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A Fuzzy Approach to Priority Queues

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Abstract

This paper investigates the queuing model of priority classes using Triangular fuzzy number with the application of fuzzy set theory. We propose a mathematical programming approach to develop the membership function of the system performance , in which the arrival rates and service rate of two priority classes are used as fuzzy numbers. Based on α-cut approach and Zadeh's extension principle, the fuzzy queues are reduced to a family of crisp queues. Triangular fuzzy numbers are used to demonstrate the validity of the proposal. An illustration is given to establish the performance measures of the characteristics of the queueing model.

Keywords: Fuzzy Sets, Membership functions, Priority queues, Mathematical programming, Triangular fuzzy number, Performance measures.

1.Introduction

Queueing models have considered the property that the unit proceed to service on a first come- first served basis. This is obviously not only the manner of service and there are many alternatives such as last come-first served, selection in random order and selection by priority.

In Priority schemes, customers with the highest priority are selected for service ahead of those with the lower priority, independent of their time of arrival in to the system . There are two further refinements possible in priority situation, namely preemption and non-preemption. In preemptive cases, the customer with the highest priority is allowed to enter service immediately even if another with lower priority is already in service when the higher priority customer arrives to the system. In addition, a decision has to be made whether to continue the preempted customer service from the point of preemption when resumed or to start a new. The priority discipline is said to be non-preemptive if there is no interruption and the highest priority customer just goes to the ahead of the queue to wait his turn.

In practical, the priority queuing model, the input data, arrival rate, service rate are uncertainly known. Uncertainty is resolved by using fuzzy set theory. Hence the classical queuing model with priority discipline will have more application if it is expanded using fuzzy models.

Fuzzy queuing models have been described by such researchers like Li and Lee[3], Kaufmann[2], Negi and Lee [4],Kao et al[1].Chen [8] has analyzed fuzzy queues using Zadeh's extension principle. Kao et al constructed the membership functions of the system characteristic for fuzzy queues using parametric linear programming. Recently Chen [8] developed (FM/FM/1): (∞/FCFS)and ($FM/FM^k/1$): (∞/FCFS) where FM denotes fuzzified exponential time based on queuing theory.

In this paper , fuzzy set theory is applied to construct the membership function of a fuzzy priority queues in which two arrival rates and single service rate are Triangular fuzzy numbers. α-cut approach and fuzzy arithmetic operations are used to derive system characteristics.

2.Fuzzy Set theory and Fuzzy Arithmetic operations, α-cuts

Definition 2.1. A fuzzy set is a set where the members are allowed to have partial membership and hence the degree of membership varies from 0 to 1. It is expressed as A={ (x,$\mu_{\tilde{A}}$(x)/x∈Z} where Z is the universe of discourse and $\mu_{\tilde{A}}$(x) is a real number, $\mu_{\tilde{A}}$(x) =0 or 1,ie,x is a non-member in A if $\mu_{\tilde{A}}$(x) =0 and x is a member in A if $\mu_{\tilde{A}}$(x) =1.

Definition 2.2 If a fuzzy set $\tilde{A}$ is defined on X , for any α∈[0 ,1], the α-cuts of the fuzzy set $\tilde{A}$ is represented by $\tilde{A}_{\alpha}$={x/$\mu_{\tilde{A}}$(x)≥α,x∈Z}={$l_{\tilde{A}}$(α), $u_{\tilde{A}}$(α)},where $l_{\tilde{A}}$(α) and $u_{\tilde{A}}$(α) represent the lower bound and upper bound of the α-cut of $\tilde{A}$ respectively.

Definition2.3. If a fuzzy set A is defined on X, for any α∈ [0 ,1], the α-cuts ${}^{\alpha}A$ is represented by the following crisp set,

Strong α-cuts: ${}^{\alpha+}A$ ={x∈X/μ_A(x)>α} ; α∈[0 ,1]

Weak α-cuts: ${}^{\alpha}A$ ={ x∈X/μ_A(x)≥ α} ; α∈[0 ,1]

Therefore, it is inferred that the fuzzy set A can be treated as crisp set ${}^{\alpha}A$ in which all the members have their membership values greater than or at least equal to α.

