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On Certain Properties of Fuzzy Supra Semi Open Sets

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Abstract

In this paper we study and investigate certain fundamental properties and relations of fuzzy supra semi open and closed sets. Counter examples have been given to support the strictness of the inequalities and relations.

Keywords: Fuzzy topological spaces, fuzzy supra topological spaces, fuzzy supra semi open sets.

1. Introduction

In 1981, K.K. Azad introduced fuzzy semi open sets as a natural generalization of classical semi open sets introduced by N. Levine in 1961. Due to the importance and need for the fuzzification of weaker forms of classical topological notions, several works has been found in this field among the various existing fuzzy topological spaces in recent years. Properties of fuzzy semi open sets were investigated by B. Ahmad et al. [4, 7]. Intuitionistic fuzzy semi open sets were investigated by V. Thiripurasundari et al. [5] and A Manimaran et al. [6]. B.C. Chetia and S. Ahmed introduced fuzzy supra semi open sets in [3].

M. E. Abd El-Monsef et al. [2] introduced Fuzzy supra topological spaces generalizing classical supra topological spaces introduced by A.S. Mashhour et al. In this paper, we investigate and establish certain properties and identities of fuzzy supra semi open and closed sets generalizing the notion of K.K Azad supported by counter examples.

2. Preliminaries

In order to make this paper self-contained, first we briefly recall certain definitions and results. Throughout the paper X is a non-empty set.

A fuzzy set in X is a function from X into the closed unit interval I= [0, 1]. The collection of all fuzzy sets of X is denoted by I^X. The basic operations on fuzzy sets are taken as usual.

Definition 2.1(see [1]): A fuzzy topology T on a set X is a family of fuzzy sets in X such that
(1) 0_x, $1_x \in T$;
(2) A, B $\in$ T $\Rightarrow$ A $\wedge$ B $\in$ T and
(3) $A_i \in T \Rightarrow \vee A_i \in T$.

The pair (X, T) is called a fuzzy topological space (FTS). The elements of T are called fuzzy open sets and the complement of fuzzy open set is called fuzzy closed set.

Definition 2.2: The Closure and the interior of a fuzzy set A in FTS (X, T) are denoted and defined respectively as

$Cl(A) = \wedge\{B: A \leq B, B \text{ is a fuzzy closed set in } X\}$,

$Int(A) = \vee\{B: B \leq A, B \text{ is a fuzzy open set in } X\}$.

Definition 2.3 (see [2]): A collection T^* of fuzzy sets in a set X is called a fuzzy supra topology on X if the following conditions are satisfied:
(1) 0_x, $1_x \in T$ and (2) $A_i \in T \Rightarrow \vee A_i \in T$.

The pair (X, T^*) is called a fuzzy supra topological space (FSTS). The elements of T^* are called fuzzy supra open sets (FSOS) and the complement of a fuzzy supra open set is called fuzzy supra closed set (FSCS). The collection of all fuzzy supra open sets (resp. fuzzy supra closed sets) of the FSTS (X, T^*) is denoted by FSOS(X) (resp. FSCS(X)).

Remark 2.4:
(1) Every FTS is a FSTS.
(2) If (X, T^*) is an associated FSTS with the FTS (X, T) (i.e. $T \subseteq T^*$), then every fuzzy open (closed) set in the FTS (X, T) is fuzzy supra open (closed) set in the FSTS (X, T^*).

Definition 2.5: Let (X, T^*) is a FSTS and A be fuzzy set in X, then the fuzzy supra closure and fuzzy supra interior are denoted and defined respectively as

$Cl^*(A) = \wedge\{B: A \leq B, B \text{ is a fuzzy supra closed set in } X\}$,

$Int^*(A) = \vee\{B: B \leq A, B \text{ is a fuzzy supra open set in } X\}$.

Remark 2.6:
(1) The fuzzy supra closure of a fuzzy set A in a FSTS is the smallest fuzzy supra closed set containing A.
(2) The fuzzy supra interior of a fuzzy set A in a FSTS is the largest fuzzy supra open set contained in A.
(3) If (X, T^*) is an associated FSTS with the FTS (X, T) and A is any fuzzy set in X, then

$Int(A) \leq Int^*(A) \leq A \leq Cl^*(A) \leq Cl(A)$.

