International Journal of Fuzzy Mathematics and Systems.
ISSN 2248-9940 Volume 13, Number 1 (2023), pp. 11-20

http://www.ripublication.com

Optimal Code Word Length Via Modified Verma *i.e.* Hybrid Shannon Information Measures for Discrete Noiseless Channel in Intuitionistic Fuzzy Environment

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Abstract

This investigation is concerned with optimal code word length via modified Verma [11, 12] *i.e.* hybrid Shannon [10] information measures in intuitionistic fuzzy set. Optimal code is that code for which the value of L_a is equal to its lower bound dependent on the parameter a. This work is validated from the previous work that is by Shannon's theorem, that has no parameter.

Keywords: Holder inequality, Intuitionistic fuzzy noiseless coding, Intuitionistic fuzzy set, Decipherable code, Information theory etc.

1. Introduction

Fuzzy set theory is one of the efficient means of researching and processing fuzzy phenomena in real world. The characteristic of imperfect information known as ambiguity or fuzziness, which is characterized by the lack of clear distinction between elements belonging to and not belonging to a set, that is, the boundaries of the set under consideration are not clearly defined. A measure of vagueness often used and cited in the literature of information theory is known as fuzzy information. The concept of fuzziness was first time introduced by Zadeh, Lotfi A. [13]. He developed a new theory to measure the uncertainty, which is also known as ambiguity, of a fuzzy set. If A be the subset of universe of discourse $i.e. X = \{x_1, \dots\dots, x_n\}$ then, A is defined as,

$$A = \{x_i/\mu_A(x_i): i = 1,2, \dots\dots\dots\dots, n\}.$$

Where $\mu_A(x_i)$ is a membership function and having the following properties:

1. If $\mu_A(x_i) = 0$, x_i does not belong to A and there is no ambiguity.
2. If $\mu_A(x_i) = 1$, x_i belong to A and there is no ambiguity.
3. If $\mu_A(x_i) = 0.5$, there is maximum ambiguity whether x_i belong to A or not.

Later, in 1983, the concept of intuitionistic fuzziness was first time introduced by Atanassov, Krassimir T. [2,3] in his original paper intuitionistic fuzzy set. He developed a new theory to measure the uncertainty. According to him, if F be a fixed set then an intuitionistic fuzzy set S in F is an object having the form

$$S = \{< x, \mu_S(x), \nu_S(x) >/x \in F\}.$$

Where the function $\mu_S(x)$ and $\nu_S(x)$ define the degree of membership and degree of non-membership of the element $x \in S$ to $S \subset F$ respectively.
The function $\mu_S(x)$ and $\nu_S(x)$ satisfy the following condition.

$$(\forall\, x \in F)\ (0 \le \mu_S(x) + \nu_S(x) \le 1).$$

Obviously, fuzzy set has the form $\{< x, \mu_S(x), 1 - \mu_S(x) >/x \in F\}$.
A measure of fuzziness $f\big(\mu_S(x), \nu_S(x)\big)$ is an Intuitionistic fuzzy set should have atleast the following conditions:
(C_1) It should be continuous in this range of $(0 \le \mu_S(x_i) + \nu_S(x_i) \le 1), (i = 1, \dots\dots, n)$.
(C_2) It should be zero when $\mu_S(x_i) = 0$ and $\nu_S(x_i) = 0$.
(C_3) It should be not changed when any of $\mu_S(x_i)$ is changed into $\nu_S(x_i)$.
(C_4) It should be defined for all $\mu_S(x_i)$ and $\nu_S(x_i)$ $(i = 1, \dots\dots, n)$ in the range of $(0 \le \mu_S(x_i) + \nu_S(x_i) \le 1), (i = 1, \dots\dots, n)$.
(C_5) It should be maximum when $\mu_S(x_i) = \frac{1}{2}$ and $\nu_S(x_i) = \frac{1}{2}$ $(i = 1, \dots\dots, n)$.
(C_6) It should be increasing function of $\mu_S(x_i)$ when $0 \le \mu_S(x_i) \le \frac{1}{2}$ and decreasing function of $\mu_S(x_i)$ when $\frac{1}{2} \le \mu_S(x_i) \le 1$ and other variable are kept fixed. It should be decreasing funcion of $\nu_S(x_i)$ when $0 \le \nu_S(x_i) \le \frac{1}{2}$ and increasing function of $\nu_S(x_i)$ when $\frac{1}{2} \le \nu_S(x_i) \le 1$ and other variable are kept fixed.
(C_7) It should be concave function of $\mu_S(x_i)$, when $\nu_S(x_i)$ set as a constant.

