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A Related Common Fixed Point Theorem in Fuzzy 3-Metric Spaces

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Abstract

In this paper we establish the existence of a related fixed point theorem for two pairs of mappings with different contraction in fuzzy 3- metric spaces.

Keywords: Fuzzy 3-metric space, Fixed Point, Related Fixed Point.

Introduction

The concept of fuzzy sets was initially investigated by Zadeh [11] in 1965 as new way to represent imprecise facts of uncertainties of vagueness in everyday life. Subsequently, it was developed extensively by many authors and used in population dynamics, chaos control, computer programming, medicine, etc. in 1975 Krammosil and Michalek [4] introduce the concept of fuzzy metric spaces . Later on it is modified and a few concepts of mathematical analysis have been developed by George and Veeramani [3] .In 1982 Fisher [2] studied some related fixed point theorems on two metric spaces. Since than many authors such as Aliouche and Fisher [1], R.K.Namdeo, S.Jain and B. Fisher [5] ,Telci [10] , Rao and Rao [6],Rao,Aliouche and Ravi Babu [7] and others proved some related fixed point theorems in two metric and fuzzy metric spaces. Recently , Samant, Mohinta and Jebril [9] established some related fixed point theorems for two pairs of mappings in fuzzy metric spaces .In this paper we extend the theorem 3.1 of Samant, Mohinta and Jebril [11] in fuzzy 3- metric spaces.

2.Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequal.

Definition 2.1: A binary operation $* :[0.1]\times[0,1] \times[0,1] \times[0,1]\to [0,1]$ is called a t-norm of $\{[0,1],*\}$ is an abelian topological monoid with unit 1 such that $a_1* b_1* c_1* d_1\leq a_2* b_2 * c_2 * d_2$whenever $a_1\leq a_2$, $b_1\leq b_2$,$c_1\leq c_2$,$d_1\leq d_2$ for all a_1, a_2, b_1, b_2 ,c_1, c_2 ,d_1, $d_2 \in[0,1]$.

Definition 2.2: The 3-triple $(X,\mu,*)$ is said to be fuzzy 3- metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in $X^4 \times [0,\infty)$ satisfying the following conditions:

For all $x,y,z,w \in X$ and $s,t > 0$

[FM-1] $\mu (x,y,z,w,0)=0$,

[FM-2] $\mu (x,y,z,w,t)=1$ for all $t > 0$ only when the three simplex $<x,y,z,w>$ degenerate,

[FM-3] $\mu (x,y,z,w,t)= \mu (y,z,w,x,t)= \mu (z,w,x,y,t) = \mu (x,w,z,y,t)$…….. symmetry about three variabls,

[FM-4] $\mu (x,y,z,u,t_1) * \mu (x,y,u,w,t_2) * \mu (x,u,z,w,t_3) * \mu (u,y,z,w,t_4) \leq \mu (x,y,z,t_1+t_2+t_3+t_4)$, $\forall\, x,y,z,w,u \in X$ and $t_1,t_2,t_3 , t_4 >0$

[FM-5] $\mu (x,y,z,w.): [0,\infty) \to[0,1]$ is left continuous ,

[FM-6] $\lim_{n\to\infty} \mu(x, y, z, w, t) = 1$

The function value $\mu(x,y,z,w,t)$ may be interpreted as the probability that the volume of tetrahedron is less than t.

Definition2.3: Let $(X, \mu,*)$ be a fuzzy 3-metric space.

A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point $x \in X$ if

$\lim\limits_{n\to\infty} \mu (x_n,x,a.b,t)=1$or all $a,b\in X$ and $t>0$

A sequence $\{x_n\}$ in fuzzy 3-metric space X is called Cauchy sequence if

$\lim\limits_{n\to\infty} \mu (x_{n+p},x_n,a.b,t)=1$ for all $a,b\in X$ and $t>0$, $p>0$

Let $(X, \mu,*)$be a Fuzzy 3-metric spaces if $k\in (0,1)$ such that $x,ya,b \in X$,

$\mu (x,y,a,b,kt) \geq \mu (x,y,a,b,t)$, $\forall\, t > 0$ then $x = y$

3.Main Result

Theorem 3.1 : Let $(X,\mu,*)$ and $(X,\sigma,*)$ be complete fuzzy 3-metric spaces. If T is a continuous mapping of X into Y and S is a mapping of Y into X satisfying the inequalities,

$k\mu(STx,STx',a,b,t) \geq \min\{\mu(x,x',a,b,t), \mu(x,STx,a,b,t),$
$\mu(x,STx',a,b,t), \mu(x',STx',a,b,t), \sigma(Tx,Tx', a',b',t)$ (i)
$k\sigma(TSx,TSx',a',b',t) \geq \min\{\sigma(y,y',a', b',t), \sigma(y,TSy,a', b',t),$
$\sigma(y,TSy',a', b',t), \sigma(y',TSy',a', b',t), \mu(Sy,Sy', a, b,t)\}$ (ii)

For all $x,x'',a,b\in X$ and $y,y',a',b'\in Y$ where $k\in (0,1]$than ST has a unique fixed point z in X and TS has a unique fixed point w in Y further Tz = w and Sw = z.