Properties of fuzzy supra closure and fuzzy supra interior which are needed in the sequel, are summarized in the following theorem.

Theorem 2.7(see [2]): For any fuzzy sets A and B in a FSTS (X, T^*),

(1) $A \in FSCS(X) \Leftrightarrow Cl^*(A) = A$,

$A \in FSOS(X) \Leftrightarrow Int^*(A) = A$;

(2) $A \le B, \Rightarrow Cl^*(A) \le Cl^*(B)$ and $Int^*(A) \le Int^*(B)$;

(3) $Cl^*(Cl^*(A)) = Cl^*(A)$ and $Int^*(Int^*(A)) = Int^*(A)$;

(4) $Cl^*(A \vee B) \ge Cl^*(A) \vee Cl^*(B)$;

(5) $Cl^*(A \wedge B) \le Cl^*(A) \wedge Cl^*(B)$;

(6) $Int^*(A \vee B) \ge Int^*(A) \vee Int^*(B)$;

(7) $Int^*(A \wedge B) \le Int^*(A) \wedge Int^*(B)$;

(8) $Cl^*(A^c) = (Int^*(A))^c$, $Int^*(A^c) = (Cl^*(A))^c$.

3. Fuzzy supra semi open and semi closed sets

Definition 3.1(see [3]): A fuzzy set A in a FSTS (X, T^*) is called a fuzzy supra semi open set (FSSOS) if there exist a fuzzy supra open set $\mu \in T^*$ such that $\mu \le A \le Cl^*(\mu)$.

The complement of a fuzzy supra semi open set is called fuzzy supra semi closed set (FSSCS). The collection of all FSSOS (resp. FSSCS) of the FSTS (X, T^*) is denoted by FSSOS(X) (resp. FSSCS(X)).

Remark 3.2:

(1) Every fuzzy supra open(closed) set is fuzzy supra semi open(closed).

(2) Union of a family of fuzzy supra semi open sets is fuzzy supra semi open.

(3) Intersection of a family of fuzzy supra semi closed sets is fuzzy supra semi closed.

Definition 3.3: Let (X, T^*) is a FSTS and A be fuzzy set in X, then the fuzzy supra semi closure and fuzzy supra semi interior are denoted and defined respectively as

$sCl^*(A) = \wedge\{B: A \le B$, B is a fuzzy supra semi closed set in X$\}$,

$sInt^*(A) = \vee\{B: B \le A$, B is a fuzzy supra semi open set in X$\}$.

Remark 3.4: For any fuzzy set A in a FSTS (X, T^*),

(1) $sCl^*(A)$ is the smallest fuzzy supra semi closed set containing A.

(2) $sInt^*(A)$ FSTS is the largest fuzzy supra semi open set contained in A.

(3) $Int^*(A) \le sInt^*(A) \le A \le sCl^*(A) \le Cl^*(A)$.

Properties of fuzzy supra semi closure and fuzzy supra semi interior are summarized in the following proposition.

Theorem 3.5: For any fuzzy sets A and B in a FSTS (X, T^*),

(1) $A \in FSSCS(X) \Leftrightarrow sCl^*(A) = A$,

$A \in FSSOS(X) \Leftrightarrow sInt^*(A) = A$;

(2) $A \le B, \Rightarrow sCl^*(A) \le sCl^*(B)$ and $sInt^*(A) \le sInt^*(B)$;

(3) $sCl^*(sCl^*(A)) = sCl^*(A)$ and $sInt^*(sInt^*(A)) = sInt^*(A)$;

(4) $sCl^*(A \vee B) \ge sCl^*(A) \vee sCl^*(B)$;

(5) $sCl^*(A \wedge B) \le sCl^*(A) \wedge sCl^*(B)$;

(6) $sInt^*(A \vee B) \ge sInt^*(A) \vee sInt^*(B)$;

(7) $sInt^*(A \wedge B) \leq sInt^*(A) \wedge sInt^*(B)$;
(8) $sCl^*(A^c) = (sInt^*(A))^c$, $sInt^*(A^c) = (sCl^*(A))^c$.

