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Some arithmetic operations on Triangular Intuitionistic Fuzzy Number and its application on reliability evaluation

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Abstract

Generally fuzzy sets are used to analyze the fuzzy system reliability. Here intuitionistic fuzzy set theory has been used for analyzing the fuzzy system reliability. To analyze the fuzzy system reliability, the reliability of each component of the system is considered as a triangular intuitionistic fuzzy number. At first triangular intuitionistic fuzzy number and their arithmetic operations are introduced. Expressions for computing the fuzzy reliability of a series system, parallel system,series-parallel and parallel-series system following triangular intuitionistic fuzzy numbers have been described. Here an imprecise failure to start of an automobile is taken. To compute the imprecise failure of the above said system, failure of each component of the systems is represented by triangular intuitionistic fuzzy numbers. Corresponding numerical example is presented.

Keyword: Fuzzy set, Intuitionistic fuzzy number, System reliability, Triangular intuitionistic fuzzy number.

1. Introduction

As it is known that the conventional reliability analysis using probabilities has been found to be inadequate to handle uncertainty of failure data and modeling. To overcome this problem, the concept of fuzzy [1] approach has been used in the evaluation of the reliability of a system. In [2] Kaufmann et al. pointed out that the discipline of the reliability engineering encompasses a number of different activities, out of which the reliability modeling is the most important activity. For a long period of time efforts have been made in the design and development of reliable large-scale systems. In that period of time considerable work has been done by researchers to build a systematic theory of reliability based on the probability theory.

In [3] Cai et al. pointed out that there are two fundamental assumptions in the conventional reliability theory, i.e. (a) Binary state assumptions: the system is precisely defined as functioning or failing. (b) Probability assumptions: the system behaviour is fuzzy characterized in the context of probability measures. Because of the inaccuracy and uncertainties of data, the estimation of precise values of probability becomes very difficult in many systems. In [4], Cai et al introduced system failure engineering and its use of fuzzy methodology. In [5], Chen presented a method for analyzing the fuzzy system reliability using fuzzy number arithmetic operations. In [6], Cheng et al. used interval of confidence for analyzing the fuzzy system reliability. In [7], Singer presented a fuzzy set approach for fault tree and the reliability analysis. Verma [8] presented the dynamic reliability evaluation of the deteriorating system using the concept of probist reliability as a triangular fuzzy number.

Intuitionistic fuzzy set (IFS) is one of the generalizations of fuzzy sets theory [9]. Out of several higher-order fuzzy sets, IFS first introduced by Atanassov [10] have been found to be compatible to deal with vagueness. The concept of IFS can be viewed as an appropriate/alternative approach to define a fuzzy set in case where available information is not sufficient for the definition of an imprecise concept by means of a conventional fuzzy set. In fuzzy sets the degree of acceptance is considered only but IFS is characterized by a membership function and a non-membership function so that the sum of both values is less than one [11]. Presently intuitionistic fuzzy sets are being studied and used in different fields of science. Among the works on these sets, Atanassov [12-14], Atanassov and Gargov [15], Szmidt and Kacrzyk [16],Cornelis, Deschrijver and Kerre[17],Buhaesku [18], Gerstenkorn and Manko [19], Stojona and Atanassov [20], Stoyanova [21], Deschrijver and Kerre [22] can be mentioned. With best of our knowledge, Burillo [23] proposed definition of intuitionistic fuzzy number and studied perturbations of intuitionistic fuzzy number and the first properties of the correlation between these numbers. Mitchell [24] considered the problem of ranking a set of intuitionistic fuzzy numbers to define a fuzzy rank and a characteristic vagueness factor for each intuitionistic fuzzy number. Here intuitionistic fuzzy number (IFN) is presented according to the approach of presentation of fuzzy number. Arithmetic operations of proposed IFN are evaluated.

This paper is organized as follows: Section 2 presents basic concept of intuitionistic fuzzy sets and intuitionistic fuzzy number. Section 3 presents arithmetic operations between two triangular intuitionistic fuzzy numbers. Section 4 presents expressions for finding the fuzzy fault of a series , parallel,parallel-series and series-parallel system using arithmetic operations on triangular intuitionistic fuzzy numbers. Section 5 presents the intuitionitic fuzzy fault tree of imprecise failure to start of an automobile. The conclusions are discussed in Section 6.

2. Basic concept of intuitionistic fuzzy sets

Fuzzy set theory was first introduced by Zadeh [9] in 1965. Let X *be* universe of discourse defined by $X = \{x_1, x_2, ..., x_n\}$. The grade of membership of an element $x_i \in X$ in a fuzzy set is represented by real value between 0 and 1. It does indicate the evidence for $x_i \in X$,but does not indicate the evidence against $x_i \in X$ .Atanassov[10]presented the concept of IFS, and pointed out that this single value combines the evidence for $x_i \in X$ and the evidence against $x_i \in X$ .An IFS $\overset{\sim i}{A}$ in X is characterized by a membership function $\mu_{\overset{\sim i}{A}}(x)$ and non -membership function $\nu_{\overset{\sim i}{A}}(x)$ .Here $\mu_{\overset{\sim i}{A}}(x)$ and $\nu_{\overset{\sim i}{A}}(x)$ are associated with each point in X, a real number in [0,1] with the value of $\mu_{\overset{\sim i}{A}}(x)$ and $\nu_{\overset{\sim i}{A}}(x)$ at X representing the grade of membership and non-membership of x in $\overset{\sim i}{A}$ .Thus closure the value of $\mu_{\overset{\sim i}{A}}(x)$ to unity and the value of $\nu_{\overset{\sim i}{A}}(x)$ to zero; higher the grade of membership and lower the grade of non-membership of x. When $\overset{\sim i}{A}$ is an ordinary set its membership function(non-membership function)can take on only two values 0 and 1.If $\mu_{\overset{\sim i}{A}}(x)=1$ and $\nu_{\overset{\sim i}{A}}(x)=0$ the element x does not belong to $\overset{\sim i}{A}$,similarly if $\mu_{\overset{\sim i}{A}}(x)=0$ and $\nu_{\overset{\sim i}{A}}(x)=1$the element x does not belong to $\overset{\sim i}{A}$ .An IFS becomes a fuzzy set $\tilde{A}$ when $\nu_{\overset{\sim i}{A}}(x)=0$ but $\mu_{\overset{\sim i}{A}}(x) \in [0,1] \forall x \in \overset{\sim i}{A}$.

Definition: Intuitionistic Fuzzy Set

Let a set X be fixed. An intuitionistic fuzzy set $\overset{\sim i}{A}$ in X is an object having the form $\overset{\sim i}{A} = \left\{\left\langle x, \mu_{\overset{\sim i}{A}}(x), \nu_{\overset{\sim i}{A}}(x)\right\rangle : x \in X\right\}$, where $\mu_{\overset{\sim i}{A}}(x) : X \to [0,1]$ and $\nu_{\overset{\sim i}{A}}(x) : X \to [0,1]$ define the degree of membership and degree of non-membership respectively, of the element $x \in X$ to the set $\overset{\sim i}{A}$, which is a subset of X, for every element of $x \in X$, $0 < \mu_{\overset{\sim i}{A}}(x) + \nu_{\overset{\sim i}{A}}(x) < 1$.