Now, let a finite set of n input symbols $X = \{x_1, \dots\dots, x_n\}$ be encoded using alphabet of D symbols, then it is shown in Feinstein [6] that there is a uniquely decipherable code with lengths $N_1, N_2, \dots, N_n$ if and only if the Kraft [7] inequality holds, that is, $\sum_{i=1}^{n} D^{-l_i} \le 1$, where D is the size of the code alphabet. Furthermore, if $L = \sum_{i=1}^{n} n_i p_i$ is the average codeword length, then for a code satisfying the condition

$\sum_{i=1}^{n} D^{-l_i} \leq 1$, the inequality $H(P) \leq L < H(P) + 1$ is also fulfilled and equality, $L = H(P)$, holds if and only if $l_i = -\ln_D p_i$ where $i = 1, \ldots, n$ and $\sum_{i=1}^{n} D^{-l_i} = 1$.
Again, if $L < H(P)$, then by suitable encoding of long input sequences, the average number of code letters per input symbol can be made arbitrarily close to $H(P)$ and this is Shannon's [10] noiseless coding theorem.
Later, generalized coding theorems by considering different information measure under the condition of unique decipherability were investigated by several authors, see for instance the papers Aczel and Daroczy [1], Ebanks [5], Kapur [8], Longo [9]

2. Our Results

Optimal Source Coding Theorems in Intuitionistic Fuzzy Environment

In this section we take the modified version of Verma [11, 12] $i.e.$ hybrid Shannon [10] information measure in intuitionistic fuzzy environment $i.e.$

$$V_a(A) = \sum_{i=1}^{n}\Big[\big\{\ln\big(1 + a\mu_A(x_i)\big) + \ln\big(1 + a\nu_A(x_i)\big)\big\} - \{\mu_A(x_i)\ln\mu_A(x_i) + \nu_A(x_i)\ln\nu_A(x_i)\} - \ln(1+a)\{\mu_A(x_i) + \nu_A(x_i)\}$$

Now, when $a \to 0$, in the above equation we achieve

$$V_0(A) = -\sum_{i=1}^{n}\{\mu_A(x_i)\ln\mu_A(x_i) + \nu_A(x_i)\ln\nu_A(x_i)\} \qquad (2.1)$$

$i.e$
$$V_a(A) = \frac{1}{1-a}\ln\sum_{i=1}^{n}\{\mu_A^a(x_i) + \nu_A^a(x_i)\}, \qquad a > 0(\neq 1). \qquad (2.2)$$

We can achieve the equation (2.1) when we apply the condition $a \to 1$ in (2.2)

Now, when the equation (2.2) is motivated by Campbell [4] then,

$$V_a(A) = \frac{1}{1-a}\ln_D\sum_{i=1}^{n}\{\mu_A^a(x_i) + \nu_A^a(x_i)\}, \qquad a > 0(\neq 1) \qquad (2.3)$$

Definition. Let $a > 0(\neq 1)$ be arbitrarily fixed, then the mean length L_a corresponding to the generalized information measure $V_a(A)$ is given by the formula

$$L_a = \frac{1}{1-a}\ln_D\sum_{i=1}^{n}\big\{f\big(\mu_A(x_i), \nu_A(x_i)\big).D^{l_i(1-a)}\big\}, \qquad (2.4)$$

where D is any positive integer so that

$$\sum_{i=1}^{n} D^{-l_i a} \leq \sum_{i=1}^{n}\Big(f\big(\mu_A(x_i), \nu_A(x_i)\big)\Big)^a \qquad (2.5)$$

and

$$\left(f\left(\mu_A(x_i),\nu_A(x_i)\right)\right)^a = \left(\mu_A(x_i)\right)^a + \left(\nu_A(x_i)\right)^a = \mu_A^a(x_i) + \nu_A^a(x_i)$$