Definition2.4. The support of a fuzzy set A is the crisp set such that it is represented as suppA(X) ={x∈X/μ_A(x)>0}. Thus, support of a fuzzy set is the set of all members with a strong α-cut , where α=0.

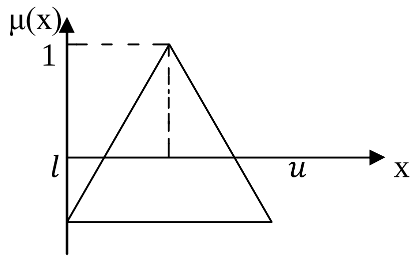
Definition2.5. The height of a fuzzy set ,h{A(x)/x∈X} is the maximum value of its membership function μ(x) such that $^{\alpha}A$ ={ x∈X/μ_A(x)≥ α} and 0∉α.
A fuzzy set Max{ μ(x)}=1 is called as a normal fuzzy set, otherwise, it is referred as subnormal fuzzy set.

Triangular fuzzy number

We define a fuzzy number M on R to be a triangular fuzzy number if its membership function μ_M(x): R→[0,1] is defined by

$$\mu_M(x) = \begin{cases} \frac{x-l}{m-l}, & \text{for } l \le x \le m \\ \frac{x-u}{m-u}, & \text{for } m \le x \le u \\ 0 & \text{otherwise} \end{cases}$$

where $l \le m \le u$, l and u stand for the lower and upper value of the support of M respectively and m for the modal value. The triangular fuzzy number can be denoted by (l, m, u). The support of M is the set of elements {x∈R/$l < x < u$}, when $l = m = u$,it is a non-fuzzy number by convention.



3.Application of fuzzy for priority queues

We analyze a fuzzy queueing model FM/FM/I with a priority .Each arrival unit is designated as a member of one of two priority classes. Further it is assumed that ,the arrivals of the first or higher priority have mean rate $\bar{\lambda}_1$ and the second or lower priority have mean rate $\bar{\lambda}_2$ such that $\bar{\lambda} = \bar{\lambda}_1 + \bar{\lambda}_2$.The first or higher priority units have the right to be served ahead of the others without preemption. It is assumed that the capacity of the system and the calling source population are infinite.

For this model $\bar{L}_1$and $\bar{L}_2$ denote the average system length of first and second priorities. $\bar{L}_{q_1}$and $\bar{L}_{q_2}$ denote the average queue length of first and second priorities. $\bar{W}_{q_1}$and
$\bar{W}_{q_2}$ denote the average waiting time of units in the queue in the first and second priorities respectively are obtained.

4.Formulation of parametric programming problem for FM/FM/I fuzzy queue with priority model

Consider a single server FM/FM/I queueing system with priority. The interarrival times $\tilde{A}_i$, i=1,2 of units in the first and second priority and service times S̃ are approximately known and are represented by the following fuzzy sets.

$$\tilde{A}_i = \{ (a, \mu_{\tilde{A}_i}(a)/ a\in X\} , i=1,2 \quad \ldots\ldots(1)$$

$$\tilde{S} = \{ (s, \mu_{\tilde{S}}(s)/ s\in Y \} \quad \ldots\ldots(2)$$

where X and Y are crisp universal sets of the interarrival times and service times and $\mu_{\tilde{A}_i}$ (a),i=1,2 , $\mu_{\tilde{S}}(s)$ are the respective membership functions. The α-cut of $\tilde{A}_i$, i=1, 2 and S̃ are

$$A_i(\alpha) = \{a\in X/\mu_{\tilde{A}_i} (a) \geq\alpha\},i=1,2 \quad \ldots\ldots(3)$$

$$S(\alpha) = \{s\in Y/\mu_{\tilde{S}}(s) \geq\alpha \} \quad \ldots\ldots (4)$$

where 0<α≤1. Both $A_i(\alpha)$,i=1,2 and S(α) are the crisp sets.

