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BI-TOPOLOGICAL PYTHAGOREAN FUZZY SOFT SET FOR DECISION MAKING

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Abstract

The primary objective of this paper is to propose a novel decision-making framework based on bi-topological Pythagorean fuzzy soft sets for handling uncertainty in complex decision environments. Pythagorean fuzzy sets, characterized by membership and non-membership degrees satisfying the Pythagorean condition, provide a more flexible and effective representation of uncertain and imprecise information than traditional fuzzy models. By integrating soft set theory with two topological structures, the proposed bi-topological approach offers a dual analytical perspective that enhances the evaluation and comparison of alternatives. In the developed methodology, relevant parameters and attributes are systematically selected and the given data are normalized to ensure consistency in the decision process. Corresponding Pythagorean fuzzy membership values are then computed and aggregated to obtain an overall assessment of each alternative. The framework supports efficient ranking and selection of optimal choices based on multiple criteria. A numerical example is presented to demonstrate the practical applicability and effectiveness of the proposed model in real-world decision-making scenarios. The obtained results show that the bi-topological Pythagorean fuzzy soft set approach provides reliable and consistent outcomes when dealing with uncertain, vague, and incomplete information. Therefore, the proposed method can be effectively applied to multi-criteria decision-making problems in various fields such as engineering, management, and applied sciences.

Keywords: Bi-topological Pythagorean Fuzzy Soft Set, Pythagorean Fuzzy Set, Soft Set Theory, Uncertainty Modeling, Multi-Criteria Decision Making.

I. INTRODUCTION

Fuzzy set theory, introduced by Zadeh [13], is an effective mathematical tool for handling uncertainty and vagueness in real-life problems. To overcome the limitations of intuitionistic fuzzy sets [2], Yager introduced Pythagorean fuzzy sets [12], in which the sum of the squares of the membership and non-membership degrees is less than or equal to one. This extension provides greater flexibility in modeling uncertain information. Soft set theory, proposed by Molodtsov [7], offers a parameterized approach for dealing with uncertainty without the restrictions of traditional fuzzy models. The concept of fuzzy soft sets was further developed by Maji P.K [6]. By combining Pythagorean fuzzy sets with soft sets and equipping the universe with two topologies, the concept of a Bi-topological Pythagorean Fuzzy Soft Set is introduced. Recent studies have also investigated several extensions of fuzzy and intuitionistic fuzzy structures and their topological properties with applications in different domains [9-11]. This structure enables the analysis of uncertainty under dual topological frameworks and provides a useful foundation for further theoretical development as well as decision-making applications.

II . BI-TOPOLOGICAL PYTHAGOREAN FUZZY SOFT SET
DEFINITION 1:

Let X be a non-empty universe of discourse and E be a set of parameters. Let (X,τ_1,τ_2) be a bi-topological space, where τ_1 and τ_2 are two topologies on X. A **Bi-topological Pythagorean Fuzzy Soft Set(BTPFSS)** over X is defined as an ordered quadruple (X,τ_1,τ_2,F), where $F:E\rightarrow PFS(X)$ and for each parameter $e\in E$, the Pythagorean fuzzy set $F(e)$ is given by $F(e)=\{(x,\mu_{F(e)}(x),\upsilon_{F(e)}(x)):x\in X\}$, here $\mu_{F(e)}(x)\in[0,1]$ denotes the membership degree of x and $\upsilon_{F(e)}(x)\in[0,1]$ denotes the non-membership degree of x such that $(\mu_{F(e)}(x))^2+(\upsilon_{F(e)}(x))^2\leq 1,\ \forall x\in X$. Then, the structure (X,τ_1,τ_2,F) is called a **Bi-topological Pythagorean Fuzzy Soft Set** over X.

APPLICATION OF BI-TOPOLOGICAL PYTHAGOREAN FUZZY SOFT SET IN DECISION MAKING:

In this section, a Bi-topological Pythagorean Fuzzy Soft Set is considered to illustrate its applicability. Let X be a universe of discourse and F be a Pythagorean fuzzy soft set defined on X. With two topologies τ_1 and τ_2, the structure (X,τ_1,τ_2,F) forms a Bi-topological Pythagorean Fuzzy Soft Space, which provides a flexible framework for handling uncertainty.

