International Journal of Fuzzy Mathematics and Systems.
ISSN 2248-9940 Volume 4, Number 1 (2014), pp. 1-5

http://www.ripublication.com

On Statistical Limit Superior, Limit Inferior and Statistical Monotonicity

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Abstract

The purpose of this paper is to study the statistical limit superior and inferior following the concept of statistical convergence and statistical cluster point of a sequence.We also study definition of statistically monotonicity and some of its properties.

Keywords: statistical convergence, statistically limit point, statistically cluster point, statistically limit superior, statistically limit inferior, statistically monotonicity, dense subsequence.

Introduction

The concept of limit and cluster point of a sequence x have been extended to statistical limit and statistical limit points and cluster points [7], [8], [9] using the concept of natural density δ [11] of a set A of positive integers. Statistical convergence has many applications in different fields of mathematics like number theory [6], summability theory [10] and in locally convex spaces [16].

Let A be a subset of IN and A(n) = { aϵA: $a \leq n\}$ then the natural density of A, denoted by δ(A), is defined by

(A) = $\lim_{n\to\infty} \frac{1}{n}$|A(n)|

If the limit exists and where the vertical bars denotes the cardinality of enclosed set. A real or complex valued sequence x = $\{\xi_n\}$ is said to converge stastically to the number ξ, if for every $\varepsilon > 0, the\ set\ A_\varepsilon\ has\ density\ zero\ where\ A_\varepsilon = \{n \in IN$: $|\xi_n - \xi| \geq \varepsilon\}$ and it is denoted by st-lim x =ξ.

The number ν is called a statistical cluster point of x = $\{\xi_k\}$ if for every $\varepsilon > 0$ the set {k: $|\xi_k - \nu| < \varepsilon\}$ does not have density zero.

Throughout in this paper IN nd IR will denote the set of natural numbers and real numbers respectively and we will consider real number sequences.

Statistical Limit Superior And Statistical Limit Inferior:

In this section we study definitions of the concepts of statistical limit superior and inferior and to develop some stastical analogues of properties of the ordinary limit superior and inferior.

If $k \subseteq IN$, then $k_n = \{k: k \leq n\}$ and $|k_n|$denotes the cardinality of k_n. For a real sequence x let B_x denote the set

$B_X = \{ b \in IR: \delta(k: \xi_k > b) \neq 0\}$

Similarily, $A_n = \{ a \in IR: \delta(k: \xi_k < a) \neq 0\}$

Note that throughout in this paper δ(k)≠0 means that either δ(k)> 0 or k does not have natural density.

Definition: If x is a real number sequence, then the statistical limit superior of x is given by

$$\text{St-lim supx} = \begin{cases} \sup B_{x,} \text{ if } B_x \neq \emptyset, \\ -\infty, \text{if } B_x \neq \emptyset. \end{cases}$$

Also, the statistical limit inferior of x is given by

$$\text{St-lim infx} = \begin{cases} \inf A_{x,} \text{ if } A_x \neq \emptyset, \\ +\infty, \text{if } A_x = \emptyset. \end{cases}$$

Following example will help to illustrate the above defined concept. Let us consider a sequence $x = \{\xi_k\}$ defined by

$$\xi_k = \begin{cases} k, \text{if k is an odd square,} \\ 2, \text{if k is an even square,} \\ 1, \text{if k is an odd nonsquare,} \\ o, \text{if k is an even nonsquare.} \end{cases}$$

Although x is unbounded above, it is "statistically bounded" because the set of squares have density zero.

Thus $B_x = (-\infty, 1)$ and st-lim supx = 1. Also, x is not statistically convergent since it has two (disjoint) subsequences of positive density that converge to 0 and 1, respectively (see [8]). Also note that the set of statistically cluster points of x is {0, 1}, and st-lim supx equals the greatest element while st-lim infx is the least element of this set. This observation leads to the main idea of the following theorem which can be proved by least upper bound argument.

Theorem 1: If β = st-lim supx is finite, then for every positive number ε

$\delta\{ k: \xi_k > \beta - \varepsilon\} \neq 0$ and $\delta\{ k: \xi_k > \beta + \varepsilon\} = 0$.

Conversely, if (1) holds for every positive ε then β = st-lim supx.

The dual statement for st-lim infx is as follows.

Theorem 2: If α = st-lim infx is finite, then for every positive number ε

$\delta\{ k: \xi_k < \alpha + \varepsilon\} \neq 0$ and $\delta\{ k: \xi_k < \alpha - \varepsilon\} = 0$.

Conversely, if (2) holds for every positive ε then α = st-lim infx.

Remark: By definition of statistical cluster point [9] we see that theorem 1 and 2 can be interpreted as saying that st-lim supx and st-lim infx are the greatest and least statistical cluster points of x.

Theorem 3: For any sequence x, st-lim infx $\leq$ st-lim supx.

Proof: first let us consider the case when st- lim sup x = $-\infty$. Then Bx = $\emptyset$, so for every b in IR, $\delta\{$ k: $\xi_k > b\}$ = 0. This implies that $\delta\{$ k: $\xi_k \leq b\}$ = 1, so for every a in IR, $\delta\{$ k: $\xi_k < a\} \neq 0$. Hence, st- lim inf x = $-\infty$.