Using α-cut, the interarrival times and service times can be represented by different levels of confidence intervals [0,1]. Hence a fuzzy queue can be reduced to a family of crisp queues with different α-cuts $\{A_i(\alpha)/0<\alpha\leq 1\}$,i=1,2 and { S(α)/ 0<α ≤1 }. These two sets represent sets of movable boundaries and they form nested structure [Zimmermann] for expressing the relationship between the crisp sets and fuzzy sets.

Let the confidence intervals of the fuzzy sets $\tilde{A}_i$,i=1,2 and S̃ be $[l_{A_i(\alpha)}, u_{A_i(\alpha)}]$,i=1,2 and $[l_{S(\alpha)}, u_{S(\alpha)}]$ respectively.Since both the interarrival times $\tilde{A}_i$, i=1, 2 and S̃ are fuzzy numbers, using Zadeh's extension principle [5,7], the membership function of the performance measure p($\tilde{A}_i$,$\tilde{S}$),i=1,2 is defined as

$$\mu_{p(\tilde{A}_i,\tilde{S})} (z) =\sup_{a\in X,s\in Y} \min \{ \mu_{\tilde{A}_i}(a), \mu_{\tilde{S}}(s)/ z=p(a,s)\} ,i=1,2 \quad \ldots\ldots (5)$$

Construction of the membership function $\mu_{p(\tilde{A}_i,\tilde{S})}$ (z) ,i=1,2 is equivalent to say that the derivation of α-cuts of $\mu_{p(\tilde{A}_i,\tilde{S})}$ .From equation (5), the equation $\mu_{p(\tilde{A}_i,\tilde{S})}$ (z) =α,i=1,2 is true only when either $\mu_{\tilde{A}_i}(a)$ =α, $\mu_{\tilde{S}}(s)\geq\alpha$ (or) $\mu_{\tilde{A}_i}(a))\geq\alpha$, $\mu_{\tilde{S}}(s)=\alpha$ is true.

The parametric programming problem have the following form,

$$l_{p(\alpha)} = \min p(a,s) \quad \ldots\ldots(6)$$

$$\text{such that} \quad l_{A_i(\alpha)} \leq a \leq u_{A_i(\alpha)} \quad ,i=1,2$$

$$l_{s(\alpha)} \leq s \leq u_{s(\alpha)}$$

and

$$u_{p(\alpha)} = \max p(a,s) \quad \ldots\ldots(7)$$

$$\text{such that} \quad l_{A_i(\alpha)} \leq a \leq u_{A_i(\alpha)} \ ,i=1,2$$

$$l_{s(\alpha)} \leq s \leq u_{s(\alpha)}$$

If both $l_{p(\alpha)}$ and $u_{p(\alpha)}$ are invertible with respect to α, then the left shape function

L(z) = $(l_{p(\alpha)})^{-1}$and the right shape function R(z) = $(u_{p(\alpha)})^{-1}$can be obtained, from which the membership function $\mu_{p(\tilde{A}_i,\tilde{S})}(z)$,i=1,2 is constructed as

$$\mu_{p(\tilde{A}_i,\tilde{S})}(z) = \begin{cases} L(z) & , \quad for\ z_1 \le z \le z_2 \\ R(z) & , \quad for\ z_2 \le z \le z_3 \\ 0 & , \quad \text{otherwise} \end{cases} \quad \ldots\ldots\ldots \qquad ..(8)$$

where $z_1 \le z_2 \le z_3$ and $L(z_1)$ =$R(z_3)$=0 for the triangular fuzzy number.
Using the concept of α-cut the FM/FM/1 queue with priority can be reduced as M/M/1 queue with priority for which

