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A Note on Fixed Point Theorems in complete new b-Metric Spaces with continuous functional

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Abstract

The purpose of this paper is to present some fixed point results for self mapping defined in complete new b-metric spaces [16] with continuous functional using iterative schemes some examples are given to support our results.

Keyword: Fixed point**;** complete new b - metric spaces, Iterative schemes.

1. INTRODUCTION:

Many generalization of the concept of metric space have been defined, and some fixed point theorems were proves in these spaces. In particular, b-metric spaces were introduced by Bakhtin [1], Bourbaki [4] and Czrewik [6] as a generalization of metric space. Many mathematicians worked on this interesting space. for more the reader Can refer [2, 3, 7, 9, 10, 11, 12, 13, 15] formally defined a b-metric space with a view of generalizing the Banach contraction mapping theorem. Later on, Fagin et al. [8] discussed some kind of relaxation in triangular inequality and called this new distance measure as non-linear Elastic Mathing (NEM). Similar type of relaxed triangle inequality was also used for trade measure [5] and to measure ice floes [14].

2. PRELIMINARIES & DEFINITION

Theorem : 1 Let (X, d) be a complete b-metric space with constant $s \geq 1$ s.t. b-metric is a continues functional. Let T: $X \rightarrow X$ be a contraction having contraction constant $k \in [0, 1)$ s.t. $ks < 1$, then T has a unique fixed point [13].

Definition 1.

Lat X be a (nonempty) set and $p \geq 1$ be a given real number. A function $d: X \times X \rightarrow [0, \infty)$ is called a b-metric on X if the following conditions hold for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x,y) \leq p[d(x, z) + d(z, y)]$ (b-triangular inequality).

Then, the pair (X,d) is called a b-metric space with parameter p.

Definition 2. Let (X, d) be a b - metric space.

(a) A sequence $\{x_n\}$in X is called b - convergent if and only if there exists $x \in X$ such that $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

(b) $\{x_n\}$ in X is said to be b-Cauchy if and only if $d(x_n, x_m) \rightarrow 0$, as $n, m \rightarrow \infty$.

(c) The b - metric space (X, d) is called complete if every b-Cauchy sequence in X is b-convergent.

Definition 3. Let (X, d_ϕ) be an New b-metric space.

(i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$, if for every $\varepsilon > 0$ there exists $N = M(\varepsilon) \in N$ such that $d_\phi(x_n, x) < \varepsilon$, for all $n \geq N$. In this case, we write $\lim_{n\rightarrow\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is said to be Cauchy, if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in N$ such that $d_\phi(x_m, x_n) < \varepsilon$, for all $m, n \geq N$.

Definition 4. Let X be s nonempty set and $\phi : X \times X \rightarrow [1, \infty)$. A function $d_\phi : X \times X \rightarrow [0, \infty)$ is called an New b-metric if for all $x, y, z \in X$ is satisfies:

($d_\phi 1$) $d_\phi (x, y) = 0$ iff $x = y$;

($d_\phi 2$) $d_\phi (x, y) = d_\phi(y, x)$;

($d_\phi 3$) $(x, z) \leq \phi(x, z) [d_\phi(x, y) + d_\phi(y, z)]$.

The pair (X, d_ϕ) is called an New b-metric space.

Definition 5. Let $T : X \to X$ and for some $x_0 \in X$, $O(x_0) = \{x_0, fx_0, f^2x_0, ...\}$ be the orbit of x_0. A function G from X into the set of real numbers is said to be T-orbitally lower semi-continuous at $t \in X$ if $\{x_n\} \subset O(x_0)$ and $x_n \to t \Rightarrow G(t) \leq \liminf_{n\to\infty} G(x_n)$.

Example 1 Let X = {2, 3, 4}. Define $\phi : X \times X \to R^+$ and $d_\phi : X \times X \to R^+$ as:

$$\phi(x, y) = 1 + x + y$$

$$d_\phi (2,2) = d_\phi (3, 3) = d_\phi (4, 4) = 0$$

$$d_\phi (2, 3) = d_\phi(3, 2) = 90$$

$$d_\phi (2, 4) = d_\phi (4, 2) = 700$$

$$d_\phi (3, 4) = d_\phi (4, 3) = 500$$

Proof ($d_\phi 1$) and ($d_\phi 2$) trivially hold for ($d_\phi 3$) we have:

$$d_\phi (2, 3) = 90, \phi (2, 3) [d_\phi (2, 4) + d_\phi (4, 3)] = 5 (700 + 500) = 6000$$

$$d_\phi (2, 4) = 700, \phi (2, 4) [d_\phi (2,3) + d_\phi (3, 4) = 7[90 + 500] = 4130$$

