

A Fixed Point Theorem For b-Metric Space

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Abstract

The aim of this paper to obtain completeness and uniqueness of fixed point theorem on b-metric space. In this paper we show that different contractive type mapping exist in b-metric space.

Keywords: contractive mapping, fixed point, b-metric space.

Introduction

In many branches of science, economics, computer science, engineering and the development of nonlinear analysis, the fixed point theory is one of the most important tool.

In 1989, Backhtin [1] introduced the concept of b-metric space. In 1993, Czerwik[5] extended the results of b-metric spaces. Using this idea many researcher presented generalization of the renowned banach fixed point theorem in the b-metric space.

Mehmet Kir[7], Boriceanu[4], Czerwik, s[5], Bota[3], Pacurar[8] extended the fixed point theorem in b-metric space. Various problems of the convergence of measurable functions with respect to measure, Czerwik [5] first presented a generalization of banach fixed point theorem in b-metric spaces.

We want to extend some well known fixed point theorems which are also valid in b-metric space.

Definition 1. 1. Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d: X \times X \rightarrow R_+$ is called a b-metric provided that for all $x, y, z \in X$

- 1) $d(x, y) = 0$ if and only if $x = y$,
- 2) $d(x, y) = d(y, x)$,
- 3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair (X, d) is called a b-metric space. It is clear that definition of b-metric space is an extension of usual metric space.

Some examples of b-metric spaces are given below:

Example 1. 2. By Boriceanu [4], The set $l_p(R)$ (with $0 < p < 1$), where $l_p(R) := \{(x_n) \subset R \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the function $d: l_p(R) \times l_p(R) \rightarrow R$,

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}$$

where $x = x_n, y = y_n \in l_p(R)$ is a b-metric space. By an elementary calculation we obtain that $d(x, z) \leq 2^{\frac{1}{p}} [d(x, y) + d(y, z)]$.

Example 1. 3. By Boriceanu [4], Let $X = \{0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \geq 2$, $d(0, 1) = d(1, 2) = d(1, 0) = d(2, 1) = 1$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$.

then $d(x, y) \leq \frac{m}{2} [d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

if $m > 2$ then the triangle inequality does not hold.

Example 1. 4. By Boriceanu[4], The space $L_p[0, 1]$ (where $0 < p < 1$) of all real functions $x(t), t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, is a b-metric space if we take

$$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{\frac{1}{p}}, \text{ for each } x, y \in L_p[0, 1].$$

Definition 1. 5. By Boriceanu [4] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

Definition 1. 6. By Boriceanu [4] Let (X, d) be a b-metric space then a sequence $\{x_n\}$ in X is called convergent sequence if and only if there exists $x \in X$ such that for all there exists $n(\varepsilon) \in N$ such that for all $n \geq n(\varepsilon)$ we have $d(x_n, x) < \varepsilon$. In this case we write $\lim_{n \rightarrow \infty} x_n = x$.

Definition 1. 6. [4] The b-metric space is complete if every Cauchy sequence convergent.

MAIN RESULT

Theorem 2. 1. Let (X, d) be a complete b-metric space. Let T be a mapping $T: X \rightarrow X$ such that

$$d(Tx, Ty) \leq a \max\{d(x, Tx), d(y, Ty), d(x, y)\} + b\{d(x, Ty) + d(y, Tx)\} \quad (1)$$

where $a, b > 0$ such that $a + 2bs \leq 1 \forall x, y \in X$ and $s \geq 1$ then T has a unique fixed point.

Proof: Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X defined by the recursion

$$x_n = Tx_{n-1} = T^n x_0 \quad n=1, 2, 3, 4 \quad (2)$$

By (1) and (2) we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq a \max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, x_n)\} \\ &\quad + b\{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})\}, \\ d(Tx_{n-1}, Tx_n) &\leq a \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} + b\{d(x_{n-1}, x_{n+1}) \\ &\quad + d(x_n, x_n)\}, \\ d(Tx_{n-1}, Tx_n) &\leq a \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + b\{d(x_{n-1}, x_{n+1})\}, \\ d(x_n, x_{n+1}) &\leq \\ a \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} &+ sb\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}, \\ d(x_n, x_{n+1}) &\leq \\ a M_1 + sb\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}, & \quad (3) \end{aligned}$$

