International Journal of Pure and Applied Mathematical Sciences (IJPAMS).
ISSN 0972-9828 Volume 14, Number 1 (2021), pp. 9-19

http://www.ripublication.com/ijpams.htm

Some Inequalities Concerning Polar Derivative of a Polynomial

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Abstract

Let $p(z) = \sum_{v=0}^{n} a_v z^v$ **be a polynomial of degree n having no zero in** $|z| < k$**,** $k \leq 1$**, then Chanam et al.** [*Far East Journal of Mathematical Sciences,* 127(1)(2020), 61-70] *proved*

$$\begin{aligned}\max_{|z|=1}|p'(z)| &\leq \frac{n}{1+k^n}\max_{|z|=1}|p(z)| \\ &- \frac{n|a_1|k}{1+\frac{1}{k^n}}\left(\frac{\frac{1}{k^n}-1}{n} - \frac{\frac{1}{k^{n-2}}-1}{n-2}\right) - |a_{n-1}|(1-k^2), \;\; \textit{if } n>2\end{aligned}$$

and

$$\max_{|z|=1}|p'(z)| \leq \frac{n}{1+k^n}\max_{|z|=1}|p(z)| - |a_{n-1}|\left(\frac{1-k^n}{1+k^n}\right), \;\; \textit{if } n=2.$$

provided $|p'(z)|$ *and* $|q'(z)|$ *attain their maxima at the same point on the circle* $|z| = 1$*,*
where

$$q(z) = z^n\overline{p\left(\frac{1}{\overline{z}}\right)}.$$

In this paper, we extend the above inequalities to polar derivative of a polynomial. Further, we also prove an improved version of above inequalities into polar derivative.

Keywords and phrases: Inequalities, Polynomials, Zeros, Maximum modulus, Polar derivative of a polynomial.

AMS Subject Classification (2020): 15A18, 30C10, 30C15, 30A10.

1. INTRODUCTION

If $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1}$$

The above inequality is the well-known Bernstein inequality [3]. Inequality (1) is best possible and equality holds for the polynomial $p(z) = \lambda z^n$, $\lambda \neq 0$ being a complex number.

If we restrict to the class of polynomials having no zero in $|z| < 1$, then inequality (1) can be sharpened. In fact, Erdós conjectured and later Lax [11] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

Inequality (2) is sharp for polynomials having their zeros on $|z| = 1$.

The polar derivative of a polynomial $p(z)$ of degree n with respect to a real or complex number α, denoted by $D_\alpha p(z)$ is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

The polynomial $D_\alpha p(z)$ is of degree at most $n - 1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

Aziz and Shah [2] extended inequality (1) to polar derivative and proved that if $p(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq n|\alpha| \max_{|z|=1} |p(z)|. \tag{3}$$

Further, Aziz [1] extended inequality (2) to polar derivative and proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then for every real or complex number α with $|\alpha| \geq 1$,

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n}{2}(|\alpha| + 1) \max_{|z|=1} |p(z)|. \tag{4}$$

It was asked by R.P. Boas that if $p(z)$ is a polynomial of degree n not vanishing in $|z| < k$, $k > 0$, then how large can

$$\left\{ \max_{|z|=1} |p'(z)| \Big/ \max_{|z|=1} |p(z)| \right\} \text{ be ?}$$

A partial answer to this problem was given by Malik [12], who proved for the case $k \geq 1$ that

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{5}$$

Equality in (5) holds for $p(z) = (z+k)^n$.

For the class of polynomials not vanishing in $|z| < k$, $k \leq 1$, the precise estimate for maximum of $|p'(z)|$ on $|z| = 1$, in general, does not seem to be easily obtainable.

For quite some time, it was believed that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then the inequality analogous to (5) should be

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|, \tag{6}$$

till E.B. Saff gave the example $p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief.

There are many extensions of inequality (5)(see, for example Bidkham and Dewan [4], Dewan and Mir [8] and Chan and Malik [5]).