Similar calculation hold for $d_\phi (3, 4)$

Hence for all $x, y, z \in X$

$$d_\phi (x, y) \leq \phi (x, y) [d_\phi (x, z) + d_\phi (z, y)$$

(X, d_ϕ) is an new b-metric space

MAIN RESULTS

Theorem 1:

Let (X, d_ϕ) be a complete new b-metric space s.t. d_ϕ is a continuous functional. Let $T : X \to X$ satisfy:

$$d_\phi(Tx, Ty) \leq l\, [d_\phi (x, Tx) + d_\phi (y, Tx)] \quad (1)$$

for all $x, y \in X$

where $l \in [0, 1)$ be s.t. for each $x_0 \in X$ $\lim_{n,m\to\infty} \phi(x_n, x_m) < \frac{1}{l}$

here $x_n = T^n x_0$, n = 1, 2,......

Then T has precisely one fixed point η. Moreover for each $y \in X$, $T^n y \to \eta$ as $n \to \infty$

Proof: We choose any $x_0 \in X$ be arbitrary, define the iterative sequence $\{x_n\}$ by:

$$x_0, Tx_0 = x_1, x_2 = Tx_1 = T(Tx_0) = T^2 x_0 ....., x_n = T^n x_0 ....$$

Then by successively applying inequality (1) we obtain

$$d_\phi(x_n, x_{n+1}) \le l\,[d_\phi(x_{n-1}, Tx_{n-1}) + d_\phi(x_n, Tx_{n-1})]$$

$$\le l\,[d_\phi(x_{n-1}, x_n) + d_\phi(x_n, x_n)]$$

$$\le l\,[d_\phi(x_{n-1}, x_n)]$$

$$d_\phi(x_n, x_{n+1}) \le l^n[d_\phi(x_0, x_1) \quad (2)$$

By triangular inequality and (2), for m > n we have:

$$d_\phi(x_n, x_m) \le \phi(x_n, x_m)\, l^n\, d_\phi(x_0, x_1) + \phi(x_n, x_m)\,\phi(x_{n+1}, x_m)\, l^{n+1}\, d_\phi(x_0, x_1)$$

$$+ \ldots + \phi(x_n, x_m)\,\phi(x_{n+1}, x_m)\,\phi(x_{n+2}, x_m)\ldots\ldots\phi(x_{m-2}, x_m)\,\phi(x_{m-1}, x_m)$$

$$l^{m-1}\, d_\phi(x_0, x_1)$$

$$\le d_\phi(x_0, x_1)\,[\phi(x_1, x_m), \phi(x_2, x_m)\ldots.\phi(x_{n-1}, x_m)\,\phi(x_n, x_m)\, l^n$$

$$+ \phi(x_1, x_m)\,\phi(x_2, x_m)\ldots.\phi(x_n, x_m)\,\phi(x_{n+1}, x_m)\, l^{n+1} + \ldots.. +$$

$$\phi(x_1, x_m)\,\phi(x_2, x_m)\ldots.\phi(x_n, x_m)\,\phi(x_{n+1}, x_m)\ldots.\phi(x_{n-2}, x_m)$$

$$\phi(x_{n-1}, x_m)\, l^{m-1}\,]$$

Since $\lim_{n,m\to\infty} \phi(x_{n+1}, x_m)\; l < 1$ so that the series $\sum_{n=1}^{\infty} l^n \prod_{i=1}^{n} \phi(x_i, x_m)$ converges by ratio test for each m $\in$ N. Let:

$$S = \sum_{n=1}^{\infty} l^n \prod_{i=1}^{n} \phi(x_i, x_m),\; S_n = \sum_{j=1}^{n} l^j \prod_{i=1}^{j} \phi(x_i, x_m)$$

Thus for m > n above inequality implies:

$$d_\phi(x_n, x_m) \le d_\phi(x_0, x_1)\,[S_{m-1}, S_n]$$

Letting $n\to\infty$ we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete let $x_n \to \eta \in X$:

$$d\phi(T\eta, \eta) \le \phi(T\eta, \eta)\,[d_\phi(T\eta, x_n) + d_\phi(x_n, \eta)]$$

$$\le \phi(T\eta, \eta)\,[l\{\, d_\phi(\eta, T\eta) + d_\phi(x_{n-1}, T\eta)\} + d_\phi(x_n, \eta)]$$

$$\le \phi(T\eta, \eta)\,[l\{\, d_\phi(\eta, \eta) + d_\phi(x_{n-1}, \eta) + d_\phi(x_n, \eta)]$$

$$\le \phi(T\eta, \eta)\,[l\, d_\phi(x_{n-1}, \eta) + d_\phi(x_n, \eta)]$$

$$d_\phi(T\eta, \eta) \to 0 \text{ as } n \to \infty$$

Hence η is a fixed point of T.

