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A Nonlocal (Two-point with Parameters) Boundary Value Problem of Differential Inclusion

A.M.A.EL-Sayed and W.G.EL-Sayed

Faculty of Science, Alexandria University, Alexandria, Egypt.

Somyya S.Amrajaa

Faculty of Science, Omar al mukhtar University, Libya

Abstract

Here, we study the existence of solutions of a nonlocal two-point with parameters boundary value problem of a first order nonlinear differential inclusion. The maximal and minimal solutions will be studied. The continuous dependence of the unique solution on the parameters of the nonlocal condition will be proved. The anti-periodic boundary value problem will be considered as an application

Key Words: Differential inclusion, two-point boundary value problem, existence of solutions, continuous dependence.
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1. INTRODUCTION

The models of the arbitrary (fractional-orders) differential and integral equations have many applications (see [1], [3]-[13] and [15]-[16]). Here, we are concerning with the nonlinear differential inclusion

$$\frac{dx}{dt} \in F_1(t, x(t), I^{\gamma} f_2(t, x(t))), \ \ \gamma \in (0,1) \quad t \in (0,T) \tag{1}$$

with the nonlocal two-point boundary condition

$$\alpha x(\tau) + \beta x(\eta) = x_0, \ \ \tau \in [0,T), \ \ \eta \in (0,T], \quad \alpha, \beta > 0. \tag{2}$$

We study the existence of solutions $x \in C[0,T]$ of (1)-(2). The maximal and minimal solutions and the continuous dependence of the unique solution will be study. The anti-periodic problem of (1)-(2) will be considered.

MAIN RESULTS

The following assumptions will be needed for our goals.

(I) (i) The set $F_1(t,x,y)$ is nonempty, closed and convex for all $(t,x,y) \in [0,T] \in R \times R$.
(ii) $F_1(t,x,y)$ is upper semicontinuous in x and y for every $t \in [0,T]$.
(iii)$F_1(t,x,y)$ is measurable in $t \in [0,T]$ for every $x, y \in R$.
(iv) There exists a bounded measurable function $a_1 : [0,T] \longrightarrow R$ and a positive constant K_1, such that

$$\|F_1(t,x,y)\| = \sup\{|f_1| : f_1 \in F_1(t,x,y)\} \leq |a_1(t)| + K_1(|x| + |y|).$$

Remark 1 From the assumptions (i)-(iv) we can deduce that(see [1], [3]-[4] and [14]) there exists $f_1 \in F_1(t,x,y)$, such that

$$|f_1(t,x,y)| \leq |a_1(t)| + K_1(|x| + |y|),$$

$$\frac{dx}{dt} = f_1(t, x(t), I^\gamma f_2(t, x(t))), \ \ \gamma \in (0,1) \quad t \in (0,T). \tag{3}$$

(II) $f_2 : I \times R \longrightarrow R$ is measurable in t for any $x \in R$ and continuous in x for $t \in [0,T]$ such that there exists a bounded measurable function $a_2(t)$ and a positive constant $K_2 > 0$ such that

$$|f_2(t,x)| \leq |a_2(t)| + K_2|x|, \ \forall t \in I, \ \ and\, x \in R.$$

(III) $|a_i(t)| \leq m_i$, $i = 1, 2$.

Remark 2 From (I) and (iv) we can deduce that every solution of (2) and (3) is a solution of (1)-(2). Now we have the following lemma.

Lemma 1.1. *If the solution of the problem (2)-(3) exists then it can be expressed by the integral equation*

$$\begin{aligned} x(t) &= \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds \\ &\quad -\beta \int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] + \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds \end{aligned} \tag{4}$$

proof Let the boundary value problem (2)-(3) be satisfied. Integrating equation (3), we obtain

$$x(t) = x(0) + \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds, \tag{5}$$

$$\alpha x(\tau) = \alpha x(0) + \alpha \int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds, \tag{6}$$

$$\beta x(\eta) = \beta x(0) + \beta \int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds. \tag{7}$$

Substituting equation (6) in the equation (7), we obtain

$$x(0) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds - \beta \int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds]$$

Substituting in (5) we get the results.

2. EXISTENCE OF SOLUTIONS

Theorem 2.1. *Assume that the assumptions (I)-(III) are valid. Then the Problem (1)-(2) has at least one continuous solution* $x \in C[0,T]$.

proof. Let Q_r be the set $Q_r = \{x \in C[0,T] : \|x\| \leq r\}$, $t \in [0,T]$,

$$r = (\frac{|x_0|}{\gamma+\beta} + 2m_1T + 2K_1M_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)})(1 - 2K_1T - 2K_1K_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)})^{-1}.$$

Define the operator F by

$$Fx(t) = \frac{1}{\alpha+\beta}[x_o - \alpha \int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] + \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds.$$

