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# A Common Fixed Point Theorem for Three Self Mappings in a Fuzzy 2-Metric Spaces

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## Abstract

The aim of this present paper is to obtain a common fixed point for three mappings in a fuzzy 2-metric spaces which generalizes a result of A. K. Sharma et. al. [9].

**Keywords:** Fuzzy sets, Fuzzy 2-metric spaces, R-weakly commuting maps, Fixed Point.

## Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequal.

**Definition1.1 [11]:** A fuzzy set A in X is function with domain X and values in [0, 1].

**Definition1.2 [11]:** A binary operation $*: [0.1] \times [0,1] \times [0,1] \to [0,1]$ is called a t-norm of $\{[0,1],*\}$ is an abelian topological monoid with unit 1 such that $a_1 * b_1 * c_1 \leq a_2 * b_2 * c_2$ whenever $a_1 \leq a_2, b_1 \leq b_2, c_1 \leq c_2$ $for\ all\ a_1, a_2, b_1, b_2, c_1, c_2 \in [0,1]$.

**Definition1.3 [11]:** The $3-triple$ $(X, M, *)$ is said to be fuzzy 2-metric space if X is an arbitrary set, * is a continuous $t-norm$ and M is a fuzzy set in $X^3 \times [0, \infty)$ satisfying the following conditions:

$For\ all\ x, y, z \in X\ and\ s, t > 0$

[FM-1] $M(x, y, z, 0) = 0,$

[FM-2] $M\,(x, y, z, t) \;=\; 1\, for\; all\; t \;>\; 0$ and when at least two of the three points are equal,

[FM-3] $M(x, y, z, t) = \; M\,(y, x, z, t) = \; M\,(z, x, y, t)$ symmetry about three variabls,

[FM-4] $M\,(x, y, u, t_1) * \; M\,(x, u, z, t_2) * \; M\,(u, y, z, t_3) \quad \leq \; M\,(x, y, z, t_1 + t_2 \;+\; t_3), \forall\, x, y, z, u \;\in\; X\; and\; t_1, t_2, t_3 \;>\; 0$

[FM-5] $M\,(x, y, z.): [0, \infty) \;\rightarrow\; [0, 1]\; is\; left\; continuous,$

[FM-6] $\lim_{t\to\infty} \mathrm{M(x, y, z, t)} = 1$

The function value $M\,(x, y, z, t)$ may be interpreted as the probability that the area of triangle is less than t.

**Definition1.4 [11]:** $Let\; (X, M,*)\; be\; a\; fuzzy\; 2 - metric\; space.$

A sequence $\{x_n\}$ in $fuzzy\; 2 - metric$ space X is said to be convergent to a point $x \;\in\; X$ if

$$\lim_{n\to\infty} \mathrm{M(x, y, z, t)} = 1 \text{ for all } a \in \; X\; and\; t \;>\; 0$$

A sequence $\{x_n\}$ in $fuzzy\; 2 - metric$ space X is called Cauchy sequence if

$$\lim_{n\to\infty} M\,(x_{n+p}, x_n, w, t) = 1 \text{ for } all\; a \;\in\; X\; and\; t \;>\; 0, p \;>\; 0$$

A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to complete.

The mapping f and g of a fuzzy metric space (X, M, *) into itself are R-weakly commuting provided there exists some positive real number R such that

$$M(fgx, gfx, a, t) \geq M(fx, gx, a, t/R) \text{ for all } x \in X$$

**Theorem 1.5:** Let $(X, M,*)$ be a complete fuzzy 2-metric space and let $f\; and\; g\; be\; R - weakly$ commuting self mappings of X satisfying the conditions:

$$M(fx, fy, w, t) \geq r\big(M(gx, gy, w, t/R)\big), for\; x, y\; in\; X$$

Where $r$: $[0, 1] \rightarrow [0, 1]$ is a continuous function such that $r(t) > t\; for\; each\; 0 < t < 1$

The sequence $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \rightarrow x\; y_n \rightarrow y, t > 0$ implies that

$$M(x_n, y_n, w, t) \rightarrow M(x, y, w, t)\; as\; n \rightarrow \infty$$

If the range of $g\; contains\; the\; range\; of\; f$ and if $either\; f\; or\; g\; is\; continuous, then\; f\; and\; g$ have a unique common fixed point in X.

Now if let $(X, M,*)$ be a complete fuzzy 2-metric space and let $f, g\; and\; h\; be\; R - weakly$ commuting mappings of X into X satisfying the conditions;

$$f(X) \cap \; g(X) \;\subset h(X) \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

$$M(fx, gy, w, t) \geq r\big(M(hx, hy, w, t)\big)\; for\; all\; x, y\; in\; X \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2)$$

$where\; r$: $[0, 1] \rightarrow [0, 1]$ is a continuous function such that $r(t) > t\; for\; each\; 0 < t < 1$

Then for any $x_0 \in X, by$ (1.1), we choose a point $x_1 \in X$ such that $fx_0 = hx_1$ and for this$x_1$; there exists a point $x_2 \in X$ such that $gx_1 = hx_2$ and so on.

Continuing in the manner, we can choose a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = hx_{2n+1,}\, y_{2n+1} = gx_{2n+1} = hx_{2n+2,}\, for\ n = 0, 1, 2 \ldots\ldots\ldots\ldots\ldots \quad (1.3)$$

The sequence $\{x_n\}$ and $\{y_n\}$ in X are such that $x_n \to x\ y_n \to y, t > 0$ implies that

$$M(x_n, y_n, w, t) \to M(x, y, w, t)\ as\ n \to \infty$$

**Lemma 1.6.** Let $(X, M,*)$ be a complete fuzzy 2-metric space and let $f, g\ and\ h\ be\ R - weakly$ commuting self mappings of X into X satisfying the conditions (1.1) and (1.2) then the sequence $\{y_n\}$ defined in (1.3) is a Cauchy sequence in X.