Definition: (α, β)-Level Intervals or (α, β)-Cuts

A set of (α, β)-cut, generated by IFS $\overset{\sim i}{A}$, where $\alpha, \beta \in [0,1]$ are fixed numbers such that $\alpha + \beta \leq 1$ is defined as

$$\overset{\sim i}{A}_{\alpha,\beta} = \left\{ \begin{array}{l} \left(x, \mu_{\overset{\sim i}{A}}(x), \nu_{\overset{\sim i}{A}}(x) \right) : x \in X, \\ \mu_{\overset{\sim i}{A}}(x) \ge \alpha, \nu_{\overset{\sim i}{A}}(x) \le \beta, \alpha, \beta \in [0,1] \end{array} \right\}$$

(α, β)-level interval or (α, β)-cut, denoted by $\overset{\sim i}{A}_{\alpha,\beta}$,is defined as the crisp set of elements x which belong to $\overset{\sim i}{A}$ atleast to the degree α and which does belong to $\overset{\sim i}{A}$ at most to the degree β .

2.1 Presentation of Intuitionistic Fuzzy Numbers and Its Properties

Intuitionistic fuzzy nuber was introduced by Burillo et al.[23] in 1994.The definition of IFN is given below

Definition: Intuitionistic Fuzzy Number

An intuitionistic fuzzy number $\overset{\sim i}{A}$ is

i. an intuitionistic fuzzy sub set of the real line

ii. normal i.e. there is any $x_0 \in R$ such that $\mu_{\overset{\sim i}{A}}(x_0) = 1$ (so $\nu_{\overset{\sim i}{A}}(x_0) = 0$)

iii. convex for the membership function $\mu_{\overset{\sim i}{A}}(x)$ i.e.

$$\mu_{\overset{\sim i}{A}}(\lambda x_1 + (1-\lambda) x_2) \ge \min\left(\mu_{\overset{\sim i}{A}}(x_1), \mu_{\overset{\sim i}{A}}(x_2) \right) \forall x_1, x_2 \in R, \lambda \in [0,1]$$

iv. concave for non-membership function $\nu_{\overset{\sim i}{A}}(x)$ i.e.

$$\nu_{\overset{\sim i}{A}}(\lambda x_1 + (1-\lambda) x_2) \le \max\left(\nu_{\overset{\sim i}{A}}(x_1), \nu_{\overset{\sim i}{A}}(x_2) \right) \forall x_1, x_2 \in R, \lambda \in [0,1]$$

$\mu_{\overset{\sim i}{A}}(x), \nu_{\overset{\sim i}{A}}(x)$

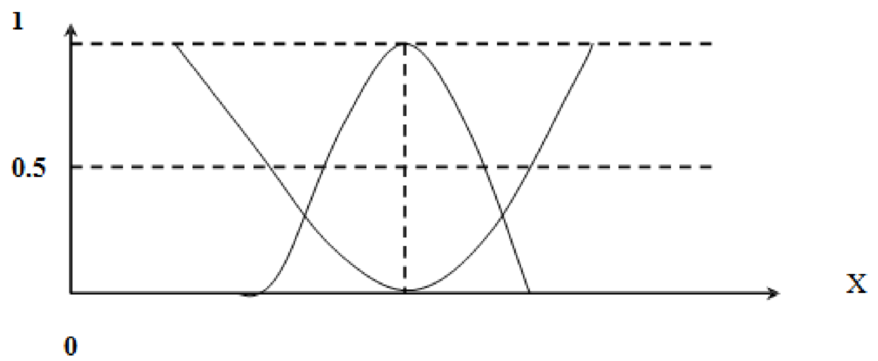

Fig.1 Membership and non membership functions of $\overset{\sim i}{A}$

Definition: Triangular Intuitionistic Fuzzy Number

A Triangular Intuitionistic Fuzzy Number (TIFN) $\overset{\sim i}{A}$ is an intuitionistic fuzzy set in R with following membership function $\left(\mu_{\overset{\sim i}{A}}(x)\right)$ and non-membership function $\left(\nu_{\overset{\sim i}{A}}(x)\right)$

$$\mu_{\overset{\sim i}{A}}(x)=\begin{cases}\dfrac{x-a_1}{b_1-a_1}, & a_1\le x\le b_1\\ \dfrac{c_1-x}{c_1-b_1}, & b_1\le x\le c_1\\ 0\quad, & otherwise\end{cases} \quad and \quad \nu_{\overset{\sim i}{A}}(x)=\begin{cases}\dfrac{b_1-x}{b_1-a'_1}, & a'_1\le x\le b_1\\ \dfrac{x-b_1}{c'_1-b_1}, & b_1\le x\le c'_1\\ 1\quad, & otherwise\end{cases}$$

Where $a'_1<a_1<b_1<c_1<c'_1$ and $\mu_{\overset{\sim i}{A}}(x),\nu_{\overset{\sim i}{A}}(x)\le 0.5$ for $\mu_{\overset{\sim i}{A}}(x)=\nu_{\overset{\sim i}{A}}(x)\ \forall\ x\in R$ .This TIFN is denoted by $\overset{\sim i}{A}_{TIFN}=\left(a_1,b_1,c_1;a'_1,b_1,c'_1\right)$.

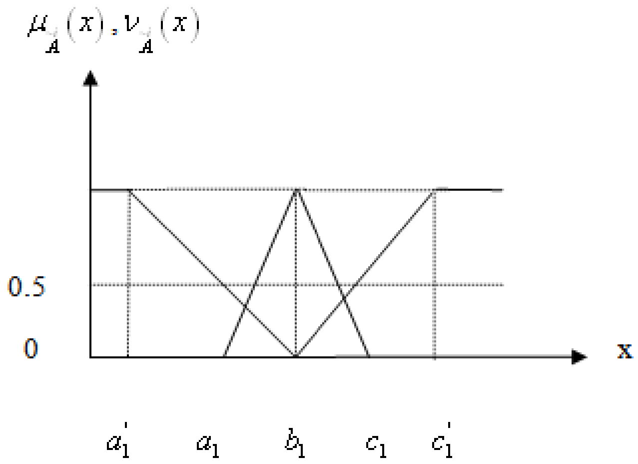

Fig.2 Membership and non-membership function of TIFN

Corollary 1. Transformation rule for the TIFN $\overset{\sim i}{A}_{TIFN}=\left(a_1,b_1,c_1;a'_1,b_1,c'_1\right)$ to triangular fuzzy number(TFN) $\tilde{A}_{TFN}=\left(a_1,b_1,c_1\right)$ is that $a_1=a'_1, c_1=c'_1$ and $\nu_{\overset{\sim i}{A}}(x)$=1-$\mu_{\overset{\sim i}{A}}(x), \forall x\in R$.