Let $x \in Q_r$, then

$$|Fx(t)| = |\frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds$$
$$-\alpha \int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] + \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds|$$
$$\leq \frac{1}{\alpha+\beta}[|x_0| + \alpha \int_0^T (|a_1(s)| + K_1(|x(s)| + |I^\gamma f_2(s, x(s))|))ds$$
$$+\beta \int_0^T (|a_1(s)| + K_1(|x(s)| + |I^\gamma f_2(s, x(s))|))ds]$$
$$+ \int_0^T (|a_1(s)| + K_1(|x(s)| + |I^\gamma f_2(s, x(s))|))ds$$
$$\leq \frac{|x_0|}{\alpha+\beta} + 2\int_0^T (|a_1(s)| + K_1(|x(s)| + |I^\gamma f_2(s, x(s))|))ds$$
$$+2K_1K_2\|x\| \int_0^T \int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)} d\theta\, ds$$

$$\leq \frac{|x_0|}{\alpha+\beta} + 2m_1T + 2K_1T\|x\| + 2K_1m_2\int_0^T I^\gamma\, ds + 2K_1K_2\|x\|\int_0^T \frac{s^\gamma}{\Gamma(\gamma+1)}\, ds$$
$$\leq \frac{|x_0|}{\alpha+\beta} + 2m_1T + 2K_1T\|x\| + 2K_1m_2\int_0^T\int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}d\theta\, ds + 2K_1K_2r\frac{T^{\gamma+1}}{\Gamma(\gamma+2)}$$
$$\leq \frac{|x_0|}{\alpha+\beta} + 2m_1T + 2K_1Tr + 2K_1m_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)} + 2K_1K_2r\frac{T^{\gamma+1}}{\Gamma(\gamma+2)} \leq r.$$

Thus the class of functions $\{Fx\}$ is uniformly bounded on Q_r and $F : Q_r \to Q_r$. Let $x \in Q_r$ and $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| < \delta$, then

$$|Fx(t_2) - Fx(t_1)| \leq \int_{t_1}^{t_2} |f_1(s, x(s), I^\gamma f_2(s, x(s)))|ds$$
$$\leq \int_{t_1}^{t_2} (|a_1(s)| + K_1(|x(s)| + I^\gamma(|a_2(s)| + K_2|x(s)|)))ds$$
$$\leq m_1(t_2-t_1) + K_1(t_2-t_1)\|x\|$$
$$+K_1m_2\int_{t_1}^{t_2}\int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}d\theta\, ds + K_1K_2\|x\|\int_{t_1}^{t_2}\int_0^s \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}d\theta\, ds$$
$$\leq m_1(t_2-t_1) + K_1(t_2-t_1)r + K_1m_2\frac{t_2^{\gamma+1} - t_1^{\gamma+1}}{\Gamma(\alpha+2)} + K_1K_2r\frac{t_2^{\gamma+1} - t_1^{\gamma+1}}{\Gamma(\gamma+2)}.$$

Thus the class of functions $\{Fx\}$ is equicontinuous on Q_r and $\{Fx\}$ is compact operator by the Arzela-Ascoli Theorem[2].
Let $x_n \subset Q_r$, be convergent sequence such that, $x_n \to x_0$, then

$$Fx_n(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha\int_0^\tau f_1(s, x_n(s), I^\gamma f_2(s, x_n(s)))ds$$
$$-\beta\int_0^\eta f_1(s, x_n(s), I^\gamma f_2(s, x_n(s)))ds] + \int_0^t f_1(s, x_n(s), I^\gamma f_2(s, x_n(s)))ds.$$

Using Lebesgue Dominated Convergence Theorem [2] we obtain

$$\lim_{n\to\infty} Fx_n(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha\lim_{n\to\infty}\int_0^\tau f_1(s, x_n(s), I^\gamma f_2(s, x_n(s)))ds$$
$$-\beta\lim_{n\to\infty}\int_0^\eta f_1(s, x_n(s), I^\gamma f_2(s, x_n(s)))ds] + \lim_{n\to\infty}\int_0^t f_1(s, x_n(s), I^\gamma f_2(s, x_n(s)))ds$$
$$\leq \frac{1}{\alpha+\beta}[x_0 - \alpha\int_0^\tau f_1(s, x_0(s), I^\gamma f_2(s, x_0(s)))ds$$
$$-\beta\int_0^\eta f_1(s, x_0(s), I^\gamma f_2(s, x_0(s)))ds] + \int_0^t f_1(s, x_0(s), I^\gamma f_2(s, x_0(s)))ds \;=\; Fx_0(t).$$

Then $F : Q_r \to Q_r$, is continuous. Now by Schauder Fixed Point Theorem[2] there exists at least one solution $x \in C[0, T]$ of (4).