**Proof:** $For\ t > 0$

$$M(y_{2n}, y_{2n+1}, w, t) = M(fx_{2n}, gx_{2n+1}, w, t)$$
$$\geq r\{M(hx_{2n}, hx_{2n+1}, w, t)\}$$
$$\geq r\{M(y_{2n-1}, y_{2n}, w, t)\} \ldots\ldots\ldots\ldots\ldots\ldots \quad (1.4)$$
$$\geq M(y_{2n-1}, y_{2n}, w, t)$$

i.e.$M(y_{2n}, y_{2n+1}, w, t) > M(y_{2n-1}, y_{2n}, w, t)\ for\ all\ n \geq 0$.

Thus $\{M(y_{2n}, y_{2n+1}, w, t)\}, n \geq 0$ is a increasing sequence of positive numbers in [0, 1] and therefore to a limit 1≤1. We claim that 1=1. For if 1<1, on letting n→∞ in (1.4), we have 1≥r(1)>1, which is a contradiction. Hence 1=1

Similarly $M(y_{2n+1}, y_{2n+2}, w, t) > M(y_{2n}, y_{2n+1}, w, t)$ ∀ n≥0 $\{M(y_{2n+1}, y_{2n+2}, w, t)\}, n \geq 0$ is a increasing sequence of positive numbers in [0, 1] and therefore to a limit 1=1.therefore for every $n \in N$

$$M(y_n, y_{n+1}, w, t) > M(y_{n-1}, y_n, w, t)\ and\ M(y_n, y_{n+1}, w, t) = 1; t > 0.$$

Now for any positive integer p,

$$M\big(y_n, y_{n+p}, w, t\big) \geq M(y_n, y_{n+1}, w, t/p) * \ldots\ldots\ldots\ldots M\big(y_{n+p-1}, y_{n+p}, w, t/p\big)$$
$$\geq M(y_n, y_{n+1}, w, t/p) * \ldots\ldots\ldots\ldots M(y_n, y_{n+1}, w, t/p)$$
$$\geq 1 * 1 * \ldots\ldots\ldots * 1$$

i.e. $M\big(y_n, y_{n+p}, w, t\big) \geq 1.\ thus\ \{y_n\}\ is\ a\ cauchy\ sequence\ in\ X.$

**Main Result**

**Theorem 2.1:** Let $f, g\ and\ h\ be\ three\ self\ mappings\ on\ a\ fuzzy\ 2 - metric\ space$ $(X, M,*)$ satisfying (1.1) and (1.2). Suppose that $h$ is continuous and pairs $(f, h)$ & $(g, h)\ are\ R - weakly$ commuting on X. Then $f, g\ and\ h$ have a unique common fixed point in X.

**Proof:** For any arbitrary point $x_0 \in X$ by (2.1), we can choose $x_1 \in X$: $fx_0 = hx_1$ and for this $x_1 \in X$, there exist $x_2 \in X$: $gx_1 = hx_2$ and so continue in this manner we can choose a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = hx_{2n+1,}\, y_{2n+1} = gx_{2n+1} = hx_{2n+2,}\, for\ n = 0, 1, 2 \dots \dots \dots$$

Then by lemma 1.2, sequence $\{y_n\}$ is a Cauchy sequence in X. But X is complete and so by completeness of X, $\{y_n\}$ is converges to some point u in X. Consequently the sequences

$\{fx_{2n}\}, \{hx_{2n+1}\ \}, \{gx_{2n+1}\}, \{hx_{2n+2}\ \}\ of\ \{y_n\}$ also converges to u in X.

Suppose that $h$ is continuous and let pairs $(f, h)\ is\ R - weakly\ commuting$ then it follows that

$$M(fhx_n, hfx_n, w, t) \geq r(M\left(fx_n, hx_n, w, \frac{t}{R}\right)\ for\ all\ x\ in\ X.$$

On letting n→∞ we get

$$M(fhx_n hu, w, t) \to M\left(u, u, w, \frac{t}{R}\right)\ hence\ fhx_n \to hu$$

From (1.2) $M(fhx_{2n}, gx_{2n+1}, w, t) \geq r\{M(hhx_{2n}, hx_{2n+1}, w, t)\}\ on\ letting\ n \to \infty, we\ get$

$$M(hu, u, w, t) \geq r\{M(hu, u, w, t)\}$$
$$>\ M(hu, u, w, t)\ which\ is\ contradiction.$$

Hence, $hu = u$

Also by (1.2) $M(fu, gx_{2n+1}, w, t) \geq r\{M(hu, hx_{2n+1}, w, t)\}$

On letting n→∞, we get $M(fu, u, w, t) \geq r\{M(hu, u, w, t)\}$

$= r\{M(u, u, w, t)$

$= r\{1\} i.e. 1$

$Hence, fu = u.$

Now consider $M(u, gu, w, t) = M(fu, gu, w, t)$

$\geq r\{M(hu, hu, w, t)$

$= r\{1\} i.e. 1$

$Hence\ gu = u, thus\ u\ is\ common\ fixed\ point\ of\ f, g\ and\ h.$

Uniqueness: Suppose that $v \neq u$ is another common fixed point of $f, g\ and\ h$. Then there exists

$t > 0$ such that $M(u, v, w, t) < 1$

$M(u, v, w, t) = M(fu, gv, w, t)$

$\geq r\{M(hu, hv, w, t)\}$

$= r\{M(u, v, w, t)\} > M(u, v, w, t)$, which is a contradiction.

Hence, $v = u\ and\ so\ u$ is unique common fixed point of $f, g\ and\ h$.

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