The case in which st-lim sup x = $+\infty$ is obvious. Next assume that st-lim supx is finite say β and let st-lim infx = α.For given $\varepsilon > 0$ we show that $\beta+\varepsilon \in A_x$, so that $\alpha \leq \beta + \varepsilon$.Since β =l.u.b. B_x so by theorem 1, $\delta\{$ k: $\xi_k > \beta + \frac{\varepsilon}{2}$ } = 0. Hence $\delta\{$ k: $\xi_k < \beta + \frac{\varepsilon}{2}\}$ = 1 and which, in turn implies that $\delta\{$ k: $\xi_k < \beta + \varepsilon\}$ = 1.Hence, $\beta + \varepsilon \in A_x$.But $\alpha =$ inf A_x by definition, so we conclude $\alpha \leq \beta + \epsilon$.Since ε is arbitrary, we have $\alpha \leq \beta$.

This completes the proof.

Definition: The real number sequence x is said to be *statistically bounded* if there is a number B such that $\delta\{$ k: $|\xi_k| > B$ } = 0.

Remark: Statistical boundedness implies that st- lim sup and st- lim inf are finite, so Properties (1) and (2) of Theorems 1 and 2 hold.

Theorem 4: The statistically bounded sequence x is statistically convergent if and only if

st- lim inf x = st-lim sup x.

Proof: Let st-lim infx = α and st-lim supx = β. First let us assume that st-lim = ξ and $\varepsilon > 0$ be given.Then $\delta\{$ k: $|\xi_k - \xi| \geq \varepsilon\}$ = 0 and thus $\delta\{$ k: $\xi_k > \xi + \varepsilon$ } = 0.This implies that $\beta \leq \xi$. Also, we have $\delta\{$ k: $\xi_k < \xi - \varepsilon$ } = 0 which implies $\xi \leq \alpha$. Hence, we have $\beta \leq \xi \leq \alpha$. i.e. $\beta \leq \alpha$. But by theorem 3, $\alpha \leq \beta$.Thus α=β. Conversely, assume $\alpha = \beta$ and let $\varepsilon > 0$ be given. Then by theorem 1 and 2 we have $\delta\{$ k: $\xi_k > \beta + \frac{\varepsilon}{2}\}$ = 0 and $\delta\{$ k: $\xi_k < \alpha - \frac{\varepsilon}{2}\}$ = 0. But $\alpha = \beta = \xi$(say) so we have $\delta\{$ k: $\xi_k > \xi + \frac{\varepsilon}{2}\}$ = 0 and $\delta\{$ k: $\xi_k < \xi - \frac{\varepsilon}{2}\}$ = 0. Hence st-lim x = ξ.

This completes the proof.

Statistical monotonicity:

In this section we study the concept of statistical monotonicity [13] and some related results.

Definition: A sequence x = (ξ_n) is statistical monotone increasing (decreasing) if there exists a subset A$\subseteq$IN with δ(A) = 1 such that the sequence x = (ξ_n) is monotone increasing (or decreasing) on A.

Definition: A sequence x = (ξ_n) is statistical monotone if it is statistical monotone increasing or statistical monotone decreasing.

Theorem: If the sequence x = (ξ_n) is bounded and statistical monotone then it is statistically convergent.

Theorem: If x = (ξ_n) is statistical monotone increasing or statistical monotone decreasing then

$$\lim_{n\to\infty} \frac{1}{n} |\{ k: k \leq n, x_{k+1} < x_k\}| = 0$$

Or

$$\lim_{n\to\infty} \frac{1}{n} |\{ k: k \leq n, x_{k+1} > x_k\}| = 0$$

respectively.

Remark: The inverse of these assertions is not correct because of the following example:

Define x = (ξ_n) by

$$\xi_n = \begin{cases} 1, \text{if } 2^k \leq n < 2^{k+1} - 1 \text{ for even k}, \\ \quad\quad 0, \text{otherwise}. \end{cases}$$

Clearly above theorem holds for this sequence but this sequence is not statistical monotone (and not statistically convergent).

Definition: (Dense Subsequence) The subsequence x' = (ξ_{n_k}) of x = (ξ_n) is called a dense subsequence, if □$(K_{\infty'})$ = 1.

Theorem: Every dense subsequence of a statistical monotone sequence is statistical monotone.

Theorem: The statistical monotone sequence x = (ξ_n) is statistically convergent if and only if x = (ξ_n) is statistical bounded.

Definition: The sequence x = (ξ_n) and y = (σ_n) are called statistical equivalent if there is a subset M of N with □(M) = 1 such that $\xi_n = \sigma_n$ for each n∈M.It is denoted by x□y.

With this definition we formulate following theorem.

Theorem: Let x = (ξ_n) and y = (σ_n) be statistical equivalent. Then x = (ξ_n) statistical monotone if and only if y = (σ_n) is statistical monotone.

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