Proof: Let x_0 be an arbitrary point in X and
$Tx_0 = y_1$, $Tx_1 = y_2$, $Tx_2 = y_3$,…………. $Tx_{n-1} = y_n$,
$Sy_1= x_1$, $Sy_2= x_2$, $Sy_3= x_3$,……………………………….. $Sy_n= x_n$.

Using inequality (i) we get
$k\mu(x_{n+2}, x_{n+1},a,b,t) = k\mu(STx_{n+1}, STx_n,a,b,t)$
$\geq \min\{\mu(x_{n+1}, x_n,a,b,t),\mu(x_{n+1}, STx_{n+1},a,b,t),\mu(x_{n+1}, STx_n,a,b,t),\mu(x_n,STx_n,a,b,t),\sigma(Tx_{n+1}, Tx_n,a',b',t)\}$
$\geq \min\{ \mu(x_{n+1}, x_n,a,b,t), \mu(x_{n+1}, x_{n+2},a,b,t), \mu(x_{n+1}, x_{n+1},a,b,t), \mu(x_n, x_{n+1},a,b,t), \sigma(y_{n+2}, y_{n+1},a',b',t)\}$
$\geq \min\{ \mu(x_{n+1}, x_n,a,b,t), \mu(x_{n+1}, x_{n+2},a,b,t), \sigma(y_{n+2}, y_{n+1},a', b',t)\}$ (1)
$\mu(x_{n+2}, x_{n+1},a,b,t) \geq \frac{1}{k} \min\{ \mu(x_{n+1}, x_n,a,b,t), \mu(x_{n+1}, x_{n+2},a,b,t),$
$\sigma(y_{n+2}, y_{n+1},a', b',t)\}$……………….. (2)

putting the value from (2) in (1)
$k\mu(x_{n+2}, x_{n+1},a,b,t) \geq \min\{ \mu(x_{n+1}, x_n,a,b,t), \sigma(y_{n+2}, y_{n+1},a',b,t), \frac{1}{k}\mu$
$(x_{n+1}, x_n,a,b,t), \frac{1}{k}\mu(x_{n+1}, x_{n+2},a,b,t), \frac{1}{k}\sigma(y_{n+2}, y_{n+1},a', b',t)\}$
$= \min\{ \mu(x_{n+1}, x_n,a,b,t), \sigma(y_{n+2}, y_{n+1},a', b',t), \frac{1}{k}\mu(x_{n+1}, x_{n+2},a,b,t)\}$

Doing m times we get
$k\mu(x_{n+2}, x_{n+1},a,b,t) \geq \min\{ \mu(x_{n+1}, x_n,a,b,t), \sigma(y_{n+2}, y_{n+1},a', b',t), \frac{1}{k^m}\mu(x_{n+1}, x_{n+2},a,b,t)\}$

Taking lim as $m \to\infty$ we have
$k\mu(x_{n+2}, x_{n+1},a,b,t) \geq \min\{ \mu(x_{n+1}, x_n,a,b,t), \sigma(y_{n+2}, y_{n+1},a', b',t)\}$ (3)

Again
$k\sigma(y_{n+2},y_{n+1},a', b',t) = k\sigma(Tx_{n+1}, Tx_n,a', b',t)$
$= k\sigma(TSTx_n, TSTx_{n-1},a', b',t)$
$= k\sigma(TSy_{n+1},TSy_n,a', b',t)$
$\geq \min\{ \sigma(y_{n+1},y_n,a', b',t), \sigma(y_{n+1},TSy_{n+1},a', b', b',t), \sigma(y_{n+1},TSy_n,a',t), \sigma(y_n,TSy_n,a', b',t), \mu(Sy_{n+1},Sy_n,a,b,t)\}$
$\geq \min\{ \sigma(y_{n+1},y_n,a', b',t), \sigma(y_{n+1},y_{n+2},a', b',t), \sigma(y_{n+1},y_{n+1},a', b',t), \sigma(y_n,y_{n+1},a', b',t), \mu(x_{n+1},x_n,a, b,t)\}$
$k\sigma(y_{n+2},y_{n+1},a', b',t) \geq \min\{ \sigma(y_{n+1},y_n,a', b',t), \mu(x_{n+1},x_n,a, b,t)\}$ (4)

By (3)

$k\mu(x_{n+1}, x_n,a,b,t) \geq \min\{\mu(x_n, x_{n-1},a,b,t), \sigma(y_{n+1}, y_n,a', b',t)\}$

$\mu(x_{n+1}, x_n,a,b,t) \geq \frac{1}{k}\min\{\mu(x_n, x_{n-1},a,b,t), \sigma(y_{n+1}, y_n,a', b',t)\}$

putting this value in (4)

