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Lattice on Fuzzy Implication

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Abstract

In this paper we present some Boolean algebras in the form of lattice by using the fuzzy implication. We show that these methods lead to various and much richer algebraic structures on the set of all fuzzy implications than obtained in the literature.

Keywords- Fuzzy logic connectives, Fuzzy implication, Algebras, Lattice.

1. INTRODUCTION

Fuzzy logic and fuzzy sets are basic framework when working with various notions. In classical logic all assertions are either true of false (i.e. have truth values 1 or 0 respectively) but in case of fuzzy logic the truth value may be any values in the interval [0, 1] connected with fuzzy logic in the notion of fuzzy sets. Fuzzy implication, commonly defined as a two place operation on the unit interval [0, 1] is an extension of the classical (Boolean) binary implication. They play an important role in approximate reasoning, fuzzy control, decision theory, control theory, expert system etc. [4][7], or the recent monograph exclusively devoted to fuzzy implications. Zadeh[6] introduced the concept of fuzzy sets later on a huge amount of work in fuzzy set theory and fuzzy logic appeared both in theoretical and applied field. Implication invented by Lukasiewicz [2] in 1958 takes the form $\min(1, 1 - x + y)$. In this paper, we discuss fuzzy implication lattice and some algebra on fuzzy implication lattice.

2. FUZZY IMPLICATION: DEFINITIONS AND BASIC PROPERTIES

In this section we give the main definitions and properties related to fuzzy implications only for the definitions of other basic fuzzy logic connectives, viz. t-norm, t-conorm, negation and classical algebra, Lattice, Sub-lattice etc. We refer [5][3].

Definition 1. [1] Let U= [0,1] be the unitary interval. A t-norm is a function $T: U \rightarrow U$ satisfying, for all $x, y, z \in U$, the following properties:

i) $T(x, y) = T(y, x)$ (commutativity);
ii) $T(x, T(y, z) = T(T(x, y), z)$ (assosiativity);
iii) If $y \leq z$ then $T(x, y) \leq T(x, z)$ (monotonicity);
iv) $T(x, 1) = x$ (boundary condition);

Definition 2. [1] Let U= [0,1] be the unitary interval. A t-conorm is a function $S: U \rightarrow U$ satisfying, for all $x, y, z \in U$, the following properties:

i) $S(x, y) = s(y, x)$ (commtativity);
ii) $S(x, S(y, z)) = S(S(x, y), z)$ (associativity);
iii) If $y \leq z$ then $S(x, y) \leq S(x, z)$ (monotonicity);
iv) $S(x, 0) = 0$ (boundarycondition).

Definition 3. [1]A function $N: U \rightarrow U$ is a fuzzy negation if

i) $N(0) = 1$ and $N(1) = 0$;
ii) If $x \geq y$ then $N(x) \geq N(y)$, for all $x, y \in U$

Definition 4.[1] A binary operator

$$I: [0,1] \times [0,1] \rightarrow [0,1]$$

Is said to be an implication function or an implication if it satisfies;

i) $I(x, y) \geq I(y, z)$ when $x \leq y$, for all $z \in [0,1]$.
ii) $I(x, y) \leq I(x, z)$ when $y \leq z$, for all $x \in [0,1]$
iii) I(0,0) = I(1,1) = I(0,1) = 1 and I(0,1)=0 etc.

Definition 5 [1] A fuzzy implication I is said to satisfy

1) The left neutrality property (NP), if
$$I(1, y) = y, y \in [0,1]$$
2) The ordering property (OP), if
$$x \leq y \leftrightarrow I(x, y) = 1, x, y \in [0,1]$$
3) The identity property (IP), if
$$I(x, y) = 1\ x, y \in [0,1]$$
4) The exchange property (EP), if
$$I(x, I(y, z)) = I(y, I(x, z))\ x, y, z \in [0,1]$$
5) The law of contraposition with respect to a fuzzy negation N, (P(N)), if
$$I(x, y) = I(N(y)N(x))\ x, y, z \in [0,1]$$

Some fuzzy implication which we use in this paper

1) Lukasiewicz Implication
$$I_{LK}(x, y) = \min(1, 1 - x + y)$$
2) Reichenbach Implication
$$I_{RC}(x, y) = 1 - x + xy$$

3) Kleen- Dienes Implication

$$I_{KD}(x, y) = \max(1 - x, y)$$

3. LATTICE ON FUZZY IMPLICATION

In this paper we are interested generating such algebraic structure they produce on fuzzy implication lattice.