$$L_{q_1} = \frac{(\rho\frac{\lambda_1}{\mu})}{(1-\frac{\lambda_1}{\mu})}$$

$$L_{q_2} = \frac{(\rho\frac{\lambda_2}{\mu})}{(1-\rho)(1-\frac{\lambda_1}{\mu})}$$

$$W_{q_1} = \frac{\lambda}{\mu(\mu-\lambda_1)}$$

$$W_{q_2} = \frac{\lambda}{(\mu-\lambda)(\mu-\lambda_1)}$$

where λ_1 and λ_2 are the arrival rates of first and second priority units respectively and μ is the service rate . Further $\lambda = \lambda_1 + \lambda_2$ and $\rho = \frac{\lambda}{\mu}$
If the functions $l_{p(\alpha)}$ and $u_{p(\alpha)}$ are not invertible with respect to α, the membership functions $\mu_{p(\tilde{A}_i,\tilde{S})}(z)$ is not derived. But we can trace the graph of $\mu_{p(\tilde{A}_i,\tilde{S})}(z)$ from the α-cuts of $[l_{p(\alpha)}, u_{p(\alpha)}]$.
This procedure can be applied to find the membership functions $\mu_{p(\tilde{A}_i,\tilde{S})}(z)$ for the queueing model with priority can be obtained.

5.Illustration:

Expected waiting time and expected number of customer in the queue for FM/FM/I queue with two priority classes.

Suppose that the arrival rates of first and second priority with the same service rate are fuzzy numbers represented by $\tilde{A}_1$=[2,3,4] , $\tilde{A}_2$=[5,6,7] and $\tilde{S}$=[16,17,18] per hour respectively. The α-cut of the membership functions $\mu_{\tilde{A}_1}(\alpha)$, $\mu_{\tilde{A}_2}(\alpha)$ and , $\mu_{\tilde{S}}(\alpha)$ are [2+α,4-α] , [5+α,7-α] and [16+α,18-α] respectively. From equations (6) and (7) the parametric programming problems are formulated to derive the membership function $\bar{L}_{q_1}$, $\bar{L}_{q_2}$, $\bar{W}_{q_1}$ and $\bar{W}_{q_2}$. They are calculated as follows.
The performance functions of

(i) $\bar{L}_{q_1}$- average queue length of first priority
(ii) $\bar{L}_{q_2}$- average queue length of second priority
(iii) $\bar{W}_{q_1}$-average waiting time of units of first priority in the queue
(iv) $\bar{W}_{q_2}$ - average waiting time of units of second priority in the queue

are derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{L_{q_1}(\alpha)} = \min\left\{\frac{(\frac{r_1+r_2}{t})(\frac{r_1}{t})}{(1-\frac{r_1}{t})}\right\}$$
such that $2+\alpha \leq r_1 \leq 4-\alpha$ ……………………………………………… (9)
$5+\alpha \leq r_2 \leq 7-\alpha$
$16+\alpha \leq t \leq 18-\alpha$

and

$$u_{L_{q_1}(\alpha)} = \max\left\{\frac{(\frac{r_1+r_2}{t})(\frac{r_1}{t})}{(1-\frac{r_1}{t})}\right\}$$
such that $2+\alpha \leq r_1 \leq 4-\alpha$ 10)
$5+\alpha \leq r_2 \leq 7-\alpha$
$16+\alpha \leq t \leq 18-\alpha$

where $0<\alpha\leq 1$

$l_{L_{q_1}(\alpha)}$ is found when r_1 and r_2 approach their lower bounds and t approaches its upper bound. Consequently the optimal solution for (9) is

$$l_{L_{q_1}(\alpha)} = \frac{14+11\alpha+2\alpha^2}{288-52\alpha+2\alpha^2} \quad \ldots \quad 11)$$

also $u_{L_{q_1}(\alpha)}$ is found when r_1 and r_2 approach their upper bounds and t approaches its lower bound. In this case, the optimal solution for (10) is

$$u_{L_{q_1}(\alpha)} = \frac{44-19\alpha+2\alpha^2}{192+44\alpha+2\alpha^2} \quad \ldots \quad (12)$$