However, for the class of polynomials not vanishing in $|z| < k$, $k \leq 1$, Govil [9] proved inequality (6) with extra condition.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)|, \tag{7}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where

$$q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}.$$

Recently, Chanam et al. [6] improved Theorem 1.1 by involving some of the co-efficients of the polynomial. In fact, they proved

Theorem 1.2. *If $p(z) = \sum_{v=0}^{n} a_n z^n$ is a polynomial of degree $n \geq 2$ having no zero in $|z| < k$, $k \leq 1$, then*

$$\begin{aligned}\max_{|z|=1} |p'(z)| \quad &\leq \quad \frac{n}{1+k^n} \max_{|z|=1} |p(z)| \\ &- \quad \frac{n|a_1|k}{1+\frac{1}{k^n}}\left(\frac{\frac{1}{k^n}-1}{n} - \frac{\frac{1}{k^{n-2}}-1}{n-2}\right) - |a_{n-1}|(1-k^2), \quad \text{if } n>2 \end{aligned} \tag{8}$$

and

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \max_{|z|=1} |p(z)| - |a_{n-1}|\left(\frac{1-k^n}{1+k^n}\right), \quad \text{if } n=2. \tag{9}$$

provided $|p'(z)|$ *and* $|q'(z)|$ *attain their maxima at the same point on the circle* $|z| = 1$, *where*

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

In this paper, we first prove the following result which extends Theorem 1.2 to polar derivative of $p(z)$. In fact, we prove

Theorem 1.3. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* $n \geq 2$ *having no zero in* $|z| < k$, $k \leq 1$, *then for every real or complex number* α *with* $|\alpha| \geq \frac{1}{k}$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| &\leq \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1 + k^n} \max_{|z|=1} |p(z)| \\ &- \frac{n|a_1|k^2(k|\alpha| - 1)}{1 + k^n} \left\{ \frac{1 - k^n}{nk^2} - \frac{1 - k^{n-2}}{(n-2)} \right\} \\ &- (1 - k^2)|n\overline{a_n} + \alpha \overline{a_{n-1}}|, \quad \text{if } n > 2, \end{aligned} \tag{10}$$

and

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| &\leq \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1 + k^n} \max_{|z|=1} |p(z)| \\ &- \frac{n|a_{n-1}|k^{n-2}(1 - k)^2(k|\alpha| - 1)}{2(1 + k^n)} \\ &- (1 - k)|n\overline{a_n} + \alpha \overline{a_{n-1}}|, \quad \text{if } n = 2, \end{aligned} \tag{11}$$

provided $|D_\alpha p(z)|$ *and* $|D_\alpha q(z)|$ *attain their maxima at the same point on the circle* $|z| = 1$, *where*

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Remark 1.4. From the hypotheses of Theorem 1.3, $|D_\alpha p(z)|$ and $|D_\alpha q(z)|$ attain their maxima at the same point on $|z| = 1$. Further, if they are divided by $|\alpha|$ and considering limit as $\alpha \to \infty$, then they become $|p'(z)|$ and $|q'(z)|$ which attain their maxima at the same point on $|z| = 1$. Hence, dividing both sides of inequalities (10) and (11) as well as the quantities $|D_\alpha p(z)|$ and $|D_\alpha q(z)|$ by $|\alpha|$ and taking respectively limit as $\alpha \to \infty$, we readily get inequalities (8) and (9) of Theorem 1.2 along with the agreement that $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$.

Next, under the same set of hypotheses, we prove a result which further improves the bounds of Theorem 1.3. More precisely, we obtain

Theorem 1.5. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n \geq 2$ having no zero in $|z| < k$, $k \leq 1$, then for every real or complex number α with $|\alpha| \geq \frac{1}{k}$,*

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1+k^n} \max_{|z|=1} |p(z)| - \frac{n(|\alpha| + k^{n-1})}{1+k^n} \min_{|z|=k} |p(z)| - \frac{n|a_1|k^2(k|\alpha|-1)}{1+k^n} \left\{ \frac{1-k^n}{nk^2} - \frac{1-k^{n-2}}{(n-2)} \right\} - (1-k^2)|n\overline{a_n} + \alpha\overline{a_{n-1}}|, \quad \textit{if } n > 2,$$

and

$$\max_{|z|=1} |D_\alpha p(z)| \leq \frac{n(|\alpha| + k^n + k^{n-1} + 1)}{1+k^n} \max_{|z|=1} |p(z)| - \frac{n(|\alpha| + k^{n-1})}{1+k^n} \min_{|z|=k} |p(z)| - \frac{n|a_{n-1}|(1-k)^n(k|\alpha|-1)}{2k^{n-2}(1+k^n)} - (1-k)|n\overline{a_n} + \alpha\overline{a_{n-1}}|, \quad \textit{if } n = 2, \tag{12}$$

provided $|D_\alpha p(z)|$ and $|D_\alpha q(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Remark 1.6. If we adopt the similar argument of Remark 1.4 in Theorem 1.5, we get the following result proved by Chanam et al. [6, Theorem 1.3].

