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Fuzzy Modular Pair in Fuzzy Lattice and Fuzzy Modular Lattice

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Abstract

In this paper,Fuzzy Modular Pair in Fuzzy Lattice-Definition of Fuzzy Modular Pair-Characterization Theorem-Fuzzy Modular Pair in Fuzzy Modular Lattice-Definition-Characterization Theorem.

Keywords: Fuzzy Lattice, Fuzzy Modular Pair, Fuzzy Modular Lattice.

Introduction

The concept of fuzzy lattice was already introduced by Ajmal,N[1], S.Nanda[4], Wilcox,L.R[5] explained modularity in the theory of lattices, G.Gratzer[2] and M.Mullai&B.Chellappa[3] explained Fuzzy L-ideal.A few definitions and results are listed that the equivalent conditions of the fuzzy lattice using in this paper,we explain Fuzzy Modular Pair in Fuzzy Lattice,Definition of Fuzzy Modular Pair,characterization theoremof Fuzzy Modular Pair,Fuzzy Modular Pair in Fuzzy Modular Lattice and some Examples are given.If L is a fuzzy lattice then $(\mu(a),\mu(b))$ is a fuzzy modular pair.

FUZZY MODULAR PAIR IN FUZZY LATTICE

Definition: 1.1

Let L be a Fuzzy lattice and$\mu(a),\mu(b)$ in L. Thus $(\mu(a),\mu(b))$ is called a Fuzzy modular pair if $\mu(c)\vee\mu(a\wedge b) =\mu\ (c\vee a)\wedge\mu(b)$, for all $\mu(c) \leq \mu(b)$ in L.

(ie)$\mu(c) \vee [\mu(a) \wedge \mu(c \vee b)] = \mu(c \vee a) \wedge \mu(c \vee b)$, for all $\mu(c)$ in L.

Example: 1.1

Consider the Fuzzy lattice N_5in the following figure 1.1.

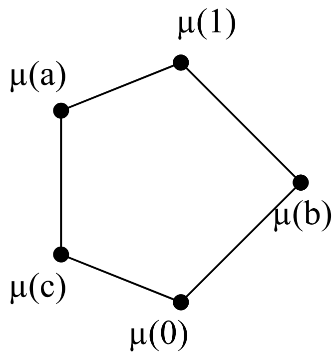

Figure:1.1

Here $(\mu(a), \mu(b))$ is a Fuzzy modular pair.

$(\mu(b),\mu(a))$ is not a Fuzzy modular pair.

Since$\mu(c) \leq \mu(a)$, $\mu(c) \vee \mu(b \wedge a) \geq \min\{\mu(c), \mu(b \wedge a)\}$

$\geq \min\{\mu(c), \mu(0)\}$

$= \mu(c)$

$\mu(c \vee b) \wedge \mu(a) \geq \min\{\mu(c \vee b), \mu(a)\}$

$\geq \min\{\mu(1), \mu(a)\}$

$= \mu(a)$

So $\mu(c) \vee \mu(b \wedge a) \neq \mu(c \vee b) \wedge \mu(a)$ for $\mu(c) \leq \mu(a)$.

Theorem: 1.1

If $\mu(a),\mu(b)$ in a Fuzzy lattice L and $\mu(a) \leq \mu(b)$ then $(\mu(a),\mu(b))$ is a Fuzzy modular pair.

Proof:

Given $\mu(a), \mu(b) \in L$ and $\mu(a) \leq \mu(b)$

$\Rightarrow \mu(a \vee b) = \mu(b), \quad \mu(a \wedge b) = \mu(a) \quad \rightarrow (1)$

To prove $(\mu(a),\mu(b))$ is a Fuzzy modular pair.

That is to prove

$\mu(c) \vee \mu(a \wedge b) = \mu(c \vee a) \wedge \mu(b)$, for all $\mu(c) \leq \mu(b)$, $\mu(c) \in L$.