Differentiating the integral Equation (4) we obtain

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}[\frac{1}{\alpha+\beta}(x_0 - \alpha \int_0^\tau f_1(s, x_n(s), I^\gamma f_2(s, x_n(s)))ds \\ &+\beta \int_0^\eta f_1(s, x_n(s), I^\gamma f_2(s, x_n(s)))ds] + \int_0^t f_1(s, x_n(s), I^\gamma f_2(s, x_n(s)))ds \\ &= f_1(s, x_n(s), I^\gamma f_2(s, x_n(s))) \quad a.e., t \in [0, T]. \end{aligned}$$

Putting $t = \tau$ and multiplying equation (5) by α, we obtain

$$\begin{aligned} \alpha x(\tau) &= \frac{\alpha}{\alpha+\beta}x_0 - \frac{\alpha^2}{\alpha+\beta}\int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds \\ &-\frac{\alpha\beta}{\alpha+\beta}\int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] + \alpha\int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds \\ &= \frac{\alpha}{\alpha+\beta}x_0 + \frac{\alpha\beta}{\alpha+\beta}\int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] \\ &-\frac{\alpha\beta}{\alpha+\beta}\int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds]. \end{aligned}$$

Also

$$\begin{aligned} \beta x(\tau) &= \frac{\beta}{\alpha+\beta}x_0 - \frac{\alpha\beta}{\alpha+\beta}\int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] \\ &-\frac{\alpha\beta}{\alpha+\beta}\int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds]. \end{aligned}$$

Then

$$\alpha x(\tau) + \beta x(\eta) = x_0, \ \ \tau \in [0, T), \ \ \eta \in (0, T], \ \ \alpha, \beta > 0.$$

This prove the equivalence between the problem (2)-(3), and the integral equation(4). Then there exists at least one solution $x \in C[0, T]$ of the problem (2)-(3). Consequently, there exist at least one solution $x \in C[0, T]$ of the problem (1)-(2).

3. MAXIMAL AND MINIMAL SOLUTIONS

Her we study the maximal and minimal solutions for problem (1)-(2). Let $u(t)$ be a solution of (4), then $u(t)$ is said to be a maximal solution of (4) satisfies the inequality

$$x(t) \leq u(t), \quad t \in [0, T].$$

A minimal solution $v(t)$ can be defined by similar way by reversing the above inequality i.e.

$$x(t) > v(t), \quad t \in [0, T].$$

Lemma 3.1. *Let the assumptions of Theorem 2.1 be satisfied. Assume that $x(t)$ and $y(t)$ are two continuous functions on $[0, T]$ satisfying.*

$$x(t) \leq \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] + \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds \;\; t \in [0, T].$$
$$y(t) \geq \frac{1}{\alpha+\beta}[x_o - \alpha \int_0^\tau f_1(s, y(s), I^\gamma f_2(s, y(s)))ds$$
$$-\beta \int_0^\eta f_1(s, y(s), I^\gamma f_2(s, y(s)))ds] + \int_0^t f_1(s, y(s), I^\gamma f_2(s, y(s)))ds \;\; t \in [0, T]$$

where one of them is strict. Let functions f_1 and f_2 are monotonic nondecreasing in x, then

$$x(t) < y(t), \quad t > 0. \tag{8}$$

Proof. Let the conclusion (8) not true, then there exists t_1 with

$$x(t_1) < y(t_1), \quad t_1 > 0 \quad \text{ and } x(t) < y(t), \quad 0 < t < t_1.$$

For f_1 and f_2 are monotonic functions in x we have

$$x(t_1) \leq \frac{1}{\alpha+\beta}[x_o - \alpha \int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] + \int_0^{t_1} f_1(s, x(s), I^\gamma f_2(s, x(s)))ds \;.$$
$$< \frac{1}{\alpha+\beta}[x_o - \alpha \int_0^\tau f_1(s, y(s), I^\gamma f_2(s, y(s)))ds$$
$$-\beta \int_0^\eta f_1(s, y(s), I^\gamma f_2(s, y(s)))ds] + \int_0^{t_1} f_1(s, y(s), I^\gamma f_2(s, y(s)))ds < y(t_1).$$

This contrasts with the fact that $x(t_1) = y(t_1)$. This complete the proof. For the existence of the continuous maximal and minimal solutions for (4), we have the following theorem.

Theorem 3.2. . *Let assumptions of Theorem 2.1, be hold. Moreover, if f_1 and f_2 are monotonic nondecreasing functions in x for each $t \in [0, T]$, then the equation (4) has maximal and minimal solutions.*

proof. First, we must demonstrate the existence of the maximal solution of (4). Let

$\epsilon > 0$ be given. Now consider the integral equation

$$\begin{aligned} x_{\epsilon_1}(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x_{\epsilon_1}(s), I^\gamma f_2(s, x_{\epsilon_1}(s)))ds \\ - \beta \int_0^\eta f_1(s, x_{\epsilon_1}(s), I^\gamma f_2(s, x_{\epsilon_1}(s)))ds] \\ + \int_0^t f_1(s, x_{\epsilon_1}(s), I^\gamma f_2(s, x_{\epsilon_1}(s)))ds \ \ t \in [0, T] \end{aligned} \tag{11}$$

where $f_2(s, x_\epsilon(s)) = f_2(s, x_\epsilon(s)) + \epsilon$

$$\begin{aligned} x_{\epsilon_2}(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x_{\epsilon_2}(s), I^\gamma f_2(s, x_{\epsilon_2}(s)))ds \\ -\beta \int_0^\eta f_1(s, x_{\epsilon_2}(s), I^\gamma f_2(s, x_{\epsilon_2}(s)))ds] + \int_0^t f_1(s, x_{\epsilon_2}(s), I^\gamma f_2(s, x_{\epsilon_2}(s)))ds. \end{aligned} \tag{12}$$