Theorem 1.7. *If $p(z) = \sum_{v=0}^{n} a_n z^n$ is a polynomial of degree $n \geq 2$ having no zero in $|z| < k$, $k \leq 1$, then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\} - \frac{n|a_1|k}{1+\frac{1}{k^n}} \left(\frac{\frac{1}{k^n} - 1}{n} - \frac{\frac{1}{k^{n-2}} - 1}{n-2} \right) - |a_{n-1}|(1-k^2), \quad \textit{if } n > 2 \tag{13}$$

and

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^n} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=k} |p(z)| \right\} - |a_{n-1}| \left(\frac{1-k^n}{1+k^n} \right), \quad \textit{if } n=2 \tag{14}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on the circle $|z| = 1$, where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

2. LEMMAS.

For the proofs of the theorems, we will use the following lemmas. The first lemma is a special case of a result due to Govil and Rahman [10].

Lemma 2.1. *If $p(z)$ is a polynomial of degree n, then on $|z| = 1$,*

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|, \tag{15}$$

where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Lemma 2.2. *If $p(z)$ is a polynomial of degree n and α is any real or complex number, then on $|z| = 1$,*

$$|D_\alpha p(z)| + |D_\alpha q(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)|, \tag{16}$$

where

$$q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Lemma 2.2 was proved by Aziz [1, Lemma 2] in more general form. However, we present a simple proof of this lemma which we think is new, simply by using definition of polar derivative of a polynomial and Lemma 2.1 due to Govil and Rahman [10].

Proof of Lemma 2.2. Let $q(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}$. Then it is easy to verify that on $|z| = 1$,

$$|q'(z)| = |np(z) - zp'(z)|. \tag{17}$$

Now, for every real or complex number α, the polar derivative of $p(z)$ with respect to α is

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z). \tag{18}$$

This implies on $|z| = 1$,

$$|D_\alpha p(z)| \leq |np(z) - zp'(z)| + |\alpha||p'(z)|. \tag{19}$$

Using (17) in (19), we have on $|z| = 1$,

$$|D_\alpha p(z)| \leq |q'(z)| + |\alpha||p'(z)|. \tag{20}$$

Similarly, on $|z| = 1$,

$$|D_\alpha q(z)| \leq |p'(z)| + |\alpha||q'(z)|. \tag{21}$$

Adding (20) and (21), we have

$$|D_\alpha p(z)| + |D_\alpha q(z)| \leq (|\alpha| + 1) \left\{|p'(z)| + |q'(z)|\right\}. \tag{22}$$

Using Lemma 2.1 in (22), we get

$$|D_\alpha p(z)| + |D_\alpha q(z)| \leq n(|\alpha| + 1) \max_{|z|=1} |p(z)|, \tag{23}$$

which completes the proof of Lemma 2.2.
The next lemma is due to Mir [13].

Lemma 2.3. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* $n \geq 2$ *having all its zeros in* $|z| \leq k$, $k \geq 1$, *then for every real or complex number* α *with* $|\alpha| \geq k$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| \ &\geq\ n\left(\frac{|\alpha| - k}{1 + k^n}\right)\left\{\max_{|z|=1} |p(z)| + \frac{|a_{n-1}|}{k}\left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2}\right)\right\} \\ &+\ \left(1 - \frac{1}{k^2}\right)|na_0 + \alpha a_1|, \ \ \textit{if } n > 2 \end{aligned}\tag{24}$$

and

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| \ &\geq\ n\left(\frac{|\alpha| - k}{1 + k^n}\right)\left\{\max_{|z|=1} |p(z)| + \frac{|a_1|}{2k}(k - 1)^2\right\} \\ &+\ \left(1 - \frac{1}{k}\right)|na_0 + \alpha a_1|, \ \ \textit{if } n = 2. \end{aligned}\tag{25}$$