Let $\mu(c) \in L$ be arbitrary and $\mu(c) \leq \mu(b)$

Then $\mu(c) \vee \mu(a \wedge b) \geq \min \{ \mu(c), \mu(a \wedge b) \}$

$\geq \min \{ \mu(c), \mu(a) \}$, by (1)

$= \mu(c) \vee \mu(a)$

We have $\mu(c) \leq \mu(b), \mu(a) \leq \mu(b)$

$\Rightarrow \mu(c \vee a) \geq \min \{ \mu(c), \mu(a) \}$

$\geq \min \{ \mu(b), \mu(b) \}$

$= \mu(b)$

$\Rightarrow \mu(c \vee a) \leq \mu(b)$

$\Rightarrow \mu(c \vee a) \wedge \mu(b) = \mu(c \vee a)$

Therefore $\mu(c) \vee \mu(a \wedge b) = \mu(c \vee a) \wedge \mu(b)$, for all $\mu(c) \leq \mu(b)$.

Hence $(\mu(a),\mu(b))$ is a Fuzzy modular pair.

Theorem: 1.2

If $\mu(a)$, $\mu(b)$ in a Fuzzy lattice L and $\mu(a) \leq \mu(b)$ then $(\mu(b),\mu(a))$ is a Fuzzy modular pair.

Proof:

Given $\mu(a),\mu(b) \in L$ and $\mu(a) \leq \mu(b)$

$\Rightarrow \mu(b \vee a) = \mu(b), \quad \mu(b \wedge a) = \mu(a) \quad \rightarrow (1)$

To prove $(\mu(b),\mu(a))$ is a Fuzzy modular pair.

That is to prove

$$\mu(c)\vee\mu(b\wedge a) = \mu(c\vee b)\wedge\mu(a), \text{ for all } \mu(c) \leq \mu(a), \mu(c)\in L.$$

Let $\mu(c)\in L$ be arbitrary and $\mu(c) \leq \mu(a)$

Then $\mu(c)\vee\mu(b\wedge a) \geq \min\{\mu(c),\mu(b\wedge a)\}$

$\geq \min\{\mu(c),\mu(a)\}$, by (1)

$= \mu(c\vee a)$

$= \mu(a)$

We have $\mu(c) \leq \mu(a), \mu(a) \leq \mu(b)$

$\Rightarrow \mu(c) \leq \mu(b)$

$\Rightarrow \mu(c\vee b) = \mu(b)$

$\Rightarrow \mu(c\vee b)\wedge\mu(a) = \mu(b)\wedge\mu(a) = \mu(a)$

Therefore $\mu(c)\vee\mu(b\wedge a) = \mu(c\vee b)\wedge\mu(a)$, for all $\mu(c) \leq \mu(b)$.

Hence $(\mu(b), \mu(a))$ is a Fuzzy modular pair.

Theorem: 1.3

Let $\mu(a)$ be an element of a Fuzzy lattice L. Then $(\mu(a),\mu(x))$ is a Fuzzy modular pair for all $\mu(x)\in L$ if and only if $\mu(y)\vee\mu(a\wedge x) = \mu(y\vee a)\wedge\mu(x)$, for any pair $(\mu(x),\mu(y))$ with $\mu(x) \geq \mu(y)$.

Proof:

(i) Assume $(\mu(a),\mu(x))$ is a Fuzzy modular pair for all $\mu(x)\in L$.

To prove $\mu(y)\vee\mu(a\wedge x) = \mu(y\vee a)\wedge\mu(x)$, for any pair $(\mu(x),\mu(y))$ with $\mu(x) \geq \mu(y)$.

Let $\mu(y)\in L$ be arbitrary such that $\mu(y) \leq \mu(x)$.

Then $\mu(y)\vee\mu(a\wedge x) \geq \min\{\mu(y),\mu(a\wedge x)\}$

$\geq \min \{\mu(y \vee a), \mu(x)\}$

$= \mu(y \vee a) \wedge \mu(x)$, for all $\mu(y)$,$\mu(y) \leq \mu(x)$.

$\Rightarrow \mu(y) \vee \mu(a \wedge x) = \mu(y \vee a) \wedge \mu(x)$, for any pair($\mu(x),\mu(y)$) with$\mu(x) \geq \mu(y)$.

.(ii) Assume$\mu(y) \vee \mu(a \wedge x) \geq \min \{ \mu(y), \mu(a \wedge x)\}$

$\geq \min \{\mu(y \vee a), \mu(x)\}$

$= \mu(y \vee a) \wedge \mu(x)$,for any pair $(\mu(x),\mu(y))$ with $\mu(x) \geq \mu(y)$.