Pythagorean Fuzzy Condition:

A Pythagorean fuzzy set (PFS) is defined by a membership degree μ and a non-membership degree υ, satisfying $\mu^2+\upsilon^2\leq 1,\ 0\leq\mu,\upsilon\leq 1$. This condition must hold for all products.

Normalization of attributes:
The normalized membership values are calculated as:

i. Profit Membership Value $\mu_{profit} = \frac{Profit\ value}{Maximum\ profit\ value}$
ii. Rating Membership Value$\mu_{rating} = \frac{Rating\ value}{Maximum\ rating\ value}$

Combined Membership Value $\mu = \frac{\mu_{profit} + \mu_{rating}}{2}$. That is $\mu = \frac{\mu_p + \mu_r}{2}$. Non-membership Value $\upsilon = \sqrt{1-\mu^2}$

Products of company A:
Let the products ofcompany A be represented as follows.
Table 1:

ITEM	μ	υ	μ^2	υ^2
HAPPY HAPPY RS10	0.955	0.297	0.912	0.088
HAPPY HAPPY RS5	0.870	0.493	0.757	0.243
HIDE & SEEK RS10	0.970	0.243	0.941	0.059
HIDE & SEEK RS30	0.836	0.549	0.669	0.301
MELODY POUCH RS100	0.800	0.600	0.640	0.360
MELODY POUCH RS50	0.952	0.307	0.906	0.094

Verification of Bi-topological Pythagorean Fuzzy Soft Set using table 1 values
Step 1: Verification of Pythagorean Fuzzy Condition
$\mu(x)^2 + \upsilon(x)^2 = 0.912 + 0.088 = 1 \leq 1$.Hence, the Pythagorean Fuzzy condition is satisfied.Similarly, all other products in the universe set X also satisfy the Pythagorean fuzzy condition.

Step 2: Construction of Pythagorean Fuzzy Soft Set
Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.Where, x_1 = HAPPY HAPPY RS10, x_2 = HAPPY HAPPY RS5, x_3= HIDE & SEEK RS10, x_4= HIDE & SEEK RS30, x_5= MELODY POUCH RS100, x_6= MELODY POUCH RS50. Let $E = \{e\}$. Define a mapping $F : E \to PFS(X)$ $F(e) = \{(x_1, 0.955, 0.297), (x_2, 0.870, 0.493), (x_3, 0.970, 0.243), (x_4, 0.836, 0.549), (x_5, 0.800, 0.600), (x_6, 0.885, 0.466)\}$. Thus, (F, E) is a Pythagorean fuzzy soft set over the universe X.

Step 3: Defining two topologies
In order to analyze the Pythagorean fuzzy soft set with respect to both acceptance and rejection information, two different topologies are defined on the universe X. One topology is constructed based on membership degrees, while the other is constructed based on non-membership degrees.
Topology τ_1 (Based on Membership Degrees)
Define a threshold value $\alpha = 0.899$. Elements whose membership degree satisfies

$\mu(x) \geq 0.899$ are considered.

From the given table:

$x_1 : \mu(x_1) = 0.955 \geq 0.899$, $x_2 : \mu(x_2) = 0.870 < 0.899$, $x_3 : \mu(x_3) = 0.970 \geq 0.899$,

$x_4 : \mu(x_4) = 0.836 < 0.899$, $x_5 : \mu(x_5) = 0.800 < 0.899$, $x_6 : \mu(x_6) = 0.885 < 0.899$

Hence, the corresponding subset is $U_1 = \{x_1, x_3\}$. Define the topology $\tau_1 = \{\phi, X, U_1\}$.

Verification that τ_1 is a topology

1. $\phi, X \in \tau_1$.
2. Arbitrary union of members of τ_1 belongs to τ_1, since $\phi \cup U_1 = U_1 \in \tau_1$, $U_1 \cup X = X \in \tau_1$
3. Finite intersection of members of τ_1 belongs to τ_1, since $\phi \cap U_1 = \phi \in \tau_1$, $U_1 \cap X = U_1 \in \tau_1$.