Lemma 2.4. *If* $p(z) = \sum_{v=0}^{n} a_v z^v$ *is a polynomial of degree* $n \geq 2$ *having all its zeros in* $|z| \leq k$, $k \geq 1$, *then for every real or complex number* α *with* $|\alpha| \geq k$,

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| \ &\geq\ n\left(\frac{|\alpha| - k}{1 + k^n}\right)\left\{\max_{|z|=1} |p(z)| + \frac{|a_{n-1}|}{k}\left(\frac{k^n - 1}{n} - \frac{k^{n-2} - 1}{n - 2}\right)\right. \\ &+\ \left.\frac{k^{n-1}|\alpha| + 1}{k^{n-1}|\alpha| - k^n} \min_{|z|=k} |p(z)|\right\} + \left(1 - \frac{1}{k^2}\right)|na_0 + \alpha a_1|, \ \ \textit{if } n > 2 \end{aligned}\tag{26}$$

and

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| \ &\geq\ n\left(\frac{|\alpha| - k}{1 + k^n}\right)\left\{\max_{|z|=1} |p(z)| + \frac{|a_{n-1}|}{2}k^{n-3}(k - 1)^n\right. \\ &+\ \left.\frac{k^{n-1}|\alpha| + 1}{k^{n-1}|\alpha| - k^n} \min_{|z|=k} |p(z)|\right\} + \left(1 - \frac{1}{k}\right)|na_0 + \alpha a_1|, \ \ \textit{if } n = 2 \end{aligned}\tag{27}$$

This result was proved by Dewan and Chanam [7].

3. PROOFS OF THEOREMS.

We first prove Theorem 1.5.
Proof of Theorem 1.5. Let $p(z) = \sum_{v=0}^{v=n} a_n z^n$ be a polynomial of degree $n \geq 2$ having no zero in $|z| < k$, $k \leq 1$. In other words, $p(z)$ has all its zeros in $|z| \geq k$, $k \leq 1$ and

hence all the zeros of $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$ lies in $|z| \leq 1/k$, $1/k \geq 1$.
Applying Lemma 2.4 on $q(z)$, for $|\alpha| \geq \frac{1}{k}$, we have

$$\begin{aligned}\max_{|z|=1}|D_\alpha q(z)| &\geq n\left(\frac{|\alpha|-\frac{1}{k}}{1+\frac{1}{k^n}}\right)\left\{\max_{|z|=1}|q(z)| + \frac{|a_1|}{\frac{1}{k}}\left(\frac{\frac{1}{k^n}-1}{n} - \frac{\frac{1}{k^{n-2}}-1}{n-2}\right)\right.\\ &+ \left.\frac{\frac{1}{k^{n-1}}|\alpha|+1}{\frac{1}{k^{n-1}}|\alpha|-\frac{1}{k^n}}\min_{|z|=k}|q(z)|\right\} + \left(1-\frac{1}{\frac{1}{k^2}}\right)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n} > 2 \end{aligned} \tag{28}$$

and

$$\begin{aligned}\max_{|z|=1}|D_\alpha q(z)| &\geq n\left(\frac{|\alpha|-\frac{1}{k}}{1+\frac{1}{k^n}}\right)\left\{\max_{|z|=1}|q(z)| + \frac{|a_1|\frac{1}{k^{n-3}}(\frac{1}{k}-1)^n}{2}\right.\\ &+ \left.\frac{\frac{1}{k^{n-1}}|\alpha|+1}{\frac{1}{k^{n-1}}|\alpha|-\frac{1}{k^n}}\min_{|z|=\frac{1}{k}}|q(z)|\right\} + \left(1-\frac{1}{\frac{1}{k}}\right)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n} = 2 \end{aligned} \tag{29}$$