To prove $(\mu(a),\mu(x))$ is a Fuzzy modular pair.

Let$\mu(y) \in L$ be arbitrary such that $\mu(y) \leq \mu(x)$.

Then by assumption

$\mu(y) \vee \mu(a \wedge x) \geq \min \{ \mu(y), \mu(a \wedge x)\}$

$\geq \min \{\mu(y \vee a), \mu(x)\}$

$= \mu(y \vee a) \wedge \mu(x)$,for any pair $(\mu(x),\mu(y))$ with $\mu(x) \geq \mu(y)$.

$\Rightarrow \mu(y) \vee \mu(a \wedge x) = \mu(y \vee a) \wedge \mu(x)$, for all $\mu(y) \leq \mu(x)$.

$\Rightarrow$ $(\mu(a),\mu(x))$ is a Fuzzy modular pair.

Theorem: 1.4

Let L is a Fuzzy lattice and then the following are equivalent.

(i) $\mu(y) \vee \mu(a \wedge x) = \mu(y \vee a) \wedge \mu(x)$ with$\mu(x) \geq \mu(y)$.

(ii) $\mu(y) \wedge \mu(a \vee x) = \mu(y \wedge a) \vee \mu(x)$ with$\mu(x) \leq \mu(y)$.

Proof:

(i) $\Rightarrow$ (ii)

Assume that $\mu(y)\vee\mu(a\wedge x) = \mu(y\vee a)\wedge\mu(x)$ with $\mu(x) \geq \mu(y)$.

To prove $\mu(y)\wedge\mu(a\vee x) = \mu(y\wedge a)\vee\mu(x)$ with $\mu(x) \leq \mu(y)$.

Suppose $\mu(x) \leq \mu(y)$

$\Rightarrow \mu(y) \geq \mu(x)$

Then $\mu(x)\vee\mu(y\wedge a) \geq \min\{\mu(x),\mu(y\wedge a)\}$

$\geq \min\{\mu(x),\mu(a\wedge y)\}$

$\geq \min\{\mu(x\vee a),\mu(y)\}$

$= \mu(x\vee a)\wedge\mu(y)$, by (i) since $\mu(y) \geq \mu(x)$.

(ii) $\Rightarrow$ (i)

Assume that $\mu(y)\wedge\mu(a\vee x) = \mu(y\wedge a)\vee\mu(x)$ with $\mu(x) \leq \mu(y)$.

To prove $\mu(y)\vee\mu(a\wedge x) = \mu(y\vee a)\wedge\mu(x)$ with $\mu(x) \geq \mu(y)$.

Suppose $\mu(x) \geq \mu(y)$

$\Rightarrow \mu(y) \leq \mu(x)$

Then $\mu(x)\wedge\mu(y\vee a) \geq \min\{\mu(x),\mu(y\vee a)\}$

$\geq \min\{\mu(x),\mu(a\vee y)\}$

$\geq \min\{\mu(x\wedge a),\mu(y)\}$

$= \mu(x\wedge a)\vee\mu(y)$, by (ii) since $\mu(y) \leq \mu(x)$.

FUZZY MODULAR PAIRS IN FUZZY MODULAR LATTICE

Definition: 1.2

A Fuzzy lattice L is said to be Fuzzy modular lattice, if

$\mu(a\vee b) \wedge\mu(a\vee c) =\mu(a)\vee[\mu(b)\wedge\mu(a\vee c)]$, for all $\mu(a),\mu(b),\mu(c)$ in L.

Example: 1.2

Consider the Fuzzy set M_4 $=\{\mu(o),\mu(a),\mu(b),\mu(c),\mu(d),\mu(l) / \mu(o)<\mu(a),\mu(b),\mu(c),\mu(d) <\mu(l)\}$ of figure 1.2. Then $M_4$is a Fuzzy modular lattice.