Proof:
(1)-(3), (8) proofs are straight forward.
(4) $A \leq A \vee B$, $B \leq A \vee B \Rightarrow sCl^*(A) \leq sCl^*(A \vee B)$, $sCl^*(B) \leq sCl^*(A \vee B)$.
Hence $sCl^*(A) \vee sCl^*(B) \leq sCl^*(A \vee B)$.
(5) $A \geq A \wedge B$, $B \geq A \wedge B \Rightarrow sCl^*(A) \geq sCl^*(A \wedge B)$, $sCl^*(B) \geq sCl^*(A \wedge B)$.
Hence $sCl^*(A) \wedge sCl^*(B) \geq sCl^*(A \wedge B)$.
proofs of (6), (7) are similar to that of (4) and (5).

In the above theorem (4)- (7), the equality may not hold as shown in the following example.

Example 3.6: Let $X = \{a, b\}$ be a set with a fuzzy supra topology $T^* = \{0_x, \{a_{.5}, b_{.1}\}, \{a_{.2}, b_{.5}\}, \{a_{.5}, b_{.5}\}, 1_x\}$. Then $FSSOS(X) = \{0_x, 1_x\} \cup \{\{a_{.5}, b_y\}: 0.1 \leq y \leq 0.5\} \cup \{\{a_x, b_{.5}\}: 0.2 \leq x \leq 0.5\}$, $FSSCS(X) = \{0_x, 1_x\} \cup \{\{a_{.5}, b_y\}: 0.5 \leq y \leq 0.9\} \cup \{\{a_x, b_{.5}\}: 0.5 \leq x \leq 0.8\}$. For fuzzy sets $A = \{a_{.5}, b_{.2}\}$, $B = \{a_{.2}, b_{.5}\}$, $C = \{a_{.5}, b_{.8}\}$, $D = \{a_{.8}, b_{.5}\}$, $E = \{a_{.5}, b_1\}$, , $F = \{a_1, b_{.5}\}$, $P = \{a_{.5}, b_0\}$, $Q = \{a_0, b_{.5}\}$ of X, calculations give
$sCl^*(C \vee D) = 1_x > sCl^*(C) \vee sCl^*(D) = \{a_{.8}, b_{.8}\}$,
$sCl^*(E \wedge F) = \{a_{.5}, b_{.5}\} < sCl^*(E) \wedge sCl^*(F) = 1_x$,
$sInt^*(P \vee Q) = \{a_{.5}, b_{.5}\} > sInt^*(P) \vee sInt^*(Q) = 0_x$,
$sInt^*(A \wedge B) = 0_x < sInt^*(A) \wedge sInt^*(B) = \{a_{.2}, b_{.2}\}$.

Proposition 3.7: For any fuzzy set A in a FSTS (X, T^*), the following are equivalent
(1) $A \in FSSOS(X)$,
(2) $A \leq Cl^*(Int^*(A))$,
(3) $A \leq sCl^*(sInt^*(A))$.

Proof:
(1)⇒(2): $A \in FSSOS(X) \Rightarrow \mu \leq A \leq Cl^*(\mu)$ for some $\mu \in T^*$
$\therefore A \leq Cl^*(\mu) = Cl^*(Int^*(\mu)) \leq Cl^*(Int^*(A))$.
(2)⇒(1): $A \leq Cl^*(Int^*(A)) \Rightarrow Int^*(A) \leq A \leq Cl^*(Int^*(A))$ where $Int^*(A) \in T^* \Rightarrow A \in FSSOS(X)$.
(1)⇒(3): $A \in FSSOS(X) \Rightarrow A \leq sCl^*(A) = sCl^*(sInt^*(A))$.
(3)⇒(1): $A \leq sCl^*(sInt^*(A))$. Since $sInt^*(A) \in FSSOS(X)$, $\mu \leq sInt^*(A) \leq Cl^*(\mu)$ for some $\mu \in T^*$
$\therefore Cl^*(\mu) = sCl^*(Cl^*(\mu)) \geq sCl^*(sInt^*(A)) \geq A \geq sInt^*(A) \geq \mu \Rightarrow A \in FSSOS(X)$.