Therefore, τ_1 is a topology on X.

Topology τ_2 (Based on Non-Membership Degrees)

Define a threshold value $\beta = 0.499$. Elements whose non- membership degree satisfies $\upsilon(x) \leq 0.499$ are considered.

From the given table:

$x_1 : \upsilon(x_1) = 0.297 \leq 0.499$, $x_2 : \upsilon(x_2) = 0.493 \leq 0.499$, $x_3 : \upsilon(x_3) = 0.243 \leq 0.499$,

$x_4 : \upsilon(x_4) = 0.549 > 0.499$, $x_5 : \upsilon(x_5) = 0.600 > 0.499$, $x_6 : \upsilon(x_6) = 0.466 \leq 0.499$

Hence, the corresponding subset is $U_2 = \{x_1, x_2, x_3, x_6\}$. Define the topology $\tau_2 = \{\phi, X, U_2\}$.

Verification that τ_2 is a topology

1. $\phi, X \in \tau_2$.
2. Arbitrary union of members of τ_2 belongs to τ_2, since $\phi \cup U_2 = U_2 \in \tau_2$, $U_2 \cup X = X \in \tau_2$
3. Finite intersection of members of τ_2 belongs to τ_2, since $\phi \cap U_2 = \phi \in \tau_2$, $U_2 \cap X = U_2 \in \tau_2$

Therefore, τ_2 is a topology on X.

Hence the quadruple (X, τ_1, τ_2, F) is a bi-topological Pythagorean fuzzy soft set.

Products of company B :

Let the products ofcompany B be represented as follows.

Table 2:

ITEM	μ	υ	μ^2	υ^2
STAYFREE 50XLL NID	0.632	0.775	0.399	0.601
STAYFREE RS55	0.612	0.791	0.375	0.626
STAYFREE RS50	0.974	0.226	0.949	0.051

STAYFREE MRP35	0.727	0.686	0.529	0.471
STAYFREE RS37	0.780	0.626	0.608	0.392
STAYFREE RS45	0.875	0.484	0.766	0.234
STAYFREE RS45 BULK	0.869	0.495	0.755	0.245
STAYFREE XL MRP42	0.869	0.495	0.755	0.245

Verification of Bi-topological Pythagorean Fuzzy Soft Set using table 2 values
Step 1: Verification of Pythagorean Fuzzy Condition
$\mu(x)^2 + \upsilon(x)^2 = 0.375 + 0.626 = 1.001 \geq 1$. Since the Pythagorean fuzzy condition $\mu(x)^2 + \upsilon(x)^2 \leq 1$ is not satisfied for company B, given data do not form a Pythagorean fuzzy set. Consequently, the corresponding Pythagorean fuzzy soft set cannot be constructed, and hence the structure cannot be extended to a Bi-topological Pythagorean Fuzzy Soft Set.

Interior and closure of Bi-topological Pythagorean Fuzzy Soft Sets:
DEFINITION 2:

Let (X, τ_1, τ_2, F) be a bi-topological Pythagorean fuzzy soft set. The interior of a Pythagorean fuzzy soft set F with respect to $\tau_i (i = 1,2)$, denoted by $Int_{\tau_i}(F)$, is defined as the union of all τ_i-open Pythagorean fuzzy soft sets contained in F.
Mathematical representation: $Int_{\tau_i}(F) = \bigcup\{G : G \subseteq F$ and G is $\tau_i - open\}$, $i = 1,2$.

DEFINITION 3:
Let (X, τ_1, τ_2, F) be a bi-topological Pythagorean fuzzy soft set. The closure of a Pythagorean fuzzy soft set F with respect to $\tau_i (i = 1,2)$, denoted by $Cl_{\tau_i}(F)$, is defined as the intersection of all τ_i-closed Pythagorean fuzzy soft sets which contain F.
Mathematical representation: $Cl_{\tau_i}(F) = \bigcap\{G : F \subseteq G$ and G is $\tau_i - closed\}, i = 1,2$.