The membership function
$$\mu_{\bar{L}_{q_1}}(z) = \begin{cases} L(z), & [l_{L_{q_1}(\alpha)}]_{\alpha=0} \leq z \leq [l_{L_{q_1}(\alpha)}]_{\alpha=1} \\ R(z), & [u_{L_{q_1}(\alpha)}]_{\alpha=1} \leq z \leq [u_{L_{q_1}(\alpha)}]_{\alpha=0} \\ 0 & \text{otherwise} \end{cases}$$

which is estimated as

$$\mu_{\bar{L}_{q_1}}(z) = \begin{cases} \frac{(52z+11)-(400z^2+3560z+9)^{\frac{1}{2}}}{4(z-1)}, & .05\leq z\leq .11 \\ \frac{-(44z+19)+(400z^2+3560z+9)^{\frac{1}{2}}}{4(z-1)}, & .11\leq z\leq .23 \\ 0 & \text{otherwise} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

Similarly the performance functions of $\bar{L}_{q_2}$, $\bar{W}_{q_1}$ and $\bar{W}_{q_2}$ are derived from the respective parametric programs. These differ only in their objective functions and are listed below.

$$l_{L_{q_2}(\alpha)} = \min\left\{\frac{\frac{r_2(r_1+r_2)}{t^2}}{\left(1-\frac{r_1+r_2}{t}\right)\left(1-\frac{r_1}{t}\right)}\right\} \qquad (14)$$

and

$$u_{L_{q_2}(\alpha)} = \max\left\{\frac{\frac{r_2(r_1+r_2)}{t^2}}{\left(1-\frac{r_1+r_2}{t}\right)\left(1-\frac{r_1}{t}\right)}\right\} \qquad (15)$$

$$l_{W_{q_1}(\alpha)} = \min\left\{\frac{r_1+r_2}{t(t-r_1)}\right\} \qquad (16)$$

and

$$u_{W_{q_1}(\alpha)} = \max\left\{\frac{r_1+r_2}{t(t-r_1)}\right\} \qquad (17)$$

$$l_{W_{q_2}(\alpha)} = \min\left\{\frac{r_1+r_2}{[t-(r_1+r_2)](t-r_1)}\right\} \qquad (18)$$

and

$$u_{W_{q_2}(\alpha)} = \max\left\{\frac{r_1+r_2}{[t-(r_1+r_2)](t-r_1)}\right\} \qquad (19)$$

The objective functions given through the equations (14) to (19) with the constraints given with the equations (9) and (10) yield the following results:

$$l_{L_{q_2}(\alpha)} = \frac{35+17\alpha+2\alpha^2}{176-70\alpha+6\alpha^2} \;;\; u_{L_{q_2}(\alpha)} = \frac{77-25\alpha+2\alpha^2}{60+46\alpha+6\alpha^2} \qquad (20)$$

$$\mu_{\bar{L}_{q_2}}(z) = \begin{cases} \frac{(70z+17)-(676z^2+4628z+9)^{\frac{1}{2}}}{4(3z-1)}, & .20 \le z \le .48 \\ \frac{-(46z+25)+(676z^2+4628z+9)^{\frac{1}{2}}}{4(3z-1)}, & .48 \le z \le 1.28 \\ 0 & \text{otherwise} \end{cases} \qquad (21)$$

$$l_{W_{q_1}(\alpha)} = \frac{7+2\alpha}{288-52\alpha+2\alpha^2} \;;\; u_{W_{q_1}(\alpha)} = \frac{11-2\alpha}{192+44\alpha+2\alpha^2} \qquad (22)$$

$$\mu_{\bar{W}_{q_1}}(z) = \begin{cases} \frac{(52z+2)-(400z^2+264z+4)^{\frac{1}{2}}}{4z}, & .024 \le z \le .038 \\ \frac{-(44z+2)+(400z^2+264z+4)^{\frac{1}{2}}}{4z}, & .038 \le z \le .057 \\ 0 & \text{otherwise} \end{cases} \qquad (23)$$