$k\sigma(y_{n+2},y_{n+1},a', b',t) \geq \frac{1}{k}\min\{\sigma(y_{n+1},y_n,a', b',t), \mu(x_n, x_{n-1},a,b,t), \sigma(y_{n+1}, y_n,a', b',t)\}$

$\geq \frac{1}{k}\min\{\sigma(y_{n+1},y_n,a', b',t), \mu(x_n, x_{n-1},a,b,t)\}$

$\vdots \qquad \vdots$

$\geq \frac{1}{k^n}\min\{\sigma(y_1,y_2,a', b',t), \mu(x, x_1,a,b,t)\}$

We now verify that $\{x_n\}$ is a Cauchy sequence Let $t_1=\frac{t}{p}$, then

$\mu(x_n, x_{n-p},a,b,t) \geq \mu(x_n, x_{n-1},a,b,t_1)* \mu(x_{n+1}, x_{n-2},a,b,t_1)*\ldots\ldots* \mu(x_{n+p-1}, x_{n+p},a,b,t_1)$

$\geq \frac{1}{k^n}\min\{\mu(x, x_1,a,b,t)* \sigma(y_1, y_2,a',b,t)*\ldots\ldots*\frac{1}{k^{n+p-1}}\min \mu(x, x_1,a,b,t), \sigma(y_1, y_2,a', b',t)\}$

$1 \geq \lim_{n\to\infty} \mu(x_n, x_{n+p},a,b,t)$

$\geq \lim_{n\to\infty}\frac{1}{k^n}\min\{\mu(x, x_1,a,b,t), \sigma(y_1, y_2,a', b',t)\}>1$

$\lim_{n\to\infty} \mu(x_n, x_{n+p},a,b,t) = 1$

Hence $\{x_n\}$ is a Cauchy sequence with limit z in X and similarly $\{y_n\}$ is a Cauchy sequence with limit w in Y

We have on using the continuity of T

$w = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Tx_n = Tz$

further

$k\mu(STz,x_{n+1},a,b,t) = k\mu(STz,STx_{n-1},a,b,t)$

$\geq \min\{\mu(z,x_{n-1},a,b,t),\mu(z,STz,a,b,t),\mu(z,STx_{n-1},a,b,t),\mu(x_{n-1},STx_{n-1},a,b,t),\sigma(Tz,Tx_{n-1},a', b',t)\}$

$\geq \min\{\mu(z,x_{n-1},a,b,t), \mu(z,STz,a,b,t), \mu(z,x_n,a,b,t), \mu(x_{n-1},x_n,a,b,t), \sigma(Tz, Tx_{n-1},a', b',t)\}$

As $\lim\ n \to \infty$

$k\mu(STz,z,a,b,t) \geq \min\{\mu(z,STz,a,b,t), \sigma(Tz, Tx_{n-1},a', b',t)\}$

$k\mu(STz,z,a,b,t) \geq \min\{\mu(z,STz,a,b,t), \sigma(Tz, w,a', b',t)\}$

$k\mu(STz,z,a,b,t) = \mu(z,STz,a, b',t)$

we follows that $STz = z$

hence $STz = Sw = z$

now suppose that ST has second fixed point z' then

$k\mu(z,z',a,b,t) = k\mu(STz,STz',a,b,t)$

$\geq \min\{\mu(z,z',a,b,t), \mu(z,STz,a,b,t), \mu(z,STz',a,b,t), \mu(z',STz',a,b,t), \sigma(Tz, Tz',a', b',t)\}$

$\geq \min\{\mu(z,z',a,b,t), 1, \mu(z,z',a,b,t), \mu(z',z',a,b,t), \sigma(Tz, Tz',a', b',t)\}$

$\geq \sigma(Tz, Tz',a', b',t)$

But

kσ(Tz, Tz′,a′, b’,t)= σ(TSTz, TSTz′,a′, b’,t)

≥ min{ σ(Tz, Tz′,a′, b’,t), σ(Tz, TSTz,a′, b’,t), σ(Tz, TSTz′,a′, b’,t), σ(Tz′, TSTz′,a′, b’,t), μ(STz, STz′,a, b,t)}

≥min{ σ(Tz, Tz′,a′, b’,t), σ(Tz, Tz,a′, b’,t), σ(Tz, Tz′,a′, b’,t), σ(Tz′, Tz′,a′, b’,t), μ(z, z′,a, b,t)}

σ(Tz, Tz′,a′, b’,t) $\geq \frac{1}{k}$ μ(z, z′,a,b,t)

hence

μ(z, z′,a,b,t) $\geq \frac{1}{k}$μ(z, z′,a,b,t) $\geq \frac{1}{k^2}$ μ(z, z′,a,b,t)………………… $\geq \frac{1}{k^n}$ μ(z, z′,a,b,t) →1 as n→∞

≥ μ(z, z′,a,b,t) ≥ $\lim \quad n \to \infty \frac{1}{k^n}$ μ(z, z′,a,b,t) > 1

μ(z, z′,a,b,t) = 1

z = z′

similarly w is the unique fixed point of TS

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