Which is equivalent to

$$\begin{aligned}\max_{|z|=1}|D_\alpha q(z)| &\geq \frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\\ &\times \left[\max_{|z|=1}|q(z)| + |a_1|k\left\{\frac{1-k^n}{nk^n} - \frac{1-k^{n-2}}{(n-2)k^{n-2}}\right\}\right.\\ &+ \left.\frac{k(|\alpha|+k^{n-1})}{k|\alpha|-1}\min_{|z|=\frac{1}{k}}|q(z)|\right] + (1-k^2)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n} > 2 \end{aligned} \tag{30}$$

and

$$\begin{aligned}\max_{|z|=1}|D_\alpha q(z)| &\geq \frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\left\{\max_{|z|=1}|q(z)| + |a_1|\frac{(1-k)^n}{2k^{2n-3}}\right.\\ &+ \left.\frac{k(|\alpha|+k^{n-1})}{k|\alpha|-1}\min_{|z|=\frac{1}{k}}|q(z)|\right\} + (1-k)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n} = 2 \end{aligned} \tag{31}$$

Now,

$$\min_{|z|=\frac{1}{k}}|q(z)| = \frac{1}{k^n}\min_{|z|=k}|p(z)|. \tag{32}$$

Using (32) in (30) and (31), we have

$$\begin{aligned}\max_{|z|=1}|D_\alpha q(z)| &\geq \frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\\ &\times \left[\max_{|z|=1}|q(z)| + |a_1|k\left\{\frac{1-k^n}{nk^n} - \frac{1-k^{n-2}}{(n-2)k^{n-2}}\right\}\right.\\ &+ \left.\frac{k(|\alpha|+k^{n-1})}{k^n(k|\alpha|-1)}\min_{|z|=k}|p(z)|\right] + (1-k^2)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ n} > 2 \end{aligned} \tag{33}$$

and

$$\begin{aligned}\max_{|z|=1}|D_\alpha q(z)| &\geq \frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\left\{\max_{|z|=1}|q(z)|+|a_1|\frac{(1-k)^n}{2k^{2n-3}}\right.\\ &+ \left.\frac{k(|\alpha|+k^{n-1})}{k^n(k|\alpha|-1)}\min_{|z|=k}|p(z)|\right\}+(1-k)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n = 2} \end{aligned} \tag{34}$$

Also, since on $|z|=1$, $|p(z)|=|q(z)|$, inequalities (33) and (34) can be written as

$$\begin{aligned}\max_{|z|=1}|D_\alpha q(z)| &\geq \frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\\ &\times \left[\max_{|z|=1}|p(z)|+|a_1|k\left\{\frac{1-k^n}{nk^n}-\frac{1-k^{n-2}}{(n-2)k^{n-2}}\right\}\right.\\ &+ \left.\frac{(|\alpha|+k^{n-1})}{k^{n-1}(k|\alpha|-1)}\min_{|z|=k}|p(z)|\right]+(1-k^2)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ n > 2} \end{aligned} \tag{35}$$

and

$$\begin{aligned}\max_{|z|=1}|D_\alpha q(z)| &\geq \frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\left\{\max_{|z|=1}|p(z)|+|a_1|\frac{(1-k)^n}{2k^{2n-3}}\right.\\ &+ \left.\frac{(|\alpha|+k^{n-1})}{k^{n-1}(k|\alpha|-1)}\min_{|z|=k}|p(z)|\right\}+(1-k)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{if n = 2} \end{aligned} \tag{36}$$

By Lemma 2.2, on $|z|=1$,

$$|D_\alpha p(z)|+|D_\alpha q(z)|\leq n(|\alpha|+1)\max_{|z|=1}|p(z)|. \tag{37}$$

Let z_0 be a point on $|z|=1$ such that $\max_{|z|=1}|D_\alpha q(z)|=|D_\alpha q(z_0)|$. Since $|D_\alpha p(z)|$ and $|D_\alpha q(z)|$ attain their maxima at the same point on $|z|=1$ with $|\alpha|\geq\frac{1}{k}$, we have

$$\max_{|z|=1}|D_\alpha p(z)|=|D_\alpha p(z_0)|.$$

Thus, in particular (37) gives

$$\max_{|z|=1}|D_\alpha q(z)|\leq n(|\alpha|+1)\max_{|z|=1}|p(z)|-\max_{|z|=1}|D_\alpha p(z)|. \tag{38}$$