$$l_{W_{q_2}(\alpha)} = \frac{7+2\alpha}{176-70\alpha+6\alpha^2} \;;\; u_{W_{q_2}(\alpha)} = \frac{11-2\alpha}{60+46\alpha+6\alpha^2} \qquad (24)$$

$$\mu_{\bar{W}_{q_2}}(z) = \begin{cases} \frac{(70z+2)-(676z^2+448z+4)^{\frac{1}{2}}}{12z}, & .0398 \le z \le .08 \\ \frac{-(46z+2)+(676z^2+448z+4)^{\frac{1}{2}}}{12z}, & .08 \le z \le .18 \\ 0 & \text{otherwise} \end{cases} \qquad (25)$$

α-cuts of arrival rates, service rate, queue length and waiting time in queue of first and second priority

Table 5.1

α	l_{x_α}	u_{x_α}	l_{y_α}	u_{y_α}	$l_{Lq_1}(\alpha)$	$u_{Lq_1}(\alpha)$	$l_{Lq_2}(\alpha)$	$u_{Lq_2}(\alpha)$	$l_{Wq_1}(\alpha)$	$u_{Wq_1}(\alpha)$	$l_{Wq_2}(\alpha)$	$u_{Wq_2}(\alpha)$
0	2	4	16	18	0.0486	0.2292	0.1989	1.2833	0.0243	0.0573	0.0398	0.1833
0.1	2.1	3.9	16.1	17.9	0.0535	0.2144	0.2172	1.1525	0.0255	0.0550	0.0426	0.1670
0.2	2.2	3.8	16.2	17.8	0.0586	0.2005	0.2372	1.0380	0.0266	0.0528	0.0456	0.1526
0.3	2.3	3.7	16.3	17.7	0.0641	0.1874	0.2590	0.9373	0.0279	0.0506	0.0489	0.1399
0.4	2.4	3.6	16.4	17.6	0.0700	0.1749	0.2828	0.8483	0.0292	0.0486	0.0524	0.1285
0.5	2.5	3.5	16.5	17.5	0.0762	0.1632	0.3088	0.7692	0.0305	0.0466	0.0561	0.1183
0.6	2.6	3.4	16.6	17.4	0.0828	0.1521	0.3373	0.6988	0.0318	0.0447	0.0602	0.1092
0.7	2.7	3.3	16.7	17.3	0.0898	0.1416	0.3685	0.6357	0.0333	0.0429	0.0646	0.1009
0.8	2.8	3.2	16.8	17.2	0.0972	0.1317	0.4028	0.5791	0.0347	0.0411	0.0694	0.0934
0.9	2.9	3.1	16.9	17.1	0.1051	0.1223	0.4405	0.5281	0.0362	0.0394	0.0747	0.0866
1	3	3	17	17	0.1134	0.1134	0.4821	0.4821	0.0378	0.0378	0.0804	0.0804