Verification:

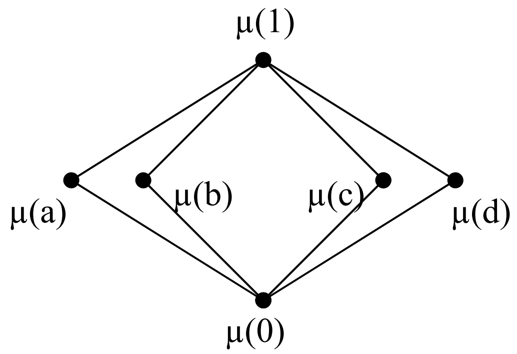

Figure:1.2

Form the tables:

$\vee$	μ(o)	μ(a)	μ(b)	μ(c)	μ(d)	μ(l)
μ(o)	μ(o)	μ(a)	μ(b)	μ(c)	μ(d)	μ(l)
μ(a)	μ(a)	μ(a)	μ(l)	μ(l)	μ(l)	μ(l)
μ(b)	μ(b)	μ(l)	μ(b)	μ(l)	μ(l)	μ(l)
μ(c)	μ(c)	μ(l)	μ(l)	μ(c)	μ(l)	μ(l)
μ(d)	μ(d)	μ(l)	μ(l)	μ(l)	μ(d)	μ(l)
μ(l)	μ(l)	μ(l)	μ(l)	μ(l)	μ(l)	μ(l)

$\wedge$	$\mu(o)$	$\mu(a)$	$\mu(b)$	$\mu(c)$	$\mu(d)$	$\mu(l)$
$\mu(o)$	$\mu(o)$	$\mu(o)$	$\mu(o)$	$\mu(o)$	$\mu(o)$	$\mu(o)$
$\mu(a)$	$\mu(o)$	$\mu(a)$	$\mu(o)$	$\mu(o)$	$\mu(o)$	$\mu(o)$
$\mu(b)$	$\mu(o)$	$\mu(o)$	$\mu(b)$	$\mu(o)$	$\mu(o)$	$\mu(o)$
$\mu(c)$	$\mu(o)$	$\mu(o)$	$\mu(o)$	$\mu(c)$	$\mu(o)$	$\mu(o)$
$\mu(d)$	$\mu(o)$	$\mu(o)$	$\mu(o)$	$\mu(o)$	$\mu(d)$	$\mu(o)$
$\mu(l)$	$\mu(o)$	$\mu(0)$	$\mu(0)$	$\mu(0)$	$\mu(0)$	$\mu(l)$

From the tables we have

i. $\mu(x\vee x) \geq \min\{\mu(x), \mu(x)\}$, $\mu(x\wedge x) \geq \min\{\mu(x), \mu(x)\}$,

$=\mu(x)$ $\quad$ $=\mu(x)$

ii. $\mu(x\vee y) \geq \min\{\mu(x), \mu(y)\}$, $\mu(x\wedge y) \geq \min\{\mu(x), \mu(y)\}$,

$\geq \min\{\mu(y), \mu(x)\}$, $\quad$ $\geq \min\{\mu(y), \mu(x)\}$,

$=\mu(y\vee x)$ $\quad$ $=\mu(y\wedge x)$

iii. $\mu(x)\vee\mu(y\vee z) \geq \min\{\mu(x),\mu(y\vee z)\}$, $\quad$ $\mu(x)\wedge\mu(y\vee z) \geq\min\{\mu(x), \mu(y\wedge z)\}$,

$\geq \min\{\mu(x\vee y), \mu(z)\}$, $\quad$ $\geq \min\{\mu(x\vee y), \mu(z)\}$,

$=\mu(x\vee y)\vee\mu(z)$ $\quad$ $= \mu(x\wedge y)\wedge\mu(z)$

iv. $\mu(x)\vee\mu(x\wedge y) \geq \min\{\mu(x),\mu(x\wedge y)\}$, $\quad$ $\mu(x)\wedge\mu(x\vee y) \geq \min\{\mu(x),\mu(x\vee y)\}$,

$=\mu(x)$ $\quad$ $=\mu(x)$

v. $\mu(x\vee y)\wedge\mu(x\vee z) \geq \min\{\mu(x\vee y),\mu(x\vee z)\}$,

$\geq \min\{\min\{\mu(x), \mu(y)\},\mu(x\vee z)\}$

$\geq \min\{\min\{\mu(x), \min\{\mu(y),\mu(x\vee z)\}\}$

$\geq \min\{\mu(x), \mu(y)\wedge\mu(x\vee z)\}$

$= \mu(x)\vee[\mu(y)\wedge\mu(x\vee z)]$, for all $\mu(x),\mu(y), \mu(z)$ in M_4

Hence M_4 is a Fuzzy modular lattice.