In the above proposition the inequalities (2) and (3) may not hold in general as shown in the following example.

Example 3.8: For fuzzy set $A = \{a_{.4}, b_{.3}\}$ of the FSTS in example 3.6, calculations give
$A \nleq Cl^*(Int^*(A)) = 0_x$ and $A \nleq sCl^*(sInt^*(A)) = 0_x$. Note that A is not a FSSOS.

Since $Int^*(A^c) = (Cl^*(A))^c$ and $sInt^*(A^c) = (sCl^*(A))^c$, we have the following proposition.

Proposition 3.9: For any fuzzy set A in a FSTS (X, T^*), the following are equivalent
(1) $A \in FSSCS(X)$,
(2) $Int^*(Cl^*(A)) \leq A$,
(3) There exist $\mu \in FSCS(X)$ such that $Int^*(\mu) \leq A \leq \mu$,
(4) $sInt^*(sCl^*(A)) \leq A$.

More properties are given in the following propositions.

Proposition 3.10: For any fuzzy set A in a FSTS (X, T^*),
(1) $A \in FSSOS(X) \Rightarrow Cl^*(A), sCl^*(A) \in FSSOS(X)$.
(2) $A \in FSSCS(X) \Rightarrow Int^*(A), sInt^*(A) \in FSSCS(X)$.

Proof:
(1) $A \in FSSOS(X) \Rightarrow \mu \leq A \leq Cl^*(\mu)$ for some $\mu \in T^* \Rightarrow \mu \leq A \leq Cl^*(A) \leq Cl^*(Cl^*(\mu)) = Cl^*(\mu) \Rightarrow Cl^*(A) \in FSSOS(X)$.

Again, $A \in FSSOS(X) \Rightarrow \mu \leq A \leq Cl^*(\mu)$ for some $\mu \in T^* \Rightarrow \mu \leq A \leq sCl^*(A) \leq sCl^*(Cl^*(\mu)) = Cl^*(\mu) \Rightarrow sCl^*(A) \in FSSOS(X)$.

Proposition 3.11: For any fuzzy set A in a FSTS (X, T^*),
(1) $Cl^*(sCl^*(A)) = Cl^*(A)$,
(2) $Int^*(Cl^*(A)) \leq sCl^*(A)$,
(3) $sInt^*(sCl^*(A)) \geq Int^*(Cl^*(A))$,
(4) $Int^*(sInt^*(sCl^*(A))) = Int^*(Cl^*(A))$,
(5) $sInt^*(sCl^*(A)) \leq Int^*(Cl^*(A)) \vee A$.

Proof:
(1) $sCl^*(A) \leq Cl^*(A) \Rightarrow Cl^*(sCl^*(A)) \leq Cl^*(A)$ and $A \leq sCl^*(A) \Rightarrow Cl^*(A) \leq Cl^*(sCl^*(A))$.
(2) Since $sCl^*(A) \in FSSCS(X)$, by (2) of proposition 3.10

$Int^*(Cl^*(A)) \leq Int^*(Cl^*(sCl^*(A))) \leq sCl^*(A)$.
(3) Since $Int^*(Cl^*(A)) \leq sCl^*(A)$ and all FSOS are FSSOS, $Int^*(Cl^*(A)) \leq sInt^*(sCl^*(A))$.
(4) Since $sInt^*(sCl^*(A)) \geq Int^*(Cl^*(A))$, $Int^*(Cl^*(A)) = Int^*(Int^*(Cl^*(A))) \leq Int^*(sInt^*(sCl^*(A)))$.