Theorem 2.1. If $a > 0(\neq 1)$ be any arbitrary fixed real numbers and $D > 1$ shows the size of the code alphabet, then for all uniquely decipherable code, the lower bound for L_a, the inequality

$$V_a(A) \leq L_a \tag{2.1.1}$$

and the equality holds if and only if

$$l_i = -\ln_D f\left(\mu_A(x_i),\nu_A(x_i)\right). \tag{2.1.2}$$

Where,

$$f\left(\mu_A(x_i),\nu_A(x_i)\right) = \mu_A(x_i) + \nu_A(x_i).$$

Proof: For the proof of the theorem we have two possibilities

Possibility-I When $a > 1$, then for all $x_i, y_i > 0$, $i = 1, \dots \dots, n$ and $\frac{1}{\gamma} + \frac{1}{\delta} = 1$, $\gamma < 1(\neq 0), \delta < 0$ or $\delta < 1(\neq 0), \gamma < 0$. The Holder inequality, that is,

$$\left(\sum_{i=1}^{n} x_i^{\gamma}\right)^{\frac{1}{\gamma}} . \left(\sum_{i=1}^{n} y_i^{\delta}\right)^{\frac{1}{\delta}} \leq \sum_{i=1}^{n} x_i . y_i \tag{2.1.3}$$

holds, and equality holds in (2.1.3) if and only if there exists a positive constant μ such that

$$x_i^{\gamma} = \mu y_i^{\delta} \tag{2.1.4}$$

Setting,

$$x_i = \left\{f\left(\mu_A(x_i),\nu_A(x_i)\right)\right\}^{-\frac{1}{u}} D^{-l_i \frac{1}{1+u}}$$

and

$$y_i = \left\{f\left(\mu_A(x_i),\nu_A(x_i)\right)\right\}^{\frac{1}{u}}$$

Also,

$$\gamma = -u \ \Rightarrow 0 < \gamma < 1$$

and

$\delta = \frac{u}{u+1} \Rightarrow \delta < 0.$

Hence from equation (2.1.3), we achieve

$$\left[\left\{\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{-\frac{1}{u}} D^{-l_i\frac{1}{1+u}}\right\}^{-u}\right]^{-\frac{1}{u}} \times \left[\left\{\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{\frac{1}{u}}\right\}^{\frac{u}{u+1}}\right]^{\frac{u+1}{u}} \leq \sum\nolimits_{i=1}^{n} D^{-l_i\frac{1}{1+u}}$$

$$i.e.\ \left[\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i), \nu_A(x_i)\right)\right\}.D^{l_i\frac{u}{1+u}}\right]^{-\frac{1}{u}} \times \left[\sum_{i=1}^{n}\left\{\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{\frac{1}{u+1}}\right\}\right]^{\frac{u+1}{u}} \leq \sum\nolimits_{i=1}^{n} D^{-l_i\frac{1}{1+u}} \tag{2.1.5}$$

On taking $a = \frac{1}{1+u}$, $u = \frac{1-a}{a}$, $a > 0$, $a \neq 1$. Then we achieve the following from (2.1.5)

$$\left[\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i), \nu_A(x_i)\right)\right\}.D^{l_i(1-a)}\right]^{\frac{a}{a-1}} \times \left[\sum_{i=1}^{n}\left\{\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{a}\right\}\right]^{\frac{1}{1-a}} \leq \sum\nolimits_{i=1}^{n} D^{-l_i a}$$

$$i.e.\ \left[\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i), \nu_A(x_i)\right)\right\}.D^{l_i(1-a)}\right]^{\frac{a}{a-1}} \times \left[\sum_{i=1}^{n}\left\{\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{a}\right\}\right]^{\frac{1}{1-a}}$$

$$\leq \sum_{i=1}^{n}\left\{\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{a}\right\}$$

$$i.e.\ \left[\sum_{i=1}^{n}\left\{f\left(\mu_A(x_i), \nu_A(x_i)\right)\right\}.D^{l_i(1-a)}\right]^{\frac{a}{a-1}} \leq \left[\sum_{i=1}^{n}\left\{\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{a}\right\}\right]^{\frac{-a}{1-a}}$$

$$i.e. \sum_{i=1}^{n}\left\{f\left(\mu_A(x_i), \nu_A(x_i)\right)\right\}.D^{l_i(1-a)} \leq \sum_{i=1}^{n}\left\{\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{a}\right\}$$