Theorem 2: Let (X, d_ϕ) be a complete new b-metric space s.t. d_ϕ is a continuous functional. Let T : X → X and there exists $x_0 \in X$ s.t.

$$d_\phi(Ty, T^2y) \le l[d_\phi(y, Tx) + d_\phi(x, Ty)] \quad \text{for each } y \in O(x_0) \tag{3}$$

where $l \in [0, 1)$ be s.t. for $x_0 \in X$, $\lim_{n,m\to\infty} \phi(x_n, x_m) < \frac{1}{l}$.

here $x_n = T^n x_0$, n = 1, 2,.....Then $T^n x_0 \to \eta \in X$ (as $n \to \infty$).

Furthermore η is a fixed point of T iff G(x) = d(x, Tx) is T-orbitally lower semi continuous at η.

Proof: For $x_0 \in X$ we define the iterative sequence $\{x_n\}$ by:

$$x_0,\ Tx_0 = x_1,\ x_2 = Tx_1 = T(Tx_0) = T^2(x_0).......,\ x_n = T^n x_0.....$$

Now for $y = Tx_0$ by successively applying inequality (3) we obtain:

$$\begin{aligned} d_\phi(T^n x_0, T^{n+1} x_0) = d_\phi(x_n\, x_{n+1}) &\le l[d_\phi(x_{n-1}, Tx_{n-1}) + d_\phi(x_n, Tx_{n-1})] \\ &\le l[d_\phi(x_{n-1}, x_n) + d_\phi(x_n, x_n)] \\ &\le l\, d_\phi(x_{n-1}, x_n) \\ &\le l^n d_\phi(x_0, x_1) \end{aligned} \tag{4}$$

Following the same procedure as in the proof of Theorem (1) we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete then $x_n = T^n x_0 \to \eta \in X$. Assume that G is orbitally lower semi continuous at $\eta \in X$, then

$$d_\phi(\eta, T\eta) \le \liminf_{n\to\infty} d_\phi(T^n x_0, T^{n+1} x_0) \tag{5}$$

$$\le \liminf_{n\to\infty} l^n d_\phi(x_0, x_1) = 0 \tag{6}$$

Conversely, let $\eta = T\eta$ and $x_n \in 0(x)$ with $x_n \to \eta$. Then:

$$G(\eta) = d(\eta, T\eta) = 0 \le \liminf_{n\to\infty} G(x_n) = d(T^n x_0, T^{n+1} x_0) \tag{7}$$

Example 2: Let X= [0, ∞). Define $d_\phi(x, y) : X \times X \to R^+$ and $\phi : X \times X \to [1, \infty)$ as :

$$d_\phi(x, y) = (x - y)^2,\ \phi(x, y) = x + y + 3$$

Then d_ϕ is a complete new b-metric on X. Define T: X → X by $Tx = \frac{x}{3}$. We have:

$$d_\phi(Tx, Ty) = \left(\frac{x}{3} - \frac{y}{3}\right)^2 \le \frac{1}{8}(x - y)^2 = l[d_\phi(x, Tx) + d_\phi(y, Tx)]$$

Note that for each $x \in X$, $T^n x = \frac{x}{3^n}$ thus we obtain :

$$\lim_{m,n\to\infty} \phi(T^m x, T^n x) = \lim_{m,n\to\infty} \left(\frac{x}{3^m} + \frac{x}{3^n} + 3\right) < 8$$

Therefore, all conditions of theorem 2 are satisfied hence T has unique fixed point.

Example 3 : Let $X= [0, \frac{1}{4})$. Define $d_\phi(x, y) : X \times X \to R^+$ and $\phi : X \times X \to [1, \infty)$ as :

$$d_\phi(x, y) = (x - y)^2,\ \phi(x, y) = x + y + 2$$

Then d_ϕ is a complete new b-metric on X. Define $T: X \to X$ by $Tx = x^2$. We have:

$$d_\phi(Tx, Ty) \le \frac{1}{4}\ d_\phi(x, y)$$

Note that for each $x \in X$, $T^n x = x^{2n}$. Thus we obtain :

$$\lim_{m,n\to\infty} \phi(T^m x, T^n x) < 4$$

Therefore, all conditions of theorem 2 are satisfied hence T has unique fixed point.

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