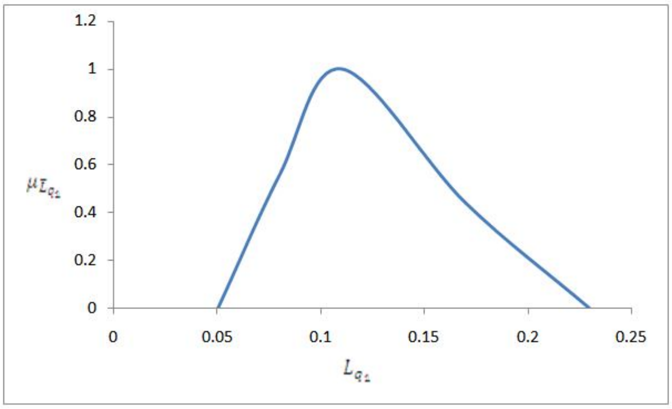


Figure 5.1: Performance measure of the average queue length of first priority

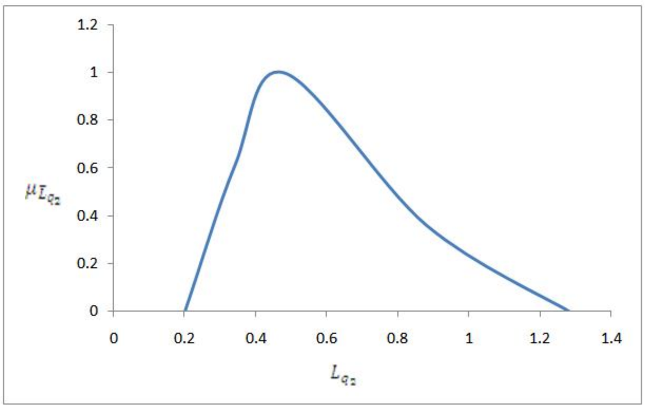


Figure 5.2: Performance measure of the average queue length of second priority

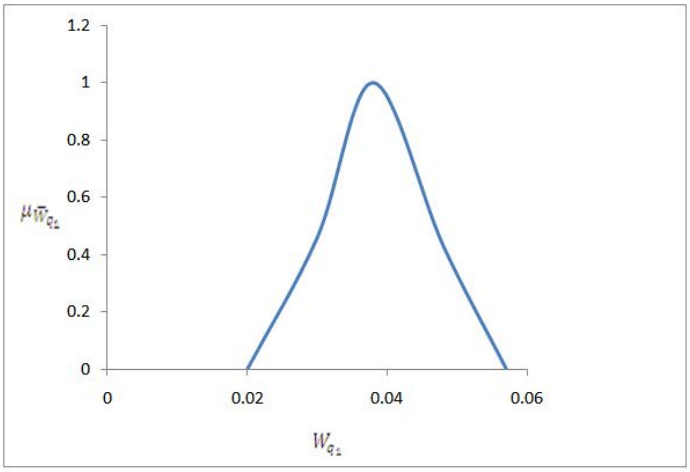


Figure 5.3: Performance measure of the average waiting time of units of first priority in the queue

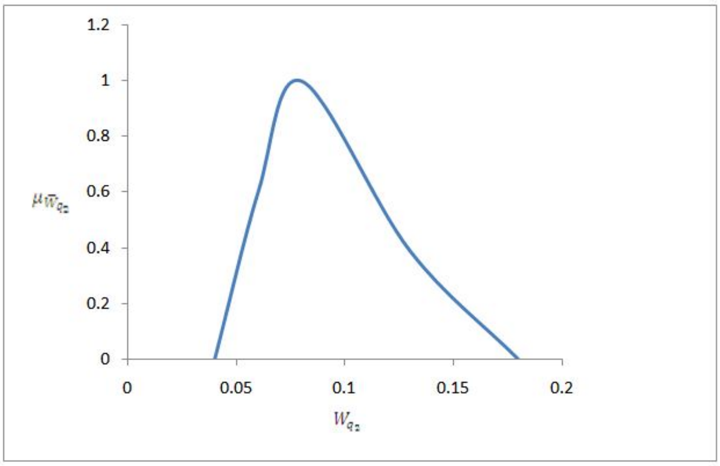


Figure 5.4: Performance measure of the average waiting time of units of second priority in the queue

Conclusion

The fuzzy set theory has been applied to investigate the queuing model of two priority classes by using triangular fuzzy numbers. Based on Zadeh's extension principle,system performances of interest for the expected waiting time in the queue of priority classes and the expected number of customers in the queue when the arrival rates of two priority classes with the same service rates are triangular fuzzy numbers. With the use of α-cuts,we have derived the membership functions of the fuzzy system performances $\bar{L}_{q_1}, \bar{L}_{q_2}, \bar{W}_{q_1}$ and $\bar{W}_{q_2}$ and graphs to the corresponding measures are obtained.

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