Theorem: 1.5

A Fuzzy lattice L is Fuzzy modular, if and only if $\mu(a) \geq \mu(b)$ and $\mu(a\vee c) = \mu(b\vee c)$, $\mu(a\wedge c) =\mu(b\wedge c)$ for any $\mu(c)$ imply that $\mu(a) = \mu(b)$.

Proof:

Part – I

Assume that L be a Fuzzy modular lattice.

To prove that: if $\mu(a) \geq \mu(b)$ and

$\mu(a\vee c) = \mu(b\vee c) \rightarrow (1)$

$\mu(a\wedge c) = \mu(b\wedge c) \rightarrow (2)$

for any $\mu(c)$ then $\mu(a) =\mu(b)$.

$\mu(a) = \mu(a\wedge (a\vee c))$,

$\geq \min\{\mu(a), \mu(a\vee c)\}$

$\geq \min\{\mu(a), \mu(b\vee c)\}$, by (1)

$\geq \min\{\mu(b\vee c), \mu(a)\}$,by commutative law

$\geq \min\{\min\{\mu(b), \mu(c)\},\mu(a)\}$

$\geq \min\{\mu(b), \min\{\mu(c),\mu(a)\}$, by commutative law

$\geq \min\{\mu(b), \mu(c\wedge a)\}$

$\geq \min\{\mu(b), \mu(b\wedge c)\}$ by (2)

$= \mu(b) \vee \mu(b \wedge c)$

$= \mu(b)$

Hence $\mu(a) = \mu(b)$.

Part: II

Assume that L is Fuzzy modular lattice and if $\mu(a) \geq \mu(b)$ and $\mu(a \vee c) = \mu(b \vee c)$, $\mu(a \wedge c) = \mu(b \wedge c)$ for any $\mu(c)$ then $\mu(a) = \mu(b)$.

To prove that L is Fuzzy modular.

We claim that $\mu(a \vee b) \wedge \mu(a \vee c) = \mu(a) \vee [\mu(b) \wedge \mu(a \vee c)]$, for all $\mu(a), \mu(b), \mu(c)$ in L

By assumption it is sufficient to prove

$\mu(a \vee b) \wedge \mu(a \vee c) \geq \mu(a) \vee [\mu(b) \wedge \mu(a \vee c)]$

$\mu(a \vee b) \geq \mu(a)$

$\mu(a \vee c) \geq \mu(a)$

$\Rightarrow \mu(a \vee b) \wedge \mu(a \vee c) \geq \min\{\mu(a \vee b), \mu(a \vee c)\}$

$\geq \min\{\mu(a), \mu(a)\}$

$\geq \mu(a)$

$\Rightarrow \mu(a \vee b) \wedge \mu(a \vee c) \geq \mu(a) \rightarrow (1)$

$\mu(a \vee b) \geq \mu(b)$

$\Rightarrow \mu(a \vee b) \wedge \mu(a \vee c) \geq \min\{\mu(a \vee b), \mu(a \vee c)\} \rightarrow (2)$

$\geq \min\{\mu(b), \mu(a \vee c)\}$

$\geq \mu(b) \wedge \mu(a \vee c) \rightarrow (2)$

(1) $\vee$ (2) gives

$[\mu(a \vee b) \wedge \mu(a \vee c)] \vee [\mu(a \vee b) \wedge \mu(a \vee c)] \geq \mu(a) \vee [$

$\mu(b)\wedge\mu(a\vee c)]$

$\Rightarrow\mu(a\vee b)\wedge \mu(a\vee c)\geq \min\{\mu(a\vee b), \mu(a\vee c)\}$

$\geq \min\{\min\{\mu(a),\mu(b)\}, \mu(a\vee c)\}$

$\geq \min\{\mu(a), \min\{\mu(b), \mu(a\vee c)\}\}$

$\geq \min\{\mu(a),\mu(b)\wedge\mu(a\vee c)\}$

$\geq \mu(a)\vee[\mu(b)\wedge\mu(a\vee c)]$

$[\mu(a\vee b)\wedge\mu(a\vee c)]\wedge\mu(b) = [\mu(a)\vee[\mu(b)\wedge\mu(a\vee c)]]\wedge\mu(b)$