Proof. If we put $a_1=a'_1, c_1=c'_1$ and consider $\nu_{\overset{\sim i}{A}}(x)$=1-$\mu_{\overset{\sim i}{A}}(x), \forall x\in R$,membership and non- membership functions of TIFN become as follows

$$\mu_{\tilde{A}^i}(x)=\begin{cases}\dfrac{x-a_1}{b_1-a_1}, & a_1\le x\le b_1\\ \dfrac{c_1-x}{c_1-b_1}, & b_1\le x\le c_1\\ 0\quad, & otherwise\end{cases}\quad and\ \nu_{\tilde{A}^i}(x)=\begin{cases}\dfrac{b_1-x}{b_1-a_1}, & a_1\le x\le b_1\\ \dfrac{x-b_1}{c_1-b_1}, & b_1\le x\le c_1\\ 1\quad, & otherwise\end{cases}$$

It is clear from the above that $\nu_{\tilde{A}^i}(x)$ is the complement of $\mu_{\tilde{A}^i}(x)$,and hence the above transformation rules transform TIFN to TFN.

Corollary 2. Transformation rule from the TIFN $\tilde{A}^i_{TIFN}=(a_1,b_1,c_1;a'_1,b_1,c'_1)$ to crisp interval $[a_1,c_1]$ is that $a_1=a'_1 and\, c_1=c'_1$.

Proof. The proof is obvious.

Corollary 3. Transformation rule from the TIFN $\tilde{A}^i_{TIFN}=(a_1,b_1,c_1;a'_1,b_1,c'_1)$ to a real number a is $a_1=a'_1=b_1=c_1=c'_1=a$.

Proof. The proof is obvious.

3. Arithmetic Operations on Intuitionistic Fuzzy Number

The arithmetic operation(*) of two intuitionistic fuzzy numbers is a mapping of an input vector $X=[x_1,x_2]^T$ which is defined in the cartesian product space $R\times R$ onto an output y defined in the real space R. If $\tilde{A}^i_1$ and $\tilde{A}^i_2$ are IFN then their resultant after arithmetic operation (*) is also an IFN $\tilde{A}^i_1*\tilde{A}^i_2$. It is defined as $\tilde{A}^i_1*\tilde{A}^i_2=\left\{\left(y,\ \vee_{y=x_1*x_2}\left[\mu_{\tilde{A}^i_1}(x_1)\wedge\mu_{\tilde{A}^i_2}(x_2)\right],\ \wedge_{y=x_1*x_2}\left[\nu_{\tilde{A}^i_1}(x_1)\vee\nu_{\tilde{A}^i_2}(x_2)\right]\right)\forall x_1,x_2,y\in R\right\}$.

To calculate the arithmetic operation of IFNs, it is sufficient to determine the membership function and non-membership function as follows $\mu_{(\tilde{A}^i_1*\tilde{A}^i_2)}(y)=\vee_{y=x_1*x_2}\left[\mu_{\tilde{A}^i_1}(x_1)\wedge\mu_{\tilde{A}^i_2}(x_2)\right]$ and $\nu_{(\tilde{A}^i_1*\tilde{A}^i_2)}(y)=\wedge_{y=x_1*x_2}\left[\nu_{\tilde{A}^i_1}(x_1)\vee\nu_{\tilde{A}^i_2}(x_2)\right]$.

3.1 Some Arithmetic Operations of Intuitionistic Fuzzy Number based on (α,β)-Cuts Method

If $\tilde{A}^i$ is an intuitionistic fuzzy number (α,β)-level intervals or (α,β)-cut is given by

$$A_{\alpha,\beta} = \left\{\left[A_1(\alpha), A_2(\alpha)\right]; \left[A'_1(\beta), A'_2(\beta)\right], \alpha + \beta < 1, \alpha, \beta \in [0,1]\right\}$$

where $A_1(\alpha)\left(A_2(\alpha)\right)$ and $A'_2(\beta)\left(A'_1(\beta)\right)$ will be an increasing (decreasing) function of $\alpha, \beta \in [0,1]$ with $A_1(1) = A_2(1); A'_1(0) = A'_2(0)$.

If $\overset{\sim i}{A}$ is a TIFN then

(i) $A_1(\alpha)$ and $A_2'(\beta)$ will be continuous, monotonic increasing function of $\alpha, \beta \in [0,1]$.

(ii) $A_2(\alpha)$ and $A'_1(\beta)$ will be continuous, monotonic decreasing function of $\alpha, \beta \in [0,1]$ and

(iii) $A_1(1) = A_2(1); A'_1(0) = A'_2(0)$.

In other form, if $\overset{\sim i}{A} = \left(a_1, b_1, c_1; a'_1, b_1, c'_1\right)$ is a TIFN then (α, β)-level intervals or (α, β)-cut is

$$\left\{\left[a_1 + \alpha\left(b_1 - a_1\right), c_1 - \alpha\left(c_1 - b_1\right)\right]; (-\infty, b_1 + \beta\left(c'_1 - b_1\right)] \cup [b_1 - \beta\left(b_1 - a'_1\right), \infty)\right\}$$

$$i.e\ \ -\infty < x \le b_1 + \beta\left(c'_1 - b_1\right)\ and\ b_1 - \beta\left(b_1 - a'_1\right) \le x < \infty\ \ \forall \alpha, \beta \in [0,1]\ and\ \alpha + \beta < 1.$$

Property 3.1 (a) If TIFN $\overset{\sim i}{A} = \left(a_1, b_1, c_1; a'_1, b_1, c'_1\right)$ and y = ka (k > 0) then $\overset{\sim i}{Y} = k\overset{\sim i}{A}$ is a TIFN (ka_1, kb_1, kc_1; ka'_1, kb_1, kc'_1).

(b) If y = ka (k < 0) then $\overset{\sim i}{Y} = k\overset{\sim i}{A}$ is a TIFN (kc_1, kb_1, ka_1; kc'_1, kb_1, ka'_1).

Proof (a) When k > 0, with the transformation y = ka, we can find the membership function.

For membership (acceptance) function of TIFN $\overset{\sim i}{Y} = k\overset{\sim i}{A}$ by α - cut method.

Left-hand and right-hand α-cut of $\overset{\sim i}{A}$ is

$\mu_{\overset{\sim i}{A}}(x) \ge \alpha \Rightarrow [a_1 + \alpha(b_1 - a_1), c_1 - \alpha(c_1 - b_1)]$ for any $\alpha \in [0, 1]$

i.e. $x \in [a_1 + \alpha(b_1 - a_1), c_1 - \alpha(c_1 - b_1)]$

so, y (= ka) $\in [ka_1 + \alpha(kb_1 - ka_1), kc_1 - \alpha(kc_1 - kb_1)]$

Thus, we get the membership function of $\overset{\sim i}{Y} = k\overset{\sim i}{A}$ as

$$\mu_{\tilde{y}}(y) = \begin{cases} \dfrac{y - ka_1}{kb_1 - ka_1} & \text{for} \quad ka_1 \le y \le kb_1 \\ \dfrac{kc_1 - y}{kc_1 - kb_1} & \text{for} \quad kb_1 \le y \le kc_1 \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

Hence the rule is proved for membership function.