Definition 6. A relation $\leq$ is defined on any fuzzy implication then relation $\leq$ is called a partial order relation if it

i) Reflexive.
ii) Antisymmetric.
iii) Transitive.

And $(FI, \leq)$ is called a poset.

Definition 7. A lattice of a fuzzy implication is a partial ordered implication$(FI, \leq)$ in which every pair of implication $I, J \in FI$ has a greatest lower bound and least upper bound

$$GLB\{I, J\} = \{I(x, y) \cap J(x, y)\} = \min\{I, J\}$$
$$LUB\{I, J\} = \{I(x, y) \cup J(x, y)\} = \max\{I, J\}$$

3. 1 Some properties of Fuzzy Implication Lattice

We shall first list some of the properties of the two binary operations of max and min on a fuzzy implication lattice $(FI, \leq)$ for and $I, J, K \in FI$ we have

i) Idempotent Law

$$\min\big(I(x, y), I(x, y)\big) = I(x, y)$$
$$\max\big(I(x, y), I(x, y)\big) = I(x, y)$$

ii) Commutative Law

$$\min\big(I(x, y), J(x, y)\big) = \min(J(x, y), I(x, y))$$
$$\max\big(I(x, y), J(x, y)\big) = \max(J(x, y), I(x, y))$$

iii) Associative Law

$$\min\Big(\min\big(I(x, y), J(x, y)\big), K(x, y)\Big) = \min\Big(I(x, y), \min\big(J(x, y), K(x, y)\big)\Big)$$
$$\max\Big(\max\big(I(x, y), J(x, y)\big), K(x, y)\Big) = \max(I(x, y), \max(J(x, y), K(x, y)))$$

iv) Absorption Law

$$\min\Big(\max\big(I(x, y), J(x, y)\big), K(x, y)\Big) = I(x, y)$$
$$\max\Big(\min\big(I(x, y), J(x, y)\big), K(x, y)\Big) = I(x, y)$$

Example 1. Let $(FI, \leq)$ be a fuzzy implication lattice and there greatest lower and least upper bounds are max and min. Let $I_{LK}(x, y), I_{RC}(x, y), I_{KD}(x, y) \in FI$ which satisfies all the above four properties where $x, y \in [0,1]$.

Solution: Let x= 0.3 and y= 0.7 then

1) Idempotent property:
Let $I_{LK} \in FI$ and for all $x, y \in [0,1]$ then
$$\min(I_{LK}(x,y), I_{LK}(x,y)) = I_{LK}(x,y)$$
This implies that
$$\min(\min(1,1-x+y), \min(1,1-x+y)) = \min(1,1-x+y)$$
$$\min(\min(1,1-0.3+0.7), \min(1,1-0.3+0.7)) = \min(1,1-0.3+0.7)$$

Similarly we can prove that
$$\max(I_{LK}(x,y), I_{LK}(x,y)) = I_{LK}(x,y)$$
2) Commutative property:
Let $I_{LK}, I_{KD} \in FI$ and for all $x, y \in [0,1]$ then
$$\min(I_{LK}(x,y), I_{KD}(x,y)) = \min(I_{KD}(x,y), I_{LK}(x,y))$$

This implies that
$$\min(\min(1,1-x+y), \max(1-x,y)) = \min(\max(1-x,y), \min(1,1-x+y))$$
$$\min(\min(1,1-0.3+0.7), \max(1-0.3,0.7)) = \min(\max(1-0.3,0.7), \min(1,1-0.3+0.7))$$