$[\mu(a\vee b)\wedge\mu (a\vee c)]\vee\mu(b) = [\mu(a)\vee[\mu(b)\wedge\mu(a\vee c)]]\vee\mu(b)$

Let $\mu(a)$, $\mu(b)$, $\mu(c)$ in L be arbitrary. Then

$[\mu(a\vee b) \wedge\mu(a\vee c)]\wedge\mu(b)\geq \min\{\mu(a\vee b)\wedge\mu(a\vee c),\mu(b)\}$

$\geq \min\{\mu(a\vee c)\wedge\mu(a\vee b),\mu(b)\}$, by commutative law

$\geq \min\{\mu(a\vee c),\mu(a\vee b)\wedge\mu(b)\}$, by associative law

$\geq \min\{\mu(a\vee c),\mu(b)\wedge\mu(a\vee b)\}$, by commutative law

$\geq \min\{\mu(a\vee c),\mu(b)\wedge\mu(b\vee a)\}$, by commutative law

$\geq \min\{\mu(a\vee c),\mu(b)\}$ by absorption law

$= \mu(a\vee c)\wedge\mu(b) \rightarrow (1)$

$\mu(a\vee c)\wedge\mu(b) = [\mu(a\vee c)\wedge\mu(b)]\wedge\mu(b)$

$\leq [\mu(a)\vee(\mu(b)\wedge\mu(a\vee c))] \wedge\mu(b)$

$$\leq [\mu(a)\vee\mu(a\vee c)] \wedge\mu(b)$$

$$=\mu(a\vee c)\wedge\mu(b)$$

$$\therefore\ \mu(a\vee c)\wedge\mu(b) = [\mu(a)\vee(\mu(b)\wedge\mu(a\vee c))] \wedge\mu(b) \rightarrow (2)$$

$$[\mu(a)\vee[\mu(b)\wedge\mu(a\vee c)]] \vee\mu(b) = \mu(a\vee b) \rightarrow (3)$$

$$\mu(a\vee b) =\mu(a\vee b) \vee\mu(b)$$

$$\geq [\mu(a\vee b) \wedge\mu(a\vee c)]\vee\mu(b)$$

$$= [\mu(b)\vee[\mu(c)\wedge\mu(a\vee b)]]\vee\mu(b)$$

$$=\mu(a\vee b)$$

$$\therefore\ [\mu(a\vee b)\wedge\mu(a\vee c)] \vee\mu(b) =\mu(a\vee b) \rightarrow (4)$$

Thus $\mu(a\vee b)\wedge\mu(a\vee c) =\mu(a)\vee [\mu(b)\wedge\mu(a\vee c)]$, for all $\mu(a),\mu(b),\mu(c)$ in L.

Hence L is a Fuzzy modular lattice.

A necessary and sufficient condition for the Fuzzy lattice L is Fuzzy modular.

Theorem: 1.6

If L is a Fuzzy modular lattice, then $(\mu(a), \mu(b))$ is a Fuzzy modular pair for all $\mu(a),\mu(b) \in L$.

Proof:

Given L is a Fuzzy modular lattice.

To prove $(\mu(a),\mu(b))$ is a Fuzzy modular pair for all $\mu(a),\mu(b) \in L$.

L is a Fuzzy modular lattice

$$\Rightarrow\mu(x\vee y) \wedge\mu(x\vee z) = \mu(x)\vee[\mu(y)\wedge\mu(x\vee z)] \text{ for all } \mu(x),\mu(y),\mu(z) \in L$$

Take $\mu(c) =\mu(x)$, $\mu(a) =\mu(y),\mu(b) =\mu(x\vee z)$ we get

$$\begin{aligned}\mu(c\vee a)\wedge\mu(b) &\geq \min\{\mu(c\vee a)\ ,\ \mu(b)\}\\ &\geq \min\{\min\{\mu(c),\mu(a)\}\ ,\ \mu(b)\}\\ &\geq \min\{\mu(c),\min\{\mu(a)\ ,\ \mu(b)\}\}\\ &\geq \min\{\mu(c),\mu(a\wedge b)\}\\ &= \mu(c)\vee\mu(a\wedge b),\text{for every } \mu(c)\leq\mu(b).\end{aligned}$$

$\Rightarrow$ $(\mu(a),\mu(b))$ is a Fuzzy modular pair.

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