For non-membership function, β-cut of $\tilde{A}^i$ is

$\upsilon_{\tilde{A}^i}(x) \le \beta \Rightarrow (-\infty, b_1 + \beta(c_1' - b_1)] \cup [b_1 - \beta(b_1 - a_1'), \infty)$ for any $\beta \in [0, 1]$

i.e. $x \le b_1 + \beta(c'_1 - b_1)$ and $x \ge b_1 - \beta(b_1 - a'_1)$.

so, y (= ka) $\in (-\infty, kb_1 + \beta(kc_1' - kb_1)] \cup [kb_1 - \beta(kb_1 - ka_1'), \infty)$

Thus, we get the non-membership function of $\tilde{Y}^i = k\tilde{A}^i$ as

$$\upsilon_{\tilde{y}}(y) = \begin{cases} \dfrac{kb_1 - y}{kb_1 - ka_1'} & \text{for} \quad ka'_1 \le y \le kb_1 \\ \dfrac{y - kb_1}{kc_1' - kb_1} & \text{for} \quad kb_1 \le y \le kc'_1 \\ 1 & \text{otherwise} \end{cases} \tag{3.2}$$

Hence rule is proved for non-membership function.

Thus we have $\tilde{Y}^i = k\tilde{A}^i = (ka_1, kb_1, kc_1; ka'_1, kb_1, kc'_1)$ is a TIFN.

(b) Similarly we can prove that, if y = ka, (k < 0) then

$$\mu_{\tilde{y}}(y) = \begin{cases} \dfrac{y - kc_1}{kb_1 - kc_1} & \text{for} \quad kc_1 \le y \le kb_1 \\ \dfrac{ka_1 - y}{ka_1 - kb_1} & \text{for} \quad kb_1 \le y \le ka_1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\upsilon_{\tilde{y}}(y) = \begin{cases} \dfrac{kb_1 - y}{kb_1 - kc_1'} & \text{for} \quad kc'_1 \le y \le kb_1 \\ \dfrac{y - kb_1}{ka_1' - kb_1} & \text{for} \quad kb_1 \le y \le ka'_1 \\ 1 & \text{otherwise} \end{cases} \tag{3.3}$$

Property 3.2. If $\overset{\sim i}{A} = (a_1, b_1, c_1; a'_1, b_1, c'_1)$ and $\overset{\sim i}{B} = (a_2, b_2, c_2; a'_2, b_2, c'_2)$ are two TIFN, then $\overset{\sim i}{C} = \overset{\sim i}{A} \oplus \overset{\sim i}{B}$ is also TIFN

$\overset{\sim i}{A} \oplus \overset{\sim i}{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2; a'_1 + a'_2, b_1 + b_2, c'_1 + c'_2)$.

Proof : With the transformation $z = x + y$, we can find the membership function of acceptance (membership) function of IFS $\overset{\sim i}{C} = \overset{\sim i}{A} \oplus \overset{\sim i}{B}$ by α-cut method.

α-cut of $\overset{\sim i}{A}$ is $[a_1 + \alpha (b_1 - a_1), c_1 - \alpha (c_1 - b_1)]$ for any $\alpha \in [0, 1]$

i.e., $x \in [a_1 + \alpha (b_1 - a_1), c_1 - \alpha (c_1 - b_1)]$

α-cut of $\overset{\sim i}{B}$ is $[a_2 + \alpha (b_2 - a_2), c_2 - \alpha (c_2 - b_2)]$ for any $\alpha \in [0, 1]$

i.e., $y \in [a_2 + \alpha (b_2 - a_2), c_2 - \alpha (c_2 - b_2)]$

So, $z\ (= x + y) \in [a_1 + a_2 + \alpha ((b_1 - a_1) + (b_2 - a_2)), c_1 + c_2 - \alpha ((c_1 - b_1) + (c_2 - b_2))]$

So, we get the membership (acceptance) function of $\overset{\sim i}{C} = \overset{\sim i}{A} \oplus \overset{\sim i}{B}$ as

$$\mu_{\tilde{c}}(z) = \begin{cases} \dfrac{z - a_1 - a_2}{(b_1 - a_1) + (b_2 - a_2)} & \text{for } a_1 + a_2 \le z \le b_1 + b_2 \\ \dfrac{c_1 + c_2 - z}{(c_1 - b_1) + (c_2 - b_2)} & \text{for } b_1 + b_2 \le z \le c_1 + c_2 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

Hence addition rule is proved for membership function.

For non-membership function, β-cut of $\overset{\sim i}{A}$ is

$\upsilon_{\overset{\sim i}{A}}(x) \le \beta \Rightarrow (-\infty, b_1 + \beta(c_1' - b_1)] \cup [b_1 - \beta(b_1 - a_1'), \infty)$ for any $\beta \in [0, 1]$

i.e. $x \le b_1 + \beta(c'_1 - b_1)$ and $x \ge b_1 - \beta(b_1 - a'_1)$.

β-cut for non-membership function of $\overset{\sim i}{B}$ is for any $\beta \in [0, 1]$

$\nu_{\tilde{B}^i}(y) \le \beta \Rightarrow (-\infty, b_2 + \beta(c'_2 - b_2)] \cup [b_2 - \beta(b_2 - a'_2), \infty)$

i.e. $y \le c_2 + \beta(d'_2 - c_2)$ and $y \ge b_2 - \beta(b_2 - a'_2)$

So, z (= x + y) $\in$ $(-\infty, b_1 + b_2 + \beta(c'_1 + c'_2 - b_1 - b_2)] \cup [b_1 + b_2 - \beta(b_1 + b_2 - a'_1 - a'_2), \infty)$

So, we have the non-membership (rejection) function of $\tilde{C}^i = \tilde{A}^i \oplus \tilde{B}^i$ as

$$\upsilon_{\tilde{C}^i}(z) = \begin{cases} \dfrac{b_1 + b_2 - z}{(b_1 - a'_1) + (b_2 - a'_2)} & \text{for } a'_1 + a'_2 \le z \le b_1 + b_2 \\ \dfrac{z - b_1 - b_2}{(c'_1 - b_1) + (c'_2 - b_2)} & \text{for } b_1 + b_2 \le z \le c'_1 + c'_2 \\ 1 & \text{otherwise} \end{cases} \tag{3.5}$$

Hence addition rule is proved for non-membership function.

Thus we have

$\tilde{A}^i \oplus \tilde{B}^i = (a_1 + a_2, b_1 + b_2, c_1 + c_2; a'_1 + a'_2, b_1 + b_2, c'_1 + c'_2)$
as a TIFN.