Taking logarithm on both sides and simplify for $\frac{1}{1-a} < 0$ we obtain as follows

$V_a(A) \leq L_a$.

Ofcourse, the equality holds when,

$$f\big(\mu_A(x_i),\nu_A(x_i)\big) = D^{-l_i}. \tag{2.1.6}$$

Which is equivalent to (2.1.2). After simplification of (2.1.6) we achieve the following result

$$f\big(\mu_A(x_i),\nu_A(x_i)\big).D^{-l_i(a-1)} = \Big(f\big(\mu_A(x_i),\nu_A(x_i)\big)\Big)^a$$

$i.e.$

$$\sum_{i=1}^{n}\{f\big(\mu_A(x_i),\nu_A(x_i)\big)\}.D^{l_i(1-a)} = \sum_{i=1}^{n}\Big\{\Big(f\big(\mu_A(x_i),\nu_A(x_i)\big)\Big)^a\Big\}.$$

Obviously,

$$L_a = V_a(A)$$

where,

$$f\big(\mu_A(x_i),\nu_A(x_i)\big) = \frac{1}{1-a}\ln_D \sum_{i=1}^{n}\{\mu_A^a(x_i) + \nu_A^a(x_i)\}.$$

Possibility-II When $a < 1$, then the proof follows on the same lines as for $a > 1$ and hence the results.

Theorem 2.2. For each a there exists a sequence of positive integers which satisfies the condition

$$\sum_{i=1}^{n} D^{-l_i a} \le \sum_{i=1}^{n}\Big\{\Big(f\big(\mu_A(x_i),\nu_A(x_i)\big)\Big)^a\Big\}.$$

Then, show that the upper bound for L_a in terms of $V_a(P)$ is

$$V_a(A) + \ln D > L_a. \tag{2.2.1}$$

Proof: Suppose l_i be the code word length for all $i = 1, \dots\dots, n$, which satisfies the condition

$$-\ln_D\Big(f\big(\mu_A(x_i),\nu_A(x_i)\big)\Big) < l_i < -\ln_D\Big(f\big(\mu_A(x_i),\nu_A(x_i)\big)\Big) + 1 \tag{2.2.2}$$

where,

$$f\big(\mu_A(x_i),\nu_A(x_i)\big) = \sum_{i=1}^{n}\{\mu_A^a(x_i) + \nu_A^a(x_i)\}.$$

Suppose the unit interval be

$$\Delta_i = \left[-\ln_D\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right), -\ln_D\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right) + 1\right] \tag{2.2.3}$$

Then, for any Δ_i, there exists exactly one positive number l_i such that

$$0 < -\ln_D\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right) \leq l_i < -\ln_D\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right) + 1 \tag{2.2.4}$$

But, the sequence of length $\{l_i\}_{i=1}^{n}$ satisfies the condition $\sum_{i=1}^{n} D^{-l_i a} \leq \sum_{i=1}^{n}\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{a}$. So from (2.2.4) we achieve

$$l_i < -\ln_D\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right) + 1$$

$i.e.$

$$-l_i > \ln_D\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right) - 1$$

$i.e.$

$$D^{-l_i} > f\left(\mu_A(x_i), \nu_A(x_i)\right). D^{-1}$$

$i.e.$

$$D^{l_i} < D.\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{-1} \tag{2.2.5}$$

Now, we have two possibilities.