$$\begin{aligned} x_{\epsilon_2}(t) &= \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau (f_1(s, x_{\epsilon_2}(s), I^\gamma f_2(s, x_{\epsilon_2}(s))) + \epsilon_2)ds \\ &\quad -\beta \int_0^\eta (f_1(s, x_{\epsilon_2}(s), I^\gamma f_2(s, x_{\epsilon_2}(s))) + \epsilon_2)\, ds] \\ &\quad + \int_0^t (f_1(s, x_{\epsilon_2}(s), I^\gamma f_2(s, x_{\epsilon_2}(s))) + \epsilon_2)\, ds, \end{aligned}$$

$$\begin{aligned} x_{\epsilon_1}(t) &= \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x_{\epsilon_1}(s), I^\gamma f_2(s, x_{\epsilon_1}(s)))ds \\ &\quad -\beta \int_0^\eta f_1(s, x_{\epsilon_1}(s), I^\gamma f_2(s, x_{\epsilon_1}(s)))ds] + \int_0^t f_1(s, x_{\epsilon_1}(s), I^\gamma f_2(s, x_{\epsilon_1}(s)))ds. \\ &\quad x_{\epsilon_1}(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau (f_1(s, x_{\epsilon_1}(s), I^\alpha f_2(s, x_{\epsilon_1}(s))) + \epsilon_1)ds \\ &\quad -\beta \int_0^\eta (f_1(s, x_{\epsilon_1}(s), I^\gamma f_2(s, x_{\epsilon_1}(s))) + \epsilon_1)\, ds] \\ &\quad + \int_0^t (f_1(s, x_{\epsilon_1}(s), I^\gamma f_2(s, x_{\epsilon_1}(s))) + \epsilon_1)\, ds. \\ &\quad > \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau (f_1(s, x_{\epsilon_2}(s), I^\gamma f_2(s, x_{\epsilon_2}(s))) + \epsilon_2)ds \\ &\quad -\beta \int_0^\eta (f_1(s, x_{\epsilon_2}(s), I^\alpha f_2(s, x_{\epsilon_2}(s))) + \epsilon_2)\, ds] \\ &\quad + \int_0^t (f_1(s, x_{\epsilon_2}(s), I^\gamma f_2(s, x_{\epsilon_2}(s))) + \epsilon_2)\, ds. \end{aligned}$$

Applying Lemma 2, we obtain $x_{\epsilon_2} < x_{\epsilon_1},\ t \in [0, T]$.

As shown before the family of function $x_\epsilon(t)$ is equi-continuous and uniformly bounded, then by Arzela Theorem, there exist decreasing sequence ϵ_n, such that $\epsilon_0 \rightarrow 0$ as $n \rightarrow \infty$, and $u(t) = \lim_{n\rightarrow\infty} x_{\epsilon_n}(t)$ exists uniformly in $[0, T]$ and denote this limit by $u(t)$. From the continuity of the functions,$f_{2\epsilon}(t, x_\epsilon(t))$, we get $f_{2\epsilon}(t, x_\epsilon(t)) \longrightarrow f_2(t, x(t))$ as $n \rightarrow \infty$ and

$$u(t) = \lim_{n\rightarrow\infty} x_{\epsilon_n}(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x(s), I^\alpha f_2(s, x(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x(s), I^\alpha f_2(s, x(s)))ds] + \int_0^t f_1(s, x(s), I^\alpha f_2(s, x(s)))ds.$$

Now we prove that $u(t)$ is the maximal solution of (4). To do this, let $x(t)$ be any solution of (4), then

$$x(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] + \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds,$$

$$x_\epsilon(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x_\epsilon(s), I^\gamma f_2(s, x_\epsilon(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x_\epsilon(s), I^\gamma f_2(s, x_\epsilon(s)))ds] + \int_0^t f_1(s, x_\epsilon(s), I^\gamma f_2(s, x_\epsilon(s)))ds.$$

$$x_\epsilon(t) = \frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau (f_1(s, x_\epsilon(s), I^\gamma f_2(s, x_\epsilon(s))) + \epsilon)\, ds$$
$$-\beta \int_0^\eta (f_1(s, x_\epsilon(s), I^\gamma f_2(s, x_\epsilon(s))) + \epsilon)\, ds] + \int_0^t f_1((s, x_\epsilon(s), I^\gamma f_2(s, x_\epsilon(s))) + \epsilon)\, ds.$$
$$> \frac{1}{\alpha+\beta}[x_o - \alpha \int_0^\tau f_1(s, x_\epsilon(s), I^\gamma f_2(s, x_\epsilon(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x_\epsilon(s), I^\alpha f_2(s, x_\epsilon(s)))ds] + \int_0^t f_1(s, x_\epsilon(s), I^\gamma f_2(s, x_\epsilon(s)))ds = x(t).$$

Applying lemma (3.1), we obtain $x(t) < x_\epsilon(t), t \in [0, T]$.
From the uniqueness of the maximal solution it clear that $x_\epsilon(t)$ tends to $u(t)$ uniformly in $[0, T]$ as $\epsilon \rightarrow 0$ by similar way as done above we can prove the existence of the minimal solution.