Note : If we have the transformation $\tilde{C}^i = k_1 \tilde{A}^i \oplus k_2 \tilde{B}^i$ (k_1, k_2 are (not all zero) real numbers) then the intuitionistic fuzzy set $\tilde{C}^i = k_1 \tilde{A}^i \oplus k_2 \tilde{B}^i$ is the following TIFN.

i. $(k_1a_1 + k_2a_2, k_1b_1 + k_2b_2, k_1c_1 + k_2c_2 ; k_1a'_1 + k_2a'_2, k_1b_1 + k_2b_2, k_1c'_1 + k_2c'_2)$ if $k_1 > 0, k_2 \ge 0$ or $k_1 \ge 0, k_2 > 0$,

ii. $(k_1a_1 + k_2 c_2, k_1b_1 + k_2b_2, k_1c_1 + k_2a_2; k_1a'_1 + k_2c'_2, k_1b_1 + k_2b_2, k_1c'_1 + k_2 a'_2)$ if $k_1 > 0, k_2 \le 0$ or $k_1 \ge 0, k_2 < 0$,

iii. $(k_1c_1 + k_2a_2, k_1b_1 + k_2b_2, k_1a_1 + k_2c_2; k_1c'_1 + k_2a'_2, k_1b_1 + k_2b_2, k_1 a'_1 + k_2 c'_2)$ if $k_1 < 0, k_2 \ge 0$ or $k_1 \le 0, k_2 > 0$,

iv. $(k_1c_1 + k_2c_2, k_1b_1 + k_2b_2, k_1a_1 + k_2a_2; k_1c'_1 + k_2c'_2, k_1b_1 + k_2b_2, k_1a'_1 + k_2a'_2)$ if $k_1 < 0, k_2 \le 0$ or $k_1 \le 0, k_2 < 0$.

Example 3.2.1: Let us consider two TIFN $\tilde{A}^i = (2, 3, 4.5; 1.5, 3, 5)$ and $\tilde{B}^i = (2.5, 4, 5; 2, 4, 6)$ using (3.4) and (3.5) addition is defined by $\tilde{A}^i \oplus \tilde{B}^i = (4.5, 7, 9.5; 3.5, 7, 11)$ with membership and non-membership function as follows

$$\mu_{\tilde{A}^i \oplus \tilde{B}^i}(x) = \begin{cases} \dfrac{x-4.5}{2.5}, if\ 4.5 \le x \le 7 \\ \dfrac{9.5-x}{2.5}, if\ 7 \le x \le 9.5 \\ 0 \quad , otherwise \end{cases} and\ \nu_{\tilde{A}^i \oplus \tilde{B}^i}(x) = \begin{cases} \dfrac{7-x}{3.5}, if\ 3.5 \le x \le 7 \\ \dfrac{x-7}{4}, if\ 7 \le x \le 11 \\ 1 \quad , otherwise \end{cases}$$

Property 3.3: If $\tilde{A}^i$ = (a_1, b_1, c_1; a'_1, b_1, c'_1) and $\tilde{B}^i$ = (a_2, b_2, c_2; a'_2, b_2, c'_2) are two TIFN, then $\tilde{P}^i = \tilde{A}^i \otimes \tilde{B}^i$ is approximated TIFN $\tilde{A}^i \otimes \tilde{B}^i$ = ($a_1 a_2$, $b_1 b_2$, $c_1 c_2$; $a'_1 a'_2$, $b_1 b_2$, $c'_1 c'_2$) (as shown in figure 3 with dotted line).

Proof: By the transformation z = x × y, we can find the membership function of acceptance (membership) IFS $\tilde{C}^i = \tilde{A}^i \otimes \tilde{B}^i$ by α-cut method.

α-cut for membership function of $\tilde{A}^i$ is $\mu_{\tilde{A}^i}(x) \ge \alpha$

$\Rightarrow [a_1 + \alpha (b_1 - a_1), c_1 - \alpha (c_1 - b_1)]\ \forall \alpha \in [0, 1]$
i.e., $x \in [a_1 + \alpha (b_1 - a_1), c_1 - \alpha (c_1 - b_1)]$

α-cut for membership function of $\tilde{B}^i$ is $\mu_{\tilde{B}^i}(x) \ge \alpha$

$\Rightarrow [a_2 + \alpha (b_2 - a_2), c_2 - \alpha (c_2 - b_2)]\ \forall \alpha \in [0, 1]$
i.e., $y \in [a_2 + \alpha (b_2 - a_2), c_2 - \alpha (c_2 - b_2)]$
So, $z (= x \times y) \in [(a_1 + \alpha (b_1 - a_1)) (a_2 + \alpha (b_2 - a_2), (c_1 - \alpha (c_1 - b_1)) (c_2 - \alpha(c_2 - b_2))]$

So, we get the membership (acceptance) function of $\tilde{C}^i = \tilde{A}^i \otimes \tilde{B}^i$ as

$$\mu_{\tilde{c}}(z) = \begin{cases} \dfrac{-B_1 + \sqrt{B_1^2 - 4A_1(a_1 a_2 - z)}}{2A_1} & \text{for } a_1 a_2 \le z \le b_1 b_2 \\ \dfrac{B_2 - \sqrt{B_2^2 - 4A_2(c_1 c_2 - z)}}{2A_2} & \text{for } b_1 b_2 \le z \le c_1 c_2 \\ 0 & \text{otherwise} \end{cases} \quad (3.7)$$

where $A_1 = (b_1 - a_1)(b_2 - a_2)$, $B_1 = a_2 (b_1 - a_1) + a_1 (b_2 - a_2)$, $A_2 = (c_1 - b_1)(c_2 - b_2)$ and $B_2 = -(c_2 (c_1 - b_1) + c_1 (c_2 - b_2))$.

For non-membership function, β-cut of $\tilde{A}^i$ is
$\upsilon_{\tilde{A}^i}(x) \le \beta \Rightarrow (-\infty, b_1 + \beta(c'_1 - b_1)] \cup [b_1 - \beta(b_1 - a'_1), \infty)$ for any $\beta \in [0, 1]$

i.e. $x \le b_1 + \beta\left(c'_1 - b_1\right)$ and $x \ge b_1 - \beta\left(b_1 - a'_1\right)$.

β-cut for non-membership function of $\tilde{B}^i$ is $\nu_{\tilde{B}^i}(y) \le \beta \Rightarrow (-\infty, b_2 + \beta\left(c'_2 - b_2\right)] \cup [b_2 - \beta\left(b_2 - a'_2\right), \infty)$ for any β∈ [0, 1]

i.e. $y \le b_2 + \beta\left(c'_2 - b_2\right)$ and $y \ge b_2 - \beta\left(b_2 - a'_2\right)$

i.e. z∈

$$(-\infty, \left(b_1 - \beta\left(b_1 - a'_1\right)\right)\left(b_2 - \beta\left(b_2 - a'_2\right)\right)] \cup [\left(b_1 + \beta\left(c'_1 - b_1\right)\right)\left(b_2 + \beta\left(c'_2 - b_2\right)\right), \infty)$$

So, we have the non-membership (rejection) function $\tilde{C}^i = \tilde{A}^i \otimes \tilde{B}^i$

$$\upsilon_{\tilde{C}}(z) = \begin{cases} \dfrac{B'_1 - \sqrt{B_1'^2 - 4A'_1(b_1 b_2 - z)}}{2A'_1} & \text{for } a'_1 a'_2 \le z \le b_1 b_2 \\ \dfrac{-B'_2 + \sqrt{B_2'^2 - 4A'_2(c'_1 c'_2 - z)}}{2A'_2} & \text{for } b_1 b_2 \le z \le c'_1 c'_2 \\ 1 & \text{Otherwise} \end{cases} \qquad (3.7)$$

where $A'_1 = (b_1 - a'_1)(b_2 - a'_2)$, $B'_1 = b_2(b_1 - a'_1) + b_1(b_2 - a'_2)$,
$A'_2 = (c'_1 - b_1)(c'_2 - b_2)$ and $B'_2 = b_2(c'_1 - b_1) + b_1(c'_2 - b_2)$,

So, $\tilde{C}^i = \tilde{A}^i \otimes \tilde{B}^i$ by (3.6) and (3.7) is a triangular shaped intuitionistic fuzzy number. It can be approximated to a TIFN $\tilde{A}^i \otimes \tilde{B}^i = (a_1a_2, b_1b_2, c_1 c_2; a'_1 a'_2, b_1 b_2, c'_1 c'_2)$ (as shown in figure 3 with dotted line).