Combining (38) with (35) and (36), we have

$$\begin{aligned}n(|\alpha|+1)\max_{|z|=1}|p(z)|-\max_{|z|=1}|D_\alpha p(z)| &\geq \frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\\ &\times \left[\max_{|z|=1}|p(z)|+|a_1|k\left\{\frac{1-k^n}{nk^n}-\frac{1-k^{n-2}}{(n-2)k^{n-2}}\right\}\right.\\ &+ \left.\frac{|\alpha|+k^{n-1}}{k^{n-1}(k|\alpha|-1)}\min_{|z|=k}|p(z)|\right]\\ &+ (1-k^2)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n > 2} \end{aligned} \tag{39}$$

and

$$\begin{aligned} n(|\alpha|+1)\max_{|z|=1}|p(z)|-\max_{|z|=1}|D_\alpha p(z)| &\geq \frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\Bigg\{\max_{|z|=1}|p(z)|+|a_1|\frac{(1-k)^n}{2k^{2n-3}}\\ &+ \frac{(|\alpha|+k^{n-1})}{k^{n-1}(k|\alpha|-1)}\min_{|z|=k}|p(z)|\Bigg\}\\ &+ (1-k)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n} = 2, \end{aligned} \tag{40}$$

which is equivalent to

$$\begin{aligned} \max_{|z|=1}|D_\alpha p(z)| &\leq n(|\alpha|+1)\max_{|z|=1}|p(z)|-\left\{\frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\right\}\max_{|z|=1}|p(z)|\\ &- \left\{\frac{n|a_1|k^n(k|\alpha|-1)}{1+k^n}\right\}\left\{\frac{1-k^n}{nk^n}-\frac{1-k^{n-2}}{(n-2)k^{n-2}}\right\}\\ &- \left\{\frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\right\}\frac{(|\alpha|+k^{n-1})}{k^{n-1}(k|\alpha|-1)}\min_{|z|=k}|p(z)|\\ &- (1-k^2)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n} > 2 \end{aligned} \tag{41}$$

and

$$\begin{aligned} \max_{|z|=1}|D_\alpha p(z)| &\leq n(|\alpha|+1)\max_{|z|=1}|p(z)|-\left\{\frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\right\}\max_{|z|=1}|p(z)|\\ &- \left\{\frac{n|a_1|k^{n-1}(k|\alpha|-1)}{1+k^n}\right\}\left\{\frac{(1-k)^n}{2k^{2n-3}}\right\}\\ &- \left\{\frac{nk^{n-1}(k|\alpha|-1)}{1+k^n}\right\}\frac{(|\alpha|+k^{n-1})}{k^{n-1}(k|\alpha|-1)}\min_{|z|=k}|p(z)|\\ &- (1-k)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n} = 2, \end{aligned} \tag{42}$$

which on simplification gives

$$\begin{aligned} \max_{|z|=1}|D_\alpha p(z)| &\leq \frac{n(|\alpha|+k^n+k^{n-1}+1)}{1+k^n}\max_{|z|=1}|p(z)|\\ &- \frac{n|a_1|k^2(k|\alpha|-1)}{1+k^n}\left\{\frac{1-k^n}{nk^2}-\frac{1-k^{n-2}}{(n-2)}\right\}\\ &- \frac{n(|\alpha|+k^{n-1})}{1+k^n}\min_{|z|=k}|p(z)|\\ &- (1-k^2)||n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n} > 2, \end{aligned} \tag{43}$$

and

$$\begin{aligned} \max_{|z|=1}|D_\alpha p(z)| &\leq \frac{n(|\alpha|+k^n+k^{n-1}+1)}{1+k^n}\max_{|z|=1}|p(z)|-\frac{n|a_1|(k|\alpha|-1)(1-k)^n}{2k^{n-2}(1+k^n)}\\ &- \frac{n(|\alpha|+k^{n-1})}{1+k^n}\min_{|z|=k}|p(z)|-(1-k)|n\overline{a_n}+\alpha\overline{a_{n-1}}|, \text{ if n} = 2, \end{aligned} \tag{44}$$

which is the proof of Theorem 1.5.

Proof of Theorem 1.3 The proof of this theorem follows on the same lines as that of Theorem 1.5 but instead of applying Lemma 2.4, we apply Lemma 2.3 and we omit it.

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