Again, $Int^*(Cl^*(A)) \leq sCl^*(A) \Rightarrow Int^*(Cl^*(A)) = sInt^*(Int^*(Cl^*(A))) \leq sInt^*(sCl^*(A))$.
(5) $Int^*(Cl^*(A)) \leq sCl^*(A) \leq Cl^*(A)$ and $A \leq Cl^*(A) \Rightarrow Int^*(Cl^*(A)) \leq Int^*(Cl^*(A)) \vee A \leq Cl^*(A) \Rightarrow Int^*(Cl^*(A)) \vee A \in FSSCS(X)$ containing $A \Rightarrow sCl^*(A) \leq Int^*(Cl^*(A)) \vee A \Rightarrow sInt^*(sCl^*(A)) \leq sCl^*(A) \leq Int^*(Cl^*(A)) \vee A$.

In the above proposition the inequalities (2), (3) and (5) are irreversible as shown in the following examples.

Example 3.12: Let X = {a, b} be a set with a fuzzy supra topology $T^* = \{0_x, \{a_{.2}, b_{.7}\}, \{a_{.6}, b_{.2}\}, \{a_{.6}, b_{.7}\}, 1_x\}$. Then $FSSOS(X) = \{0_x, 1_x\} \cup \{\{a_x, b_y\}: 0.2 \leq x \leq 0.4, 0.7 \leq y \leq$

$0.8\}\cup\{\{a_x, b_y\}: 0.6\leq x \leq 0.8, 0.2\leq y \leq 0.3\}\cup\{\{a_x, b_y\}: 0.6\leq x \leq 1, 0.7\leq y \leq 1\}$, $FSSCS(X) = \{0_x, 1_x\}\cup\{\{a_x, b_y\}: 0.6\leq x \leq 0.8, 0.2\leq y \leq 0.3\}\cup\{\{a_x, b_y\}: 0.2\leq x \leq 0.4, 0.7\leq y \leq 0.8\}\cup\{\{a_x, b_y\}: 0\leq x \leq 0.4, 0\leq y \leq 0.3\}$. For fuzzy set $A = \{a_{.6}, b_{.3}\}$ in X, calculations give

$Int^*(Cl^*(A)) =\{a_{.6}, b_{.2}\} \ngeq sCl^*(A) =\{a_{.6}, b_{.3}\}$ and $sInt^*(sCl^*(A)) =\{a_{.6}, b_{.3}\} \nleq Int^*(Cl^*(A)) =\{a_{.6}, b_{.2}\}$.

Example 3.13: Let $X = \{a, b\}$ be a set with a fuzzy supra topology $T^* = \{0_x, \{a_{.2}, b_{.6}\}, \{a_{.5}, b_{.4}\}, \{a_{.5}, b_{.6}\}, 1_x\}$. Then $FSSOS(X) = \{0_x, 1_x\} \cup \{\{a_x, b_{.6}\}: 0.2\leq x \leq 0.5\}\cup\{a_{.5}, b_{.4}\}\}$, $FSSCS(X) = \{0_x, 1_x\}\cup\{\{a_x, b_{.4}\}: 0.5\leq x \leq 0.8\}\cup\{a_{.5}, b_{.6}\}$. For fuzzy set $A = \{a_{.8}, b_0\}$, calculations give

$sInt^*(sCl^*(A)) =\{a_{.5}, b_{.4}\} \ngeq Int^*(Cl^*(A)) \vee A =\{a_{.8}, b_{.4}\}$.

Since $Int^*(A^c) = (Cl^*(A))^c$ and $sInt^*(A^c) = (sCl^*(A))^c$, we have the following proposition.

Proposition 3.14: For any fuzzy set A in a FSTS (X, T^*),

(1) $Int^*(sInt^*(A)) = Int^*(A)$,

(2) $Cl^*(Int^*(A)) \geq sInt^*(A)$,

(3) $sCl^*(sInt^*(A)) \leq Cl^*(Int^*(A))$,

(4) $Cl^*(sCl^*(sInt^*(A))) = Cl^*(Int^*(A))$,

(5) $sCl^*(sInt^*(A)) \geq Cl^*(Int^*(A)) \wedge A$.

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