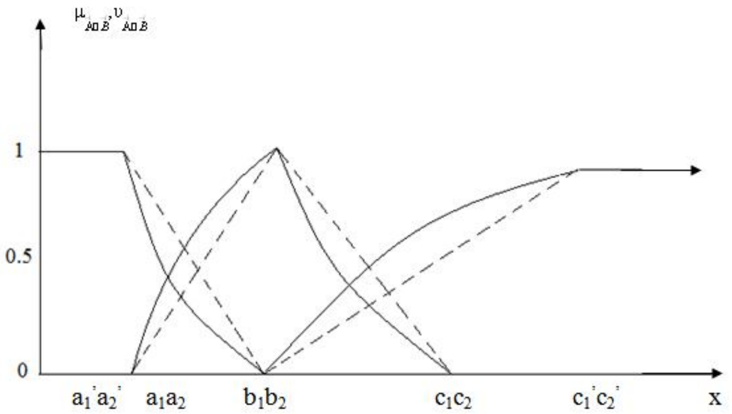

Fig.3 Membership and non-membership for product of two TIFN

Example 3.3.1 Let us consider two TIFN $\tilde{A}^i$ = (3, 4, 5; 2.5, 4, 5.5) and $\tilde{B}^i$ = (4, 5, 6; 3, 5,6.5) then $\tilde{A}^i \otimes \tilde{B}^i$ is defined by $\tilde{A}^i \otimes \tilde{B}^i$ = (12, 20, 30; 7.5, 20,35.75) with membership and non-membership function as follows

$$\mu_{\tilde{A}^i \otimes \tilde{B}^i}(x) = \begin{cases} \dfrac{-7+\sqrt{49-4(12-x)}}{2} & \text{for } 12 \le x \le 20 \\ \dfrac{-11-\sqrt{121-4(30-x)}}{2} & \text{for } 20 \le x \le 30 \\ 0 & \text{Otherwise} \end{cases}$$

$$\upsilon_{\tilde{A}^i \otimes \tilde{B}^i}(x) = \begin{cases} \dfrac{15.5-\sqrt{240.25-12(20-x)}}{6} & \text{for } 7.5 \le x \le 20 \\ \dfrac{-13.5+\sqrt{182.25-9(35.75-x)}}{4.5} & \text{for } 20 \le x \le 35.75 \\ 1 & \text{otherwise} \end{cases}$$

Above $\tilde{A}^i \otimes \tilde{B}^i$ is a triangular shaped intuitionistic fuzzy number. It can be approximated to TIFN as $\tilde{A}^i \otimes \tilde{B}^i$ = (12, 20, 30; 7.5, 20, 35.75) with membership and non-membership function as follows:

$$\mu_{\tilde{A}^i \otimes \tilde{B}^i}(x) = \begin{cases} \dfrac{x-12}{8} & for\, 12 \le x \le 20 \\ \dfrac{20-x}{10} & for\, 20 \le x \le 30 \\ 0 & otherwise \end{cases}, \quad \nu_{\tilde{A}^i \otimes \tilde{B}^i}(x) = \begin{cases} \dfrac{20-x}{12.5} & for\, 7.5 \le x \le 20 \\ \dfrac{x-20}{15.25} & for\, 20 \le x \le 30.25 \\ 1 & otherwise \end{cases}$$

4. Imprecise reliability of series , parallel, parallel-series and series-parallel system using arithmetic operations on triangular intuitionistic fuzzy numbers.

Here we present the expressions for evaluation of the imprecise reliability of a series , parallel, parallel-series and series-parallel system where the reliability of each component of the system is represented by a triangular intuitionistic fuzzy number.

(i)Series System

Let us consider a series system consisting of n components as shown in Fig. 4. The intuitionistic fuzzy reliability $\tilde{R}^i_{ss}$ of the series system shown below can be evaluated by using the expression as follows:

$\tilde{R}^{i}_{ss} = \tilde{R}^{i}_{1} \otimes \tilde{R}^{i}_{2} \otimes \ldots\ldots \otimes \tilde{R}^{i}_{n} = \{ (r_{11}, r_{12}, r_{13}; r'_{11}, r_{12}, r'_{13}) \otimes (r_{21}, r_{22}, r_{23}; r'_{21}, r_{22}, r'_{23}) \otimes \ldots\ldots\ldots \otimes (r_{n1}, r_{n2}, r_{n3}; r'_{n1}, r_{n2}, r'_{n3})\}$

It can be approximated to a TIFN as

$$\cong \left(\prod_{j=1}^{n} r_{j1}, \prod_{j=1}^{n} r_{j2}, \prod_{j=1}^{n} r_{j3}; \prod_{j=1}^{n} r'_{j1}, \prod_{j=1}^{n} r_{j2}, \prod_{j=1}^{n} r'_{j3} \right)$$

where $\tilde{R}^{i}_{j} = (r_{j1}, r_{j2}, r_{j3} ; r'_{j1}, r_{j2}, r'_{j3})$ is intuitionistic fuzzy reliability of the jth component for j = 1, 2,….., n.

In $\rightarrow \tilde{R}^{i}_{1}\ \tilde{R}^{i}_{2}\ \tilde{R}^{i}_{n} \rightarrow$ out

Fig.4 Diagram of series system

(ii) Parallel System

Let us consider a parallel system consisting of n components as shown in Fig. 5. The fuzzy reliability $\tilde{R}^{i}_{ps}$ of the parallel system shown below can be evaluated by using the expression as follows:

$$\tilde{R}^{i}_{ps} = 1 \ominus \prod_{j=1}^{n} (1 \ominus \tilde{R}^{i}_{j}) = 1 \ominus [(1 \ominus (r_{11}, r_{12}, r_{13} ; r'_{11}, r_{12}, r'_{13})) \otimes \ldots\ldots\ldots\ldots \otimes (1 \ominus (r_{n1}, r_{n2}, r_{n3} ; r'_{n1}, r_{n2}, r'_{n3}))]$$

$$= [1 - \prod_{j=1}^{n} (1 - r_{j1}), 1 - \prod_{j=1}^{n} (1 - r_{j2}), 1 - \prod_{j=1}^{n} (1 - r_{j3}) ;$$

$$1 - \prod_{j=1}^{n} (1 - r'_{j1}), 1 - \prod_{j=1}^{n} (1 - r_{j2}), 1 - \prod_{j=1}^{n} (1 - r'_{j3})]$$

It is an approximated TIFN, where $\tilde{R}^{i}_{j} = (r_{j1}, r_{j2}, r_{j3} ; r'_{j1}, r_{j2}, r'_{j3})$ is an intuitionistic fuzzy reliability of the jth component for j = 1, 2,…., n.