4. UNIQUENESS OF THE SOLUTION

Here we study the sufficient conditions for the uniqueness of the solution $x \in C[0, T]$, of problems (1)-(2). Consider the following assumptions

(I) (i) The set $F_1(t,x,y)$ is nonempty, compact and convex for all $(t,x,y) \in [0,T] \in R \times R$.
(ii) F_1 is continuous and satisfies Lipschitz condition with a positive constant K_1 such that

$$H(F_1(t,x_1,y_1), F_1(t,x_2,y_2)| \leq K_1(|x_1 - x_2| + |y_1 - y_2|)$$

where $H(A,B)$ is the Hausdorff metric between the two subsets $A, B \in I \times E$ (see[13])
Remark 3 From this assumptions we can deduce that there exists a function $f_1 \in F_1(t,x,y)$, such that
(iii) $f_1 : I \times R \times R \to R$ is continuous and satisfies Lipschitz condition with a positive constant K_1 such that (see [2],[3] and [14])

$$|f_1(t,x_1,y_1) - f_1(t,x_2,y_2)| \leq K_1(|x_1 - x_2| + |y_1 - y_2|),$$

$$\frac{dx}{dt} = f_1(t, x(t), I^{\gamma} f_2(t, x(t))), \ \ \gamma \in (0,1) \quad t \in (0,T).$$

(II) $f_2 : I \times R \longrightarrow R$, is continuous and satisfies Lipschitz condition with positive constant K_2, such that

$$|f_2(t,x) - f_2(t,y)| \leq K_2|x - y|.$$

From this assumption (I), we have

$$|f_1(t,x,y)| - |f_1(t,0,0)| \leq |f_1(t,x,y) - f_1(t,0,0)| \leq K_1(|x| + |y|).$$

Then

$$|f_1(t,x,y)| \ \leq \ K_1(|x| + |y|) + |f_1(t,0,0)| \leq K_1(|x| + |y|) + |\, a_1(t)\, |$$

where $|a_1(t)| = \sup_{t \in I} |f_1(t,0,0)|$ and from assumption (II), we have

$$|f_2(t,x)| - |f_2(t,0)| \leq |f_2(t,x) - f_2(t,0)| \leq K_2|x|.$$

Then

$$|f_2(t,x)| \ \leq \ K_2|x| + |f_2(t,0)| \leq K_2|x| + |\, a_2(t)\, |, \quad |a_2(t)| = \sup_{t \in I} |f_1(t,0)|.$$

Theorem 4.1. *Let the assumptions (I)-(II) be satisfies. Then the solution of the Problem (2)-(3) is a unique.*

Proof *Let $x_1(t)$ and $x_2(t)$ be solution of the problem (2)-(3), then*

$$
\begin{aligned}
&|x_1(t) - x_2(t)| = |\frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^{\tau} f_1(s, x_1(s), I^{\gamma} f_2(s, x_1(s)))ds \\
&-\beta \int_0^{\eta} f_1(s, x_1(s), I^{\gamma} f_2(s, x_1(s)))ds] + \int_0^{t} f_1(s, x_1(s), I^{\gamma} f_2(s, x_1(s)))ds \\
&-(\frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^{\tau} f_1(s, x_2(s), I^{\gamma} f_2(s, x_2(s)))ds \\
&-\beta \int_0^{\eta} f_1(s, x_2(s), I^{\gamma} f_2(s, x_2(s)))ds] + \int_0^{t} f_2(s, x_2(s), I^{\gamma} f_2(s, x_2(s)))ds)| \\
&\leq \frac{\alpha}{\alpha+\beta} \int_0^{\tau} |f_1(s, x_2(s), I^{\gamma} f_2(s, x_2(s))) - f_1(s, x_1(s), I^{\gamma} f_2(s, x_1(s)))|ds \\
&+\frac{\beta}{\alpha+\beta} \int_0^{\eta} |f_1(s, x_2(s), I^{\gamma} f_2(s, x_2(s))) - f_1(s, x_1(s), I^{\gamma} f_2(s, x_1(s)))|ds \\
&+\int_0^{t} |f_1(s, x_1(s), I^{\gamma} f_2(s, x_1(s))) - f_1(s, x_2(s), I^{\gamma} f_2(s, x_2(s)))|ds \\
&\leq 2 \int_0^{T} |f_1(s, x_1(s), I^{\gamma} f_2(s, x_1(s))) - f_1(s, x_2(s), I^{\gamma} f_2(s, x_2(s)))|ds \\
&\leq 2K_1 \int_0^{T} (|x_2(s) - x_1(s)| + |I^{\gamma} f_2(s, x_2(s)) - I^{\gamma} f_2(s, x_1(s))|)\, ds \\
&\leq 2K_1 \|x_1 - x_2\| T + 2K_1 \int_0^{T} I^{\gamma} |f_2(s, x_2(s)) - f_2(s, x_1(s))|\, ds \\
&\leq 2K_1 \|x_1 - x_2\| T + 2K_1 K_2 \int_0^{T} I^{\gamma} |x_2(s) - x_1(s)|\, ds \\
&\leq 2K_1 \|x_1 - x_2\| T + 2K_1 K_2 \|x_1 - x_2\| \int_0^{T} \int_0^{s} \frac{(s-\theta)^{\gamma-1}}{\Gamma(\gamma)}\, d\theta\, ds \\
&\leq 2K_1 \|x_1 - x_2\| T + 2K_1 K_2 \|x_1 - x_2\| \int_0^{T} \frac{s^{\gamma}}{\Gamma(\gamma+1)}\, ds \\
&\leq 2K_1 \|x_1 - x_2\| T + 2K_1 K_2 \frac{T^{\gamma+1}}{\Gamma(\gamma+2)} \|x_1 - x_2\| \\
&\leq (2K_1 T + \frac{2K_1 K_2 T^{\gamma+1}}{\Gamma(\gamma+2)}) \|x_1 - x_2\|.
\end{aligned}
$$

