International Journal of Fuzzy Mathematics and Systems.
ISSN 2248-9940 Volume 2, Number 4 (2012), pp. 401-406

http://www.ripublication.com

Strongly Generalized Closed Sets with Respect to the Ideal

[1]S.Maragathavalli and [2]E.Kungumaraj

[1]Department of Mathematics, Sree Saraswathi Thyagaraja College,
Pollachi, Tamilnadu, India
Email: smvalli@rediffmail.com
[2]Department of Mathematics, Sree Saraswathi Thyagaraja College,
Pollachi, Tamilnadu, India
Email: kungumaraje@yahoo.co.in

Abstract

Here we have introduced strongly generalized closed sets with respect to the ideal. Also we have studied the properties of the strongly generalized closed sets with respect to an ideal.

Key words and Phrases: Topological spaces, generalized closed set, strongly generalized closed set and Ideal.

1. Introduction

Nowadays ideals are playing very important role in General Topology. It was the works of Newcomb[10], Rancin [13], Samuels [14] and Hamlet and Jankovic ([2, 3, 4, 5, 6]) which motivated the research in applying topological ideals to generalize the most basic properties in General Topology. A nonempty collection I of subsets on a topological space (X , τ) is called a topological ideal [8] if it satisfies the following two conditions:

(i). If $A \in I$ and $B \subset A$ implies $B \in I$ (heredity)

(ii). If $A \in I$ and $B \in I$, then $A \cup B \in I$ (finite additivity)

Throughout this paper (X, τ) will denote topological space. For a subset A of a topological space (X, τ). The closure of A (denoted as cl(A)) is defined as the intersection of all closed sets containing A and the interior of A (denoted as int(A)) is defined as the union of all open sets contained in A. Let $A \subset B \subset X$. Then $cl_B(A)$ (resp.$int_B(A)$) denotes closure of A (resp. interior of A) with respect to B.Jafari and

Rajesh introduced the concept Generalized closed sets with respect to an ideal(briefly Ig-closed sets).

In this paper, we introduce and study the concept of g*-closed sets with respect to an ideal, which is the extension of the concept of Ig-closed sets.

2. Preliminaries

Definition 2.1: A subset of a topological space (X, τ) is called a generalizedclosed set(briefly g-closed) if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

Definition 2.2: Let (X, τ)be a topological space and A is a subset of X, the generalized closure operator (briefly cl*)[1] is defined by the intersection of all g-closed sets containing A. The interior operator (briefly int*) is defined by union of all g-open sets contained in A.

Definition 2.3: A subset of a topological space (X, τ) is called a strongly generalized closed set(briefly g*-closed) if cl(A) ⊆ U whenever A ⊆ U and U is g-open in (X, τ).

Definition 2.4: Let (X, τ) be a topological space and I be an ideal on X. A subset A of X is said to be generalized closed with respect to an ideal (briefly Ig-closed)[7] if and only if cl(A) – B ∈ I whenever A⊂ B and B is open.

3. Strongly Generalized closed sets with Respect to an Ideal

Definition 3.1: Let (X, τ) be a topological space and I be an ideal on X. A subset A of X is said to be strongly generalized closed with respect to an ideal (briefly Ig*-closed) if and only if cl*(A) – B ∈ I, whenever A ⊂ B and B is g-open.

Remark 3.2: Every g-closed set is Ig*-closed, but the converse need not be true, as this may be seen from the following example.

Example 3.3: Let X={a, b, c} with topology τ={∅,{a},{a, c}, X} and I={∅, {b}, {c}, {b,c}}.Clearly, the set {a, c}is Ig*-closed but not g-closed in (X , τ).

The following theorem gives a characterization of Ig*-closed sets.

Theorem 3.4: A set A is Ig*-closed in (X, τ) if and only if F⊂cl*(A) – A and F is g-closed in X implies F ∈ I.

Proof: Assume that A is Ig*-closed. Let F⊂cl*(A) – A. Suppose F is g-closed. Then, A ⊂ X – F. By our assumption, cl*(A) – (X – F) ∈ I. But F⊂ cl*(A) – (X – F) and hence F ∈ I.

Conversely, assume that F ⊂ cl*(A) – A and F is g-closed in X implies that F ∈ I. Suppose A ⊂ U and U is g-open. Then cl*(A) – U = cl*(A) ∩ (X – U) is a g-closed set in X, that is contained in cl*(A) – A. By assumption, cl*(A) – U ∈ I. This implies that A is Ig*-closed.

Theorem 3.5: If A and B are Ig*-closed sets of (X, τ) then their union A ∪ B is also Ig*-closed. Proof:Suppose A and B are Ig*-closed sets in (X, τ). If A ∪ B ⊂ U and U is g-open, then A ⊂ U and B ⊂ U. By definition n of Ig*-closed cl*(A) – U ∈I and cl*(B) – U ∈I. Hence cl*(A∪B) – U= (cl*(A) – U) ∪ (cl*(B) – U) ∈ I. Therefore, A ∪ B is Ig*-closed.