Possibility-I When $0 < a < 1$, then raising power $(1 - a)$ on both sides of (2.2.5), we get the following result

$$D^{l_i(1-a)} < D^{(1-a)}\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{a-1}. \tag{2.2.6}$$

Now, multiplying (2.2.6) throughout by $f\left(\mu_A(x_i), \nu_A(x_i)\right)$ and then summing up from $i = 1$ to $i = n$, we have as follows

$$\sum_{i=1}^{n} f\left(\mu_A(x_i), \nu_A(x_i)\right). D^{l_i(1-a)} < D^{(1-a)} \sum_{i=1}^{n}\left(f\left(\mu_A(x_i), \nu_A(x_i)\right)\right)^{a}.$$

Taking logarithm on both sides and we achieve the result (2.2.1) after simplification for $\frac{1}{1-a} > 0$.
$i.e.$

$V_a(A) + \ln D > L_a$.

Possibility-II When $a > 1$, then raising power $(1-a)$ on both sides of (2.2.5), we get the following result

$$D^{l_i(1-a)} > D^{(1-a)}\left(f\big(\mu_A(x_i), \nu_A(x_i)\big)\right)^{a-1}. \tag{2.2.7}$$

Now, multiplying (2.2.7) throughout by $f\big(\mu_A(x_i), \nu_A(x_i)\big)$ and then summing up from $i = 1$ to $i = n$, we have as follows

$$\sum_{i=1}^{n} f\big(\mu_A(x_i), \nu_A(x_i)\big) \,.\, D^{l_i(1-a)} > D^{(1-a)} \sum_{i=1}^{n}\left(f\big(\mu_A(x_i), \nu_A(x_i)\big)\right)^{a}.$$

Taking logarithm on both sides and we achieve the result (2.2.1) after simplification for $\frac{1}{1-a} < 0$.

$i.e.$

$V_a(A) + \ln D > L_a$.

Hence the result.

Theorem 2.3. For $a > 1$ and for every code word lengths $\{l_i\}_{i=1}^{n}$, show that L_a can be made to satisfy the inequality

$$(1-a)V_a^b(A) + \ln D < (1-a)V_a(A) \leq (1-a)L_a \tag{2.3.1}$$

Proof: Letting,

$$L_i = -\ln_D \left(f\big(\mu_A(x_i), \nu_A(x_i)\big)\right) \tag{2.3.2}$$

Ofcourse, L_i and $L_i + 1$ satisfy the equality in Holder’s inequality. On the other hand L_i satisfies the condition $\sum_{i=1}^{n} D^{-L_i a} \leq \sum_{i=1}^{n}\left(f\big(\mu_A(x_i), \nu_A(x_i)\big)\right)^{a}$. Obviously, l_i also satisfies this condition because l_i lies between L_i and $L_i + 1$. Now, since $a > 1$. So

$$D\sum_{i=1}^{n}\big(\mu_A(x_i) + \nu_A(x_i)\big)\, D^{L_i(1-a)} > \sum_{i=1}^{n}\big(\mu_A(x_i) + \nu_A(x_i)\big)\, D^{L_i(1-a)}$$

$$\geq \sum_{i=1}^{n}\big(\mu_A(x_i) + \nu_A(x_i)\big)\, D^{l_i(1-a)} \tag{2.3.3}$$

But, $\sum_{i=1}^{n}\big(\mu_A(x_i)+\nu_A(x_i)\big)\,D^{L_i(1-a)}=\sum_{i=1}^{n}\{\mu_A^a(x_i)+\nu_A^a(x_i)\}$. So, (2.3.3) is now becomes

$$D\sum_{i=1}^{n}\big(\mu_A^a(x_i)+\nu_A^a(x_i)\big)>\sum_{i=1}^{n}\big(\mu_A^a(x_i)+\nu_A^a(x_i)\big)\geq\sum_{i=1}^{n}\big(\mu_A(x_i)+\nu_A(x_i)\big)\,D^{l_i(1-a)}$$

after simplification we shall achieve (2.3.1).

Conclusion:

As we know that optimal code word length is the length for which the value of code length equalize with its lower bound. In this communication we derived not only the lower boundary for the discussed information measures, but also the upper boundary in intuitionistic fuzzy environment. All this results are dependent on sinle parameter a. While in the case of Shannon's work it does not depend on any parameter. So it can be reduced significantly by taking some suitable value of the parameter.

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