where, $M_1 = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$

Now two cases arises,

Case I: If suppose that $M_1 = d(x_n, x_{n+1})$ then we have,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq a d(x_n, x_{n+1}) + sb\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}, \\ d(x_n, x_{n+1}) &\leq a d(x_n, x_{n+1}) + sb d(x_{n-1}, x_n) + sbd(x_n, x_{n+1}), \\ (1 - a - bs)d(x_n, x_{n+1}) &\leq bs d(x_{n-1}, x_n), \\ d(x_n, x_{n+1}) &\leq \frac{bs}{(1-a-bs)} d(x_{n-1}, x_n), \\ d(x_n, x_{n+1}) &\leq k d(x_{n-1}, x_n), \\ \text{where } k &= \frac{bs}{(1-a-bs)} < 1 \\ \Rightarrow d(x_n, x_{n+1}) &\leq k d(x_{n-1}, x_n), \\ \Rightarrow d(x_n, x_{n+1}) &\leq k^2 d(x_{n-2}, x_{n-1}), \\ \text{continuing this process, we get} \\ \Rightarrow d(x_n, x_{n+1}) &\leq k^n d(x_0, x_1). \end{aligned}$$

Case II: If suppose that $M_1 = d(x_{n-1}, x_n)$ then we have,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq a d(x_{n-1}, x_n) + sb\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}, \\ d(x_n, x_{n+1}) &\leq a d(x_{n-1}, x_n) + sb d(x_{n-1}, x_n) + sbd(x_n, x_{n+1}), \\ (1-bs)d(x_n, x_{n+1}) &\leq (a + bs)d(x_{n-1}, x_n), \\ d(x_n, x_{n+1}) &\leq \frac{(a+bs)}{(1-bs)} d(x_{n-1}, x_n), \\ \text{where } k &= \frac{(a+bs)}{(1-bs)} < 1 \\ \Rightarrow d(x_n, x_{n+1}) &\leq k d(x_{n-1}, x_n), \\ \Rightarrow d(x_n, x_{n+1}) &\leq k^2 d(x_{n-2}, x_{n-1}), \\ \text{continuing this process, we get} \\ \Rightarrow d(x_n, x_{n+1}) &\leq k^n d(x_0, x_1). \end{aligned}$$

Thus T is a contractive mapping.

Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m, n \in N, m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq s\{d(x_n, x_{n+1}) + d(x_{n+1}, x_m)\}, \\ &\leq sd(x_n, x_{n+1}) + s^2\{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)\}, \end{aligned}$$

$$\begin{aligned}
&\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_m), \\
&\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^3d(x_{n+2}, x_{n+3}) + \dots \dots \\
&\leq sk^n d(x_0, x_1) + s^2k^{n+1}d(x_0, x_1) + s^3k^{n+2}d(x_0, x_1) + \dots \dots \\
&\leq sk^n d(x_0, x_1)[1 + sk + (sk)^2 + (sk)^3 + \dots \dots], \\
&\leq \frac{sk^n}{1-sk} d(x_0, x_1).
\end{aligned}$$

Then $\lim_{n \rightarrow \infty} d(x_n, x_m) = 0$, as $n, m \rightarrow \infty$, since $k < 1$, $\lim_{n \rightarrow \infty} \frac{sk^n}{1-sk} d(x_0, x_1) = 0$

as $n, m \rightarrow \infty$.

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X . Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{x_n\}$ converges to $x^* \in X$.

Now, we show that x^* is the fixed point of T .

$$\begin{aligned}
d(x^*, Tx^*) &\leq s\{d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)\}, \\
&\leq s d(x^*, x_{n+1}) + s d(Tx_n, Tx^*), \\
d(x^*, Tx^*) &\leq s d(x^*, x_{n+1}) + sa \max\{d(x_n, Tx_n), d(x^*, Tx^*), d(x_n, x^*)\} \\
&\quad + sb\{d(x_n, Tx^*) + d(x^*, Tx_n)\}, \\
d(x^*, Tx^*) &\leq s d(x^*, x_{n+1}) + sa \max\{d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, x^*)\} \\
&\quad + sb\{d(x_n, Tx^*) + d(x^*, x_{n+1})\}, \\
&\leq s d(x^*, x_{n+1}) + sa \max\{d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, x^*)\} + s^2b\{d(x_n, x^*) + \\
&\quad d(x^*, Tx^*)\} + sb d(x^*, x_{n+1}), \\
(1 - s^2b)d(x^*, Tx^*) &\leq s(1 + b)d(x^*, x_{n+1}) \\
&\quad + sa \max\{d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, x^*)\} + s^2b d(x_n, x^*), \\
(1 - s^2b)d(x^*, Tx^*) &\leq s(1 + b)d(x^*, x_{n+1}) + sa M_2 + s^2b d(x_n, x^*), \\
\text{where } M_2 &= \max\{d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, x^*)\}.
\end{aligned}$$