Hence $(1 - (T + K_2 \frac{T^{\gamma+1}}{\Gamma(\gamma+2)}) 2K_1) < 1$, *then* $x_1(t) = x_2(t)$.
The solution of (2)-(3) is unique. This completes the proof.

5. CONTINUOUS DEPENDENCE OF THE SOLUTION

Definition 1. The unique solution of the problem (1)-(2) depends continuously on initial data x_0, if $\epsilon > 0$,$\exists\delta > 0$ such that

$$|x_0 - x_0^*| \leq \delta \quad \Rightarrow \|x - x^*\| \leq \epsilon$$

where x^* is the unique solution of

$$x^*(t) = \frac{1}{\alpha+\beta}[x_0^* - \alpha \int_0^\tau f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds] + \int_0^t f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds \quad (9)$$

Theorem 5.1. *Let the assumption of Theorem 2.1 be satisfied, then the unique solution of (1)-(2) depends continuously on* x_0

Proof. Let $x(t)$ and $x^*(t)$ be the solutions of problem (1)-(2), then

$$|x(t) - x^*(t)| = |\frac{1}{\alpha+\beta}[x_0 - \alpha \int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] + \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds$$
$$-(\frac{1}{\alpha+\beta}[x_0^* - \alpha \int_0^\tau f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds$$
$$-\beta \int_0^\eta f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds] + \int_0^t f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds)|$$
$$\leq \frac{1}{\alpha+\beta} \mid x_0 - x_0^* \mid$$
$$+\frac{\alpha}{\alpha+\beta} \int_0^\tau |f_1(s, x(s), I^\gamma f_2(s, x(s)) - f_1(s, x^*(s), I^\gamma f_2(s, x^*(s))| \, ds$$
$$+\frac{\beta}{\alpha+\beta} \int_0^\eta |f_1(s, x(s), I^\gamma f_2(s, x(s)) - f_1(s, x^*(s), I^\gamma f_2(s, x^*(s))| \, ds$$
$$+\int_0^t |f_1(s, x(s), I^\gamma f_2(s, x(s)) - f_1(s, x^*(s), I^\gamma f_2(s, x^*(s))| \, ds$$
$$\leq \frac{1}{\alpha+\beta} \mid x_0 - x_0^* \mid +2 \int_0^T |f_1(s, x(s), I^\gamma f_2(s, x(s)) - f_1(s, x^*(s), I^\gamma f_2(s, x^*(s))| \, ds$$
$$\leq \frac{1}{\alpha+\beta} \mid x_0 - x_0^* \mid +2 \int_0^T (K_1|x(s) - x^*(s)| + K_1|I^\gamma f_2(s, x(s)) - I^\gamma f_2(s, x^*(s))|) \, ds$$
$$\leq \frac{1}{\alpha+\beta} \mid x_0 - x_0^* \mid +2K_1\|x - x^*\|T + 2K_1K_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)}\|x - x^*\|$$
$$\leq \frac{\delta}{\alpha+\beta} + (2K_1T + 2K_1K_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)})\|x - x^*\|.$$

Hence

$$\|x - x^*\| \leq \frac{\delta}{\alpha+\beta}(1 - (2K_1T + 2K_1K_2\frac{T^{\gamma+1}}{\Gamma(\alpha+2)}))^{-1} = \epsilon.$$

Definition 2. The unique solution of (1)-(2) depends continuously on initial α and β if $\epsilon > 0$,$\exists \delta > 0$, such that

$$|\alpha - \alpha^*| \leq \delta, |\beta - \beta^*| \leq \delta \quad \Rightarrow \|x - x^*\| \leq \epsilon$$

Theorem 5.2. *Let the assumption of Theorem 2.1 be satisfied, then the solution of (1)-(2), depends continuously on α and β.*