Similarly we can prove that
$$\max(I_{LK}(x,y), I_{KD}(x,y)) = \max(I_{KD}(x,y), I_{LK}(x,y))$$
3) Associative property:
Let $I_{LK}, I_{KD}, I_{RC} \in FI$ and for all $x, y \in [0,1]$
$$\min\left(\min(I_{LK}(x,y), I_{KD}(x,y)), I_{RC}(x,y)\right) = \min(I_{LK}(x,y), \min(I_{KD}(x,y), I_{RC}(x,y)))$$
$$\min(\min(\min(1,1-x+y), \max(1-x,y)), 1-x+xy)$$
$$= \min(\min(1,1-x+y), \min(\max(1-x,y), 1-x+xy))$$

This implies that
$$\min(\min(\min(1,1-0.3+0.7), \max(1-0.3,0.7)), 1-0.3+(0.3).(0.7))$$
$$= \min(\min(1,1-0.3+0.7), \min(\max(1-0.3,0.7), 1-0.3$$
$$+(0.3).(0.7)))$$

Similarly we can show that
$$\max\left(\max(I_{LK}(x,y), I_{KD}(x,y)), I_{RC}(x,y)\right) = \max(I_{LK}(x,y), \max(I_{RC}(x,y), I_{KD}(x,y)))$$

4) Absorption property:
Let $I_{LK}, I_{KD}, I_{RC} \in FI$ and $x = y = 0$ and 1 then
$$\mathrm{Min}(I_{LK}(x,y), \max(I_{KD}(x,y), I_{RC}(x,y))) = I_{LK}(x,y)$$

Theorem 1. Let $(FI, \leq)$ be a fuzzy implication lattice in which max and min denote the operations. Then for all $I, J \in FI$ such that
$$I \leq J \leftrightarrow \min(I(x,y), J(x,y)) = I(x,y) \leftrightarrow \max(I(x,y), J(x,y)) = J(x,y)$$

Proof: We shall first prove that

$I \leq J \leftrightarrow \min(I(x,y), J(x,y)) = I(x,y)$ let $I \leq J$ and then by reflexivity

$I \leq J$ and by definition we know that $\min(I(x,y), J(x,y)) = \text{glb}\{I, J\}$
i.e. $\min(I(x,y), J(x,y)) \leq I(x,y)$ and $\min(I(x,y), J(x,y)) \leq J$ so $I \leq I$ and $I \leq J$
now taking min in both side we get

$$\min(I(x,y), I(x,y)) \leq \min(I(x,y), J(x,y))$$

Then by idempotent law we get

$$I \leq \min(I, J) \text{ hence we get } \min(I, J) = I$$

Similarly we can show that if $\min(I, J) = I$ then $I \leq J$
Hence it is clear that

$$I \leq J \leftrightarrow \min(I(x,y), J(x,y)) = I(x,y)$$

Now we shall prove the second part of the theorem i.e.
If $I \leq J \leftrightarrow \max(I(x,y), J(x,y)) = J(x,y)$ let $\min(I(x,y), J(x,y)) = I(x,y)$
This implies that $\max(J(x,y), \min(I(x,y), J(x,y))) = \max(J(x,y), I(x,y))$ then by commutativity we get $\max(J(x,y), \min(J(x,y), I(x,y))) = \max(I(x,y), J(x,y))$ this implies that $J(x,y) = \max(I(x,y), J(x,y))$ by absorption law so
$\min(I(x,y), J(x,y)) = I(x,y) \rightarrow \max(I(x,y), J(x,y)) = J(x,y)$ similarly we can show that

$$\max(I(x,y), J(x,y)) = J(x,y) \rightarrow \min(I(x,y), J(x,y)) = I(x,y) \text{ i.e.}$$

$$\max(I(x,y), J(x,y)) = J(x,y) \leftrightarrow \min(I(x,y), J(x,y)) = I(x,y)$$

Hence we get

$$I \leq J \leftrightarrow \min(I(x,y), J(x,y)) = I(x,y) \leftrightarrow \max(I(x,y), J(x,y)) = J(x,y)$$

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