)$$

for $F = \pounds(f)$. Thus,

$$\begin{aligned}\pounds(D) &= \frac{\partial^2}{\partial x^2}F\cdot(s^2F - sf(x,0) - f_y(x,0)) - (s\frac{\partial}{\partial x}F - f_x(x,0))^2\\ &= (s^2F^2 - sf(x,0)F - f_y(x,0)F)\,\frac{\partial^2}{\partial x^2} - s^2F^2\,(\frac{\partial}{\partial x})^2\\ &\quad +2sf_x(x,0)F\,\frac{\partial}{\partial x} - (f_x(x,0))^2.\end{aligned}$$

Thus, the coefficients of

$$\frac{\partial^2}{\partial x^2},\ (\frac{\partial}{\partial x})^2,\ \frac{\partial}{\partial x},$$

and 1 are

$$s^2F^2 - sf(x,0)F - f_y(x,0)F,\ -s^2F^2,\ 2sf_x(x,0)F,$$

and $-(f_x(x,0))^2$, respectively. We have dealt with the Laplace transform of the integer-order derivative. Consider generalizing this as a noninteger-order derivative. We consider

$$D^q = \frac{d^{2q}f}{dx^{2q}}\cdot\frac{d^{2q}f}{dy^{2q}} - \frac{d^q f}{dx^q}\cdot\frac{d^q f}{dy^q},$$

all $q$. Then

$$\begin{aligned}\pounds(D^q) &= (s^{2q}F - \sum_{k=0}^{n-1} s^k\frac{d^{2q-1-k}f}{dx^{2q-1-k}}(0))(s^{2q}F - \sum_{k=0}^{n-1} s^k\frac{d^{2q-1-k}f}{dy^{2q-1-k}}(0))\\ &\quad -(s^qF - \sum_{k=0}^{n-1\ /2} s^k\frac{d^{q-1-k}f}{dx^{q-1-k}}(0))(s^qF - \sum_{k=0}^{n-1\ /2} s^k\frac{d^{q-1-k}f}{dy^{q-1-k}}(0)),\end{aligned}$$

where $n$ is the integer such that $n-1 < 2q \le n$. If $n$ is odd, there is no problem. If $n$ is even, we can change $\frac{n-1}{2}$ to $\frac{n}{2}$.

Note that

$$\frac{\partial^2}{\partial x^2} \neq (\frac{\partial}{\partial x})^2$$

in the above equation. For example, we consider $f(x,y) = x^2 - y$. Then

$$\frac{\partial^2 f}{\partial x^2} = 2,\ but\ (\frac{\partial f}{\partial x})^2 = 4x^2.$$

**Conflict of interest** The authors declare no conflicts of interest.

**Acknowledgements** This research was supported by Kyungdong University Research Fund, 2021.

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