Proof. Let $x(t)$ and $x^*(t)$ be the solutions of problem (1)-(2), then

$$\begin{aligned}
&|x(t) - x^*(t)| = |\frac{x_0}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}\int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds \\
&-\frac{\beta}{\alpha+\beta}\int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds] + \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds \\
&-\frac{x_0}{\alpha_*+\beta_*} + \frac{\alpha^*}{\alpha_*+\beta_*}\int_0^\tau f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds \\
&+\frac{\beta^*}{\alpha_*+\beta_*}\int_0^\eta f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds - \int_0^t f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds| \\
&\leq |\frac{x_0}{(\alpha+\beta)} - \frac{x_0}{(\alpha^*+\beta^*)}| \\
&+|\frac{\alpha^*}{(\alpha^*+\beta^*)}\int_0^\tau f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds - \frac{\alpha}{\alpha+\beta}\int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds| \\
&+|\frac{\beta^*}{(\alpha^*+\beta^*)}\int_0^\eta f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds - \frac{\beta}{\alpha+\beta}\int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds| \\
&+\int_0^t |f_1(s, x^*(s), I^\gamma f_2(s, x^*(s))) - f_1(s, x(s), I^\gamma f_2(s, x(s)))|ds \\
&\leq \frac{|x_0|(|\alpha - \alpha_*| + |\beta - \beta^*|)}{(\alpha+\beta)(\alpha^*+\beta^*)} \\
&+|\frac{\alpha^*}{(\alpha^*+\beta^*)}\int_0^\tau |f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))|ds - \frac{\alpha}{\alpha+\beta}\int_0^\tau f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds| \\
&+|\frac{\alpha}{\alpha+\beta}\int_0^\tau f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds - \frac{\alpha}{\alpha+\beta}\int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds| \\
&+|\frac{\beta^*}{(\alpha^*+\beta^*)}\int_0^\eta f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds - \frac{\beta}{\alpha+\beta}\int_0^\eta f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds| \\
&+|\frac{\beta}{\alpha+\beta}\int_0^\eta f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))ds - \frac{\beta}{\alpha+\beta}\int_0^\eta f_1(s, x(s), I^\gamma f_2(s, x(s)))ds| \\
&+\int_0^t |f_1(s, x^*(s), I^\gamma f_2(s, x^*(s))) - f_1(s, x(s), I^\gamma f_2(s, x(s)))|\, ds \\
&\leq \frac{|x_0|(|\alpha - \alpha_*| + |\beta - \beta^*|)}{(\alpha+\beta)(\alpha^*+\beta^*)} \\
&+|\frac{\alpha^*}{(\alpha^*+\beta^*)} - \frac{\alpha}{(\alpha+\beta)}|\int_0^\tau |f_1(s, x^*(s), I^\gamma f_2(s, x^*(s)))|\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha}{\alpha+\beta}\int_0^{\tau}|f_1(s,x^*(s),I^{\gamma}f_2(s,x^*(s)))-f_1(s,x(s),I^{\gamma}f_2(s,x(s)))|\,ds\\
&+|\frac{\beta^*}{(\alpha^*+\beta^*)}-\frac{\beta}{(\alpha+\beta)}|\int_0^{\tau}|f_1(s,x^*(s),I^{\gamma}f_2(s,x^*(s)))|\,ds\\
&+\frac{\beta}{\alpha+\beta}\int_0^{\eta}|f_1(s,x^*(s),I^{\gamma}f_2(s,x^*(s)))-f_1(s,x(s),I^{\gamma}f_2(s,x(s)))|\,ds\\
&+\int_0^{t}|f_1(s,x^*(s),I^{\gamma}f_2(s,x^*(s)))-f_1(s,x(s),I^{\gamma}f_2(s,x(s)))|\,ds\\
&\leq\frac{|x_0|(\delta+\delta)}{(\alpha+\beta)(\alpha^*+\beta^*)}+[|\frac{\alpha^*}{(\alpha^*+\beta^*)}-\frac{\alpha}{\alpha+\beta}|\\
&+|\frac{\beta^*}{(\alpha^*+\beta^*)}-\frac{\beta}{\alpha+\beta}|]\int_0^{T}f_1(s,x^*(s),I^{\gamma}f_2(s,x^*(s)))ds\\
&+2\int_0^{T}|f_1(s,x^*(s),I^{\gamma}f_2(s,x^*(s)))-f_1(s,x(s),I^{\gamma}f_2(s,x(s)))|\,ds\\
&\leq\frac{(2\delta)|x_0|}{(\alpha+\beta)(\alpha^*+\beta^*)}\\
&+(\frac{|\alpha^*\beta-\alpha\beta^*|}{(\alpha+\beta)(\alpha^*+\beta^*)}+\frac{|\alpha^*\beta-\alpha\beta^*|}{(\alpha+\beta)(\alpha^*+\beta^*)})\int_0^{T}f_1(s,x^*(s),I^{\gamma}f_2(s,x^*(s)))ds\\
&+(2K_1T+2K_1K_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)})\|x-x^*\|\leq\frac{(2\delta)|x_0|}{(\alpha+\beta)(\alpha^*+\beta^*)}\\
&+[\frac{|\alpha^*\beta-\alpha\beta|+|\alpha\beta-\alpha\beta^*|}{(\alpha+\beta)(\alpha^*+\beta^*)}\\
&+\frac{|\alpha^*\beta-\alpha\beta|+|\alpha\beta-\alpha\beta^*|}{(\alpha+\beta)(\alpha^*+\beta^*)}]\int_0^{T}(K_1|x^*(s)|+K_1|I^{\gamma}f_2(s,x^*(s)))|+|f_1(s,0,0)|)ds\\
&+(2K_1T+2K_1K_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)})\|x-x^*\|\leq\frac{(2\delta)|x_0|}{(\alpha+\beta)(\alpha^*+\beta^*)}\\
&+(\frac{2\delta}{(\alpha^*+\beta^*)}(K_1rT+(K_1\int_0^{T}|I^{\gamma}f_2(s,x^*(s)))|ds+m_1T)\\
&+(2K_1T+2K_1K_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)})\|x-x^*\|\leq\frac{(2\delta)|x_0|}{(\alpha+\beta)(\alpha^*+\beta^*)}\\
&+(\frac{2\delta}{(\alpha^*+\beta^*)})(K_1rT+K_1\int_0^{T}I^{\gamma}(K_2|x(s)|+|f_2(s,0)|)ds+m_1T)\\
&+(2K_1T+2K_1K_2\frac{T^{\alpha+1}}{\Gamma(\alpha+2)})\|x-x^*\|\leq\frac{(2\delta)|x_0|}{(\alpha+\beta)(\alpha^*+\beta^*)}\\
&+(\frac{2\delta}{(\alpha^*+\beta^*)})(K_1rT+(K_1K_2r+K_1m_2)\int_0^{t}\int_0^{s}\frac{(s-\tau)^{\gamma-1}}{\Gamma\gamma}d\tau\,ds+m_1T)\\
&+(2K_1T+2K_1K_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)})\|x-x^*\|\leq\frac{(2\delta)|x_0|}{(\alpha+\beta)(\alpha^*+\beta^*)}\\
&+(\frac{2\delta}{(\alpha+\beta)(\alpha^*+\beta^*)})(K_1rT+(K_1K_2r+K_1m_2)\frac{T^{\gamma+1}}{\Gamma(\gamma+2)}+m_1T)\\
&+(2K_1T+2K_1K_2\frac{T^{\gamma+1}}{\Gamma(\gamma+2)})\|x-x^*\|,
\end{aligned}
$$