Remark 3.6: The intersection of Ig*-closed sets need not be an Ig*-closed set as shown by the following example.

Example 3.7: Let X = {a,b,c} with topology τ = {∅, {a}, {a, c}, X}. If A = {a, b}, B = {a, c} and I = {∅, {b}, {b, c}}, then A and B are Ig*-closed sets but their intersection A ∩ B = {a} is not Ig*-closed.

Theorem 3.8: If A is Ig*-closed and A ⊂ B ⊂ cl*(A) and in (X, τ),then B is Ig*-closed in (X,τ). Proof:Suppose A is Ig*-closed and A ⊂ B ⊂ cl*(A) in (X, τ). Suppose B⊂ U and U is g-open.Then A⊂ U. Since A is Ig*-closed, we have cl*(A) – U∈ I. Now B⊂ cl*(A). This implies thatcl*(B)–U⊂ cl*(A) – U∈ I. Hence B is Ig*-closed in (X, τ).

Theorem 3.9: Let A ⊂ Y ⊂ X and suppose that A is Ig*- closed in (X, τ). Then A is Ig*- closed relative to the subspace Y of X, with respect to the ideal I_y = {F ⊂ Y: F ∈ I}.

Proof: Suppose A ⊂ U ∩ Y and U is g-open in (X , τ), then A ⊂ U. Since A is Ig*-closed in (X , τ), we have cl*(A) –U ∈ I. Now (cl*(A) ∩Y) – (U ∩Y) = (cl*(A) – U) ∩Y∈ I, whenever A ⊂ U ∩Y and U is g-open. Hence A is Ig*-closed relative to the subspace Y.

Theorem 3.10: Let A be an Ig*-closed set and F be a g-closed set in (X, τ), then A ∩ F is an Ig*-closed set in (X, τ).

Proof: Let A ∩ F⊂ U and U is g-open. Then A ⊂ U ∪ (X–F). Since A is Ig*-closed, we have

cl*(A) – (U ∪ (X–F)) ∈ I. Now, cl*(A∩ F) ⊂ cl*(A) ∩ F = (cl*(A) ∩ F) – (X–F). Therefore,

cl*(A ∩ F) – U ⊂ (cl*(A) ∩ F) – (U ∩ ((X – F))⊂ cl*(A) – (U ∪ (X – F)) ∈ I. Hence A ∩ F is Ig*- closed in (X, τ).

4. Strongly Generalized open sets with Respect to an Ideal

Definition 4.1: Let (X, τ) be a topological space and I be an ideal on X. A subset A of X is said to be strongly generalized open with respect to an ideal (briefly Ig*-open) if and only if X–A is Ig*-closed.

Theorem 4.2: A set A is Ig*-open in (X, τ) if and only if F–U ⊂ int*(A), for some U∈ I, whenever F ⊂ A and F is g-closed.

Proof: Suppose A is Ig*-open. Suppose F ⊂ A and F is g-closed. We have X – A ⊂ X – F. By assumption, cl*(X – A) ⊂ (X – F) ∪ U, for some U ∈ I. This implies X– ((X–F) ∪ U) ⊂ X – (cl*(X–A)) and hence F – U⊂ int*(A).

Conversely, assume that F ⊂ A and F is g-closed imply F– U ⊂ int*(A), for some U∈ I. Consider an g-open set G such that X – A ⊂ G. Then X– G ⊂ A. By assumption, (X–G) – U ⊂ int*(A) = X–cl*(X–A). This gives that X – (G ∪ U) ⊂ X–cl*(X–A). Then cl*(X–A) ⊂ G ∪ U, for some U ∈ I. This shows that cl*(X–A) – G ∈ I. Hence X–A is Ig*-closed.

Theorem 4.3. If A and B are separated Ig*-open sets in (X , τ), then A∪ B is Ig*-open.

Proof: Suppose A and B are separated Ig*-open sets in (X , τ) and F be a g-closed subset of A ∪ B. Then F ∩ cl*(A) ⊂ A and F ∩ cl*(B) ⊂ B. By assumption, (F ∩ cl*(A)) – U_1 ⊂ int*(A) and (F ∩ cl*(B)) – U_2 ⊂ int*(B), for some U_1,U_2∈ I. This means that (F ∩ cl*(A)) – int*(A)) ∈ I and (F ∩ cl*(B)) – int*(B) ∈ I. Then ((F ∩ cl*(A)) – int*(A)) ∪ ((F ∩ cl*(B)) – int*(B)) ∈ I. Hence (F ∩ (cl*(A) ∪ cl*(B)) – (int*(A) ∪ int*(B))) ∈I. But F = F∩(A∪B) ⊂ F ∩ cl*(A∪B), and we have F–int*(A∪ B) ⊂ (F∩cl*(A∪ B)) – int*(A∪ B)⊂(F∩ cl*(A∪B)) – (int*(A) ∪ int*(B)) ∈ I. Hence, F – U ⊂ int*(A∪ B), for some U ∈ I. This proves that A ∪ B is Ig*-open.

Corollary 4.4. Let A and B are Ig*-closed sets and suppose X–A and X–B are separated in (X ,τ). Then A ∩ B is Ig*-closed.