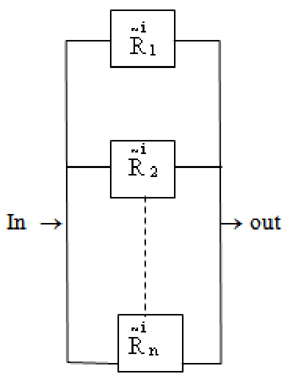

Fig. 5 Diagram of parallel system

(iii) Parallel-series System

Consider a parallel-series system consisting of m branches connected in parallel and each branch contains n components as shown in Fig.6.The fuzzy reliability $\tilde{R}^{i}_{pss}$ of the parallel-series system shown in Fig.6 can be evaluated as follows:

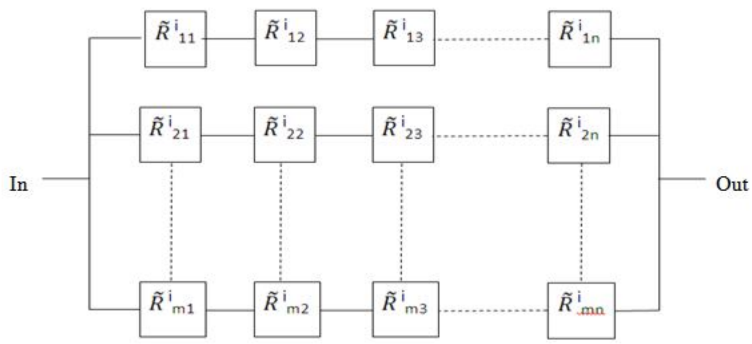

Fig. 6: Parallel-Series system

$$\tilde{R}^{i}_{pss} = 1\Theta \underset{k=1}{\overset{m}{\square}} (1\ \Theta\ (\underset{i=1}{\overset{n}{\square}} \tilde{R}^{i}_{ki}))$$

$$= \left[1-\prod_{k=1}^{m}\left(1-\left(\prod_{i=1}^{n} x_{ki}\right)\right), 1-\prod_{k=1}^{m}\left(1-\left(\prod_{i=1}^{n} y_{ki}\right)\right), 1-\prod_{k=1}^{m}\left(1-\left(\prod_{i=1}^{n} z_{ki}\right)\right); 1-\prod_{k=1}^{m}\left(1-\left(\prod_{i=1}^{n} x'_{ki}\right)\right), 1-\prod_{k=1}^{m}\left(1-\left(\prod_{i=1}^{n} y_{ki}\right)\right), 1-\prod_{k=1}^{m}\left(1-\left(\prod_{i=1}^{n} z'_{ki}\right)\right)\right]$$

Where $\tilde{R}^{i}_{ki} = \left(x_{ki}, y_{ki}, z_{ki}; x'_{ki}, y_{ki}, z'_{ki}\right)$ represents the reliability of the ith components at kth branch.

(iv) Series-parallel System

Consider a series-parallel system consisting of n stages connected in series and each stage contains m components as shown in Fig.7.The fuzzy reliability $\tilde{R}^{i}_{sps}$ of the series-parallel system shown in Fig.5 can be evaluated as follows:

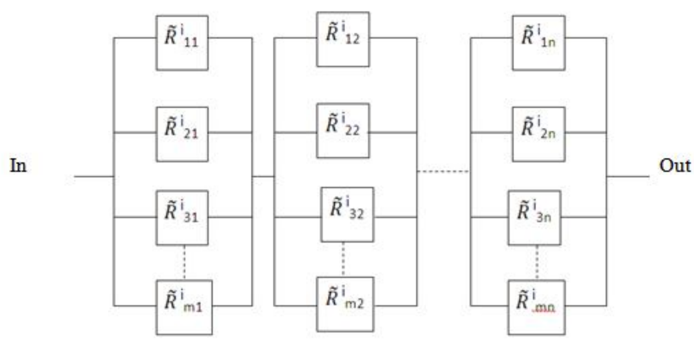

Fig. 7: Series-Parallel system

$$\tilde{R}^{i}_{sps} = \prod_{k=1}^{n} (1 \ominus (\prod_{i=1}^{m} \tilde{R}^{i}_{ik}))$$

$$= \left[\left(\prod_{k=1}^{n}\left(1-\left(\prod_{i=1}^{m} x_{ik}\right)\right), \prod_{k=1}^{n}\left(1-\left(\prod_{i=1}^{m} y_{ik}\right)\right), \prod_{k=1}^{n}\left(1-\left(\prod_{i=1}^{m} z_{ik}\right)\right)\right); \left(\prod_{k=1}^{n}\left(1-\left(\prod_{i=1}^{m} x'_{ik}\right)\right), \prod_{k=1}^{n}\left(1-\left(\prod_{i=1}^{m} y'_{ik}\right)\right), \prod_{k=1}^{n}\left(1-\left(\prod_{i=1}^{m} z'_{ik}\right)\right)\right)\right]$$

where $\tilde{R}^{i}_{ik} = \left(x_{ik}, y_{ik}, z_{ik}; x'_{ik}, y_{ik}, z'_{ik}\right)$ represent the reliability of the ith component at kth stage.

5.1 Calculation of system failure using triangular intuitionistic fuzzy number

Failure to start of an automobile depends on different facts. The facts are battery low charge, ignition failure and fuel supply failure. There are two sub-factors of each of facts. The fault- tree of failure to start of automobile is shown in the figure 8.

$\tilde{F}^{i}_{fs}$ represents the system failure to start of automobile.

$\tilde{F}^{i}_{blc}$ represents the failure to start of automobile due to battery low charge.

$\tilde{F}^{i}_{if}$ represents the failure to start of automobile due to ignition failure.

$\tilde{F}^{i}_{fsf}$ represents the failure to start of automobile due to fuel supply failure.

$\tilde{F}^{i}_{lbf}$ represents the failure to start of automobile due to low battery fluid.

$\tilde{F}^{i}_{bis}$ represents the failure to start of automobile due to battery internal short.

$\tilde{F}^{i}_{whf}$ represents the failure to start of automobile due to wire harness failure.

$\tilde{F}^{i}_{spf}$ represents the failure to start of automobile due to spark plug failure.

$\tilde{F}^{i}_{fif}$ represents the failure to start of automobile due to fuel injector failure.

$\tilde{F}^{i}_{fpf}$ represents the failure to start of automobile due to fuel pump failure.

The intuitionistic fuzzy failure to start of an automobile can be calculated when the failures of the occurrence of basic fault events are known. Failure to start of an automobile can be evaluated by using the following steps.