Case-I: If suppose that $M_2 = d(x_n, x_{n+1})$ then we have,

$$\begin{aligned}
(1 - s^2b)d(x^*, Tx^*) &\leq s(1 + b)d(x^*, x_{n+1}) + sa d(x_n, x_{n+1}) + s^2b d(x_n, x^*), \\
&\leq s(1 + b)d(x^*, x_{n+1}) + s^2a \{d(x_n, x^*) + d(x^*, x_{n+1})\} + s^2b d(x_n, x^*), \\
(1 - s^2b)d(x^*, Tx^*) &\leq s(1 + b + as)d(x^*, x_{n+1}) + s^2(a + b)d(x_n, x^*), \\
d(x^*, Tx^*) &\leq \frac{s(1 + b + as)}{(1 - s^2b)} d(x^*, x_{n+1}) + \frac{s^2(a + b)}{(1 - s^2b)} d(x_n, x^*).
\end{aligned}$$

Taking $\lim n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} d(x^*, Tx^*) = 0,$$

$$\Rightarrow x^* = Tx^*,$$

Therefore x^* is the fixed point of T .

Case-II: If suppose that $M_2 = d(x_n, x^*)$ then,

$$\begin{aligned}
(1 - s^2b)d(x^*, Tx^*) &\leq s(1 + b)d(x^*, x_{n+1}) + sa d(x_n, x^*) + s^2b d(x_n, x^*), \\
(1 - s^2b)d(x^*, Tx^*) &\leq s(1 + b)d(x^*, x_{n+1}) + (sa + s^2b)d(x_n, x^*), \\
d(x^*, Tx^*) &\leq \frac{s(1 + b)}{(1 - s^2b)} d(x^*, x_{n+1}) + \frac{(sa + s^2b)}{(1 - s^2b)} d(x_n, x^*),
\end{aligned}$$

Taking $\lim n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} d(x^*, Tx^*) = 0,$$

$$\Rightarrow x^* = Tx^*.$$

Therefore x^* is the fixed point of T .

Case-III: If suppose that $M_2 = d(x^*, Tx^*)$ then

$$(1 - s^2b)d(x^*, Tx^*) \leq s(1 + b)d(x^*, x_{n+1}) + sa d(x^*, Tx^*) + s^2b d(x_n, x^*),$$

$$(1 - sa - s^2b)d(x^*, Tx^*) \leq s(1 + b)d(x^*, x_{n+1}) + s^2b d(x_n, x^*),$$

$$d(x^*, Tx^*) \leq \frac{s(1 + b)}{(1 - sa - s^2b)} d(x^*, x_{n+1}) + \frac{s^2b}{(1 - sa - s^2b)} d(x_n, x^*).$$

Taking $\lim n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} d(x^*, Tx^*) = 0,$$

$$\Rightarrow x^* = Tx^*.$$

Therefore x^* is the fixed point of T.

Uniqueness of Fixed Point:

We have to show that x^* is unique fixed point of T.

Assume that x' is another fixed point of T then we have

$$Tx' = x' \text{ and } d(x^*, x') = d(Tx^*, Tx')$$

$$\leq a \max \{d(x^*, Tx^*), d(x', Tx'), d(x^*, x')\}$$

$$+ b\{d(x^*, Tx'), d(x', Tx^*)\},$$

$$\leq a \max \{d(x^*, x^*), d(x', x'), d(x^*, x')\}$$

$$+ b\{d(x^*, x'), d(x', x^*)\},$$

$$\leq ad(x^*, x') + b\{d(x^*, x'), d(x', x^*)\},$$

$$\leq ad(x^*, x') + 2bd(x^*, x'),$$

$$d(x^*, x') \leq (a + 2b) d(x^*, x').$$

This is contradiction. Therefore $x^* = x'$.

This completes the proof. Hence x^* is the unique fixed point.

Corollary

Let (X, d) be a complete b-metric space. Let T be a mapping $T: X \rightarrow X$ such that

$$d(Tx, Ty) \leq a\{d(x, Tx) + d(y, Ty)\} + b\{d(x, Ty) + d(y, Tx)\} \tag{1}$$

where $a, b > 0$ such that $2a + b(2s + 1) < 1 \forall x, y \in X$ and $s \geq 1$ then T has a unique fixed point.

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