Corollary 4.5. If A and B are Ig*-open sets in (X , τ), then A ∩ B is Ig*-open.

Proof: If A and B are Ig*-open, then X – A and X – B are Ig*-closed. By Theorem 3.5, X– (A ∩ B) is Ig*-closed, which implies A ∩ B is Ig*-open.

Theorem 4.6. If A ⊂ B ⊂ X, A is Ig*-open relative to B and B is Ig*-open relative to X, then A is Ig*-open relative to X.

Proof: Suppose $A \subset B \subset X$, A is Ig*-open relative to B and B is Ig*-open relative to X. Suppose $F \subset A$ and F is g-closed. Since A is Ig*-open relative to B, by theorem 4.2, $F - U_1 \subset int_B^*(A)$, for some $U_1 \in I$. This implies there exists an open set G_1 such that $F - U_1 \subset G_1 \cap B \subset A$, for some $U_1 \in I$.Since B is Ig*-open, $F \subset B$ and F is g-closed; we have $F - U_2 \subset int_A^*(B)$, for some $U_2 \in I$. This implies there exists an open set G_2 such that $F - U_2 \subset G_2 \subset B$, for some $U_2 \in I$. Now

$F - (U_1 \cup U_2) \subset (F - U_1) \cap (F - U_2) \subset G_1 \cap G_2 \subset G_1 \cap B \subset A$. This implies that

$F - (U_1 \cup U_2) \subset int^*(A)$, for some $U_1 \cup U_2 \in I$ and hence A is Ig*-open relative to X.

Theorem 4.7: If $int^*(A) \subset B \subset A$ and if A is Ig*-open in (X, τ), then B is Ig*-open in X.

Proof: Suppose $int^*(A) \subset B \subset A$ and A is Ig*-open. Then $X - A \subset X - B \subset cl^*(X - A)$ and $X - A$ is Ig*-closed. By Theorem 3.8, $X–B$ is Ig*-closed and hence B is Ig*-open.

Theorem 4.8: A set A is Ig*-closed in (X, τ) if and only if $cl^*(A) - A$ is Ig*-open.

Proof: Necessity: Suppose $F \subset cl^*(A) - A$ and F be g-closed. Then $F \in I$. This implies that $F - U = \emptyset$, for some $U \in I$. Clearly, $F - U \subset int^*(cl^*(A)–A)$. By Theorem 4.2 $cl^*(A)–A$ is Ig*-open.

Sufficiency: Suppose $A \subset G$ and G is open in (X, τ). Then $cl^*(A) \cap (X–G) \subset cl^*(A) \cap (X–A) = cl^*(A) - A$. By hypothesis, $(cl^*(A) \cap (X–G)) - U \subset int^*(cl^*(A)–A) = \emptyset$, for some $U \in I$.This implies that $cl^*(A) \cap (X–G) \subset U \in I$ and hence $cl^*(A)–G \in I$. Thus, A is Ig*-closed.

References

[1] W.Dunham. A new closure operator for non-T_1 topologies, Kyunpook Math.J.22(1982),55-60

[2] T. R. Hamlett and D. Jankovic, Compactness with respect to an ideal, Boll. Un. Mat. Ita.,(7), 4-B, 849-861. 1990.

[3] T. R. Hamlett and D. Jankovic, Ideals in topological spaces and the set operator, Boll.Un. Mat. Ita., 7, 863-874. 1990.

[4] T. R. Hamlett and D. Jankovic, Ideals in General Topology and Applications (Midletown,CT, 1988), 115-125, Lecture Notes in Pure and Appl. Math. Dekker, New York, 1990.

[5] T. R. Hamlett and D. Jankovic, Compatible extensions of ideals, Boll. Un. Mat. Ita., 7,453-465. 1992.

[6] D. Jankovic and T. R. Hamlett, New topologies from old via ideals, Amer. Math. Month.,97, 295-310. 1990

[7] S. Jafari, N. Rajesh Generalized Closed Sets with Respect to an Ideal European Journal of Pure and Applied Mathematics vol.4, No. 2, 2011,147-151
[8] K. Kuratowski, Topologies I, Warszawa, 1933.
[9] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo, 19(2), 89-96.1970
[10] R. L. Newcomb, Topologies which are compact modulo an ideal, Ph.D. Dissertation, Univ.Cal.at Santa Barbara, 1967.
[11] A. Pushpalatha and K.Anitha, g*s-closed sets in Topological Spaces Int. J. Contemp. Math. Sciences, Vol. 6, 2011, no. 19,917-219
[12] A. Pushpalatha, S.Eswaran and P.Rajarubi, τ*- Generalized Closed Sets in Topological Spaces,Proceedings on the World Congress on Engineering 2009 Vol II, WCE 2009, July 1-3, 2009.London, U.K.
[13] D. V. Rancin, Compactness modulo an ideal, Soviet Math. Dokl., 13, 193-197. 1972.
[14] P. Samuels, A topology from a given topology and ideal, J. London Math. Soc. (2)(10), 409-416. 1975