EXAMPLE: 1
Let (X, τ_1, τ_2, F) be a Bi-topological Pythagorean Fuzzy Soft Set. Universe: $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Parameter set: $E = \{e\}$. Let $A = \{x_1, x_3, x_4\}$. Definition of F : $F(e) = \{(x_1, 0.955, 0.297), (x_2, 0.870, 0.493), (x_3, 0.970, 0.243), (x_4, 0.836, 0.549), (x_5, 0.800, 0.600), (x_6, 0.885, 0.466)\}$. For all $x \in X$, the Pythagorean condition $\mu^2 + \upsilon^2 \leq 1$ is satisfied.
Interior: $Int_{\tau_1}(F) = \{(x_1, 0.955, 0.297), (x_3, 0.970, 0.243)\}, Int_{\tau_2}(F) = \phi.$
Closure: $Cl_{\tau_1}(F) = F(e), Cl_{\tau_2}(F) = F(e).$
EXAMPLE: 2

Let (X,τ_1,τ_2,F) be a Bi-topological Pythagorean Fuzzy Soft Set. Universe: $X=\{x_1,x_2,x_3,x_4\}$ Parameter set: $E=\{e\}$. Let $A=\{x_1,x_3\}$. Definition of F: $F(e)=\{(x_1,0.632,0.775),(x_2,0.612,0.791),(x_3,0.974,0.226),(x_4,0.727,0.686)\}$. The Pythagorean condition $\mu^2+\upsilon^2\le 1$ is not satisfied for some elements, then the given set is not a valid Pythagorean fuzzy soft set. Hence, such elements are excluded from the fuzzy topology. As a result, the interior of the set becomes empty and the closure reduces to the null Pythagorean fuzzy soft set.

ALGORITHM: Construction of a Bi-topological Pythagorean Fuzzy Soft Set

Step1: Create a universe set with appropriate elements and assign corresponding membership and non-membership values.

Step2: Initialize the Pythagorean fuzzy parameters, namely the membership function μ and non-membership function υ.

Step3: For each element of the universe, compute the value $\mu(x)^2+\upsilon(x)^2$.

Step4: Verify whether the computed values satisfy the Pythagorean fuzzy condition $\mu(x)^2+\upsilon(x)^2\le 1$.

Step5: If the condition is satisfied, construct the Pythagorean fuzzy soft set F. Otherwise, stop the process.

Step6: Choose suitable threshold values α and β.

Step7: Construct the first topology τ_1 based on the membership values using the threshold α.

Step8: Construct the second topology τ_2 based on the non-membership values using the threshold β.

Step9: Verify that τ_1 and τ_2 satisfy the axioms of topology.

Step10: Conclude that (X,τ_1,τ_2,F) is a BTPFSS.

FINAL DECISION:

The Bi-topological Pythagorean Fuzzy Soft Set model is applicable only to the Company A data, and among them, HIDE & SEEK RS10 is selected as the optimal product. The Company B dataset is excluded since it does not satisfy the Pythagorean fuzzy condition.

CONCLUSION:

In this paper, the concept of a **Bi-topological Pythagorean Fuzzy Soft Set** framework has been introduced by integrating Pythagorean fuzzy set theory with soft set theory under two distinct topological structures. The Pythagorean fuzzy condition was rigorously verified, and suitable Pythagorean fuzzy soft sets were constructed. Further, two topologies were defined and shown to satisfy the axioms of a topological space, which confirms the validity of the proposed bi-topological framework. The developed model provides a more effective mathematical tool for dealing with uncertainty, imprecision, and dual evaluation criteria compared to existing fuzzy and soft set models. The study also indicates that Bi-topological Pythagorean Fuzzy Soft Sets can be effectively applied in decision-making and related real-world problems. Future work may focus on extending this framework by introducing new topological properties and operators, and by exploring applications in areas such as medical diagnosis, data analysis, and multi-criteria decision-making.

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