Step1.

$$\tilde{F}^{i}_{blc} = 1 \ominus (1 \ominus \tilde{F}^{i}_{lbf})(1 \ominus \tilde{F}^{i}_{bis})$$

$$\tilde{F}^{i}_{if} = 1 \ominus (1 \ominus \tilde{F}^{i}_{whf})(1 \ominus \tilde{F}^{i}_{spf}) \qquad (5.1.1)$$

$$\tilde{F}^{i}_{fsf} = 1 \ominus (1 \ominus \tilde{F}^{i}_{fif})(1 \ominus \tilde{F}^{i}_{fpf})$$

Step2.

$$\tilde{F}^{i}_{fs} = 1 \ominus (1 \ominus \tilde{F}^{i}_{blc})(1 \ominus \tilde{F}^{i}_{if})(1 \ominus \tilde{F}^{i}_{fsf}) \qquad (5.1.2)$$

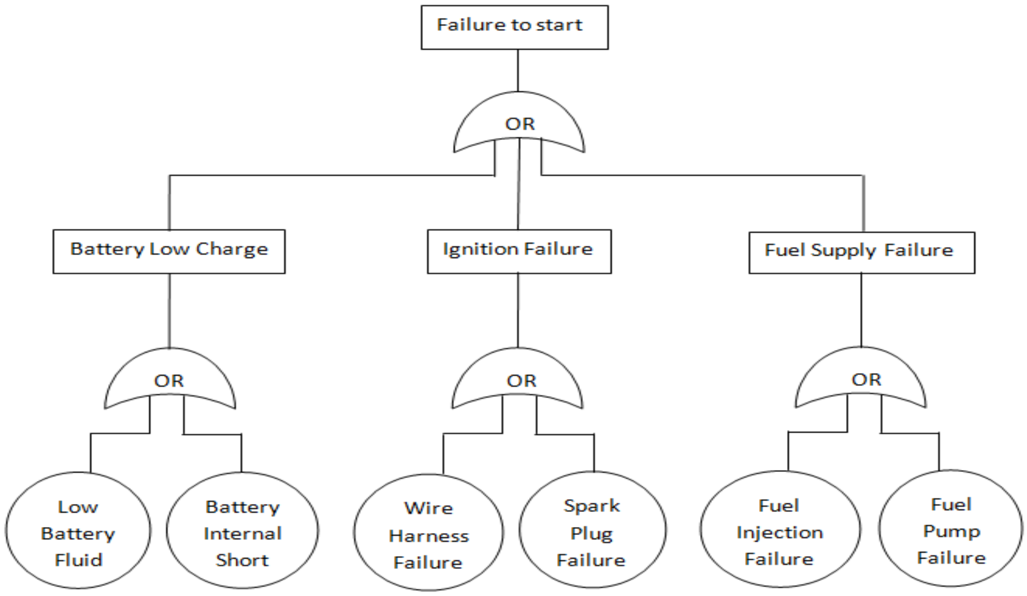

Fig 8: Fault-tree of failure to start of an automobile

5.2 Result of start to failure of automobile using TIFN

Numerical of starting failure of an automobile using fault tree analysis with intuitionistic fuzzy failure rate.The components failure rates as TIFN are given by

$\tilde{F}^{i}_{lbf} = (0.03,0.04,0.05;0.02,0.04,0.06)$, $\tilde{F}^{i}_{bis} = (0.03,0.05,0.06;0.02,0.05,0.07)$,

$\tilde{F}^{i}_{whf} = (0.02,0.03,0.04;0.01,0.03,0.05)$, $\tilde{F}^{i}_{spf} = (0.04,0.06,0.07;0.02,0.06,0.09)$,

$\tilde{F}^{i}_{fif} = (0.06,0.07,0.09;0.04,0.07,0.1)$ and $\tilde{F}^{i}_{fpf} = (0.05,0.07,0.08;0.03,0.07,0.09)$

Using (5.1.1) in the step-1 we have the following results

$\tilde{F}^{i}_{blc} = (0.0591, 0.088, 0.107; 0.0396, 0.088, 0.1258)$,

$\tilde{F}^{i}_{if} = (0.0592, 0.0882, 0.1072; 0.0298, 0.0882, 0.1355)$,

$\tilde{F}^{i}_{fsf} = (0.107, 0.1351, 0.1628; 0.0688, 0.1351, 0.181)$

Using (5.1.2) in the second and final step , we get the failure to start of automobile, calculated fuzzy failure to start of an automobile as shown in figure 9, represented by the following TIFN

$\tilde{F}^{i}_{fs} = (0.2095175, 0.2807824, 0.3325252; 0.1323264, 0.2807824, 0.3810441)$

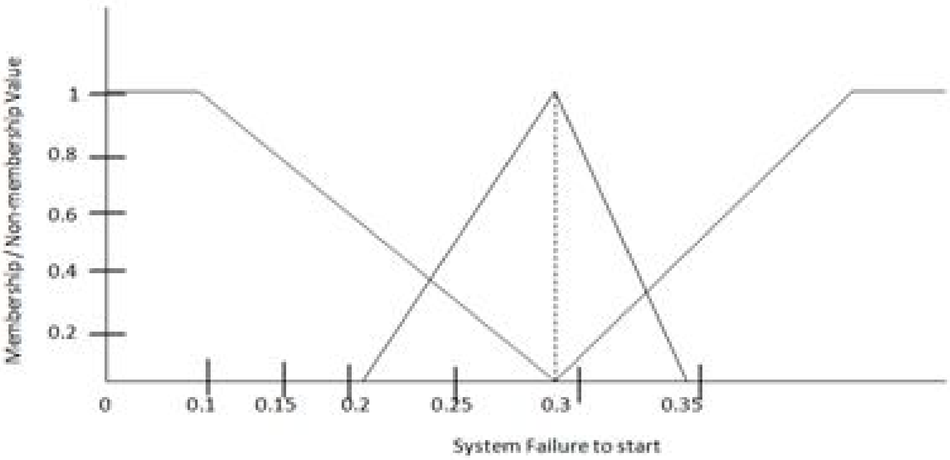

Fig 9: TIFN representing the system failure to start of an automobile

Here system failure to start an automobile is about 0.2807824 with tolerance level of acceptance [0.2095175, 0.3325252] and tolerance level of rejection [0.1323264, 0.3008815].

Note: - We have considered here failure rates as TIFN. Here left and right divergences are not significant since elements of these TIFN are very small, so to calculate system failure we have considered TIFN instead of trapezoidal shaped intuitionistic fuzzy number.

6. Conclusion

In this paper, a definition of IFN according to the approach of fuzzy number presentation is proposed. Also some arithmetic operation of proposed TIFN is evaluated based on intuitionistic fuzzy (α,β) cut method. Here, a method to analyze system reliability which is based on intuitionistic fuzzy set theory has been presented, where the components of the system are represented by triangular intuitionistic fuzzy number. Some Arithmetic operations over triangular intuitionistic fuzzy number are used to analyze the fuzzy reliability of the series system , parallel system, series-parallel and parallel-series system. Intuitionistic fuzzy fault- tree are used to analyze the imprecise failure to start of automobile with desired event. In imprecise situation several real life Operations Research and scientific models may be described and evaluated by triangular intuitionistic fuzzy number. The major advantage of using intuitionistic fuzzy sets over fuzzy sets is that intuitionistic fuzzy sets separate the positive and the negative evidence for the membership of an element in a set.

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