then

$$\|x - x^*\| \leq \frac{(2\delta)|x_0| + 2\delta(K_1 r T + (K_1 K_2 r + K_1 m_2)\frac{T^{\gamma+1}}{\Gamma(\gamma+2)}) + m_1 T)}{(\alpha^* + \beta^*)[1 - (2K_1 T + 2K_1 K_2 \frac{T^{\gamma+1}}{\Gamma(\gamma+2)})]} \leq \epsilon.$$

6. ANTI-PERIODIC NONLOCAL BOUNDARY VALUE PROBLEM.

Consider the nonlocal boundary value problem of (2)-(3) with the anti-periodic nonlocal condition

$$x(\tau) = -x(1-\tau) \quad \tau \in [0,T].$$

Corollary 1. If $\alpha = 1$, $\beta = 1$, and $\eta = 1 - \tau$, and $x_0 = 0$, in Theorem (1.2), then the anti-periodic boundary value problem.

$$\frac{dx}{dt} = f_1(t, x(t), I^\gamma f_2(t, x(t))), \quad \alpha \in (0,1) \quad t \in (0,T) \tag{10}$$

$$x(\tau) = -x(1-\tau) \quad \tau \in [0,T]$$

has the at least one solution $x \in AC[0,T]$

$$\begin{aligned} x(t) &= -\int_0^\tau f_1(s, x(s), I^\gamma f_2(s, x(s)))ds - \int_0^{1-\tau} f_1(s, x(s), I^\gamma f_2(s, x(s)))ds. \\ &+ \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds. \end{aligned}$$

Now, let $\tau = \frac{1}{2}$, then

$$\begin{aligned} & x(t) = -\int_0^{\frac{1}{2}} f_1(s, x(s), I^\gamma f_2(s, x(s)))ds - \int_0^{\frac{1}{2}} f_1(s, x(s), I^\gamma f_2(s, x(s)))ds. \\ & + \int_0^t f_1(s, x(s), I^\alpha f_2(s, x(s)))ds. \\ & = \int_0^t f_1(s, x(s), I^\gamma f_2(s, x(s)))ds - 2\int_0^{\frac{1}{2}} f_1(s, x(s), I^\gamma f_2(s, x(s)))ds. \end{aligned}$$

CONCLUSIONS

We proved here, under certain conditions, the existence of at least one continuous solution $x \in C[0,T]$ of the nonlocal two-point, with parameters (α, β and x_0) boundary value problem (1)-(2). The maximal and minimal solutions of the problem (1)-(2) have been proved. The continuous dependence of the unique solution on the parameters (α, β and x_0) have been also proved. The anti-periodic boundary value problem have been considered as an application.

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