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A Novel Approach to G-metric Spaces by Using Ternary Relations

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Abstract

Recently, a novel variant of Banach contraction principle was presented by Alam and Imdad on a complete metric space endowed with the binary relation. In this paper, we prove some common fixed point theorems in G-metric spaces endowed with the ternary relations. Several fixed point theorems in G-metric spaces can be weakened to the extent of an arbitrary ternary relation.

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1. INTRODUCTION

The standard Banach contraction principle [2] continues to be very important and helpful tool in theory as well as applications inside and outside Mathematics, which guarantees the existence and uniqueness of the fixed points of contraction self-mappings defined on complete metric spaces. Moreover a constructive method to calculate the fixed point of the underlying mapping. In the current past, many authors extended Banach theorem by applying various general contractive mappings on different types of spaces.

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In this paper, we extended the standard Banach contraction principle to a complete G-metric space endowed with ternary relation. But the contraction condition is relatively weaker than the usual contraction as it is required to work only on those elements which are related under the fundamental relation rather than the whole space. Particularly, under the universal relation, our result reduces to Banach contraction principle.

2. PRELIMINARIES

In this section, we introduce notations, definitions and some preliminary facts which are used throughout this paper.
Metric spaces are sets in which there is a defined a notion of '*distance between pair of points*'. The concept of metric spaces was formulated in 1906 by M.Frechet [5] ,though the definition presently in use given by the German mathematician, Felix Hausdorff.

Definition 2.1. Let M be a non empty arbitrary set and d be a real function called metric from $M \times M$ into R^+ such that for all $l, m, n \in M$ we have

1. $d(l, m) \geq 0$,
2. $d(l, m) = 0 \Leftrightarrow l = m$,
3. $d(l, m) = d(m, l)$ and
4. $d(l, n) \leq d(l, m) + d(m, n)$,

Here (M, d) is called a metric in R and (R, d) is a metric space.

Mustafa and Sims introduced a new class of generalized metric space called G-metric space in 2005 which is a generalization of metric sapce (M, d).

Definition 2.2. [8] Let M be a non empty set, and $G : M \times M \times M \to R^+$ be a function satisfying the following properties:

1. $G(l, m, n) = 0$ if $l = m = n$,
2. $0 < G(l, l, m)$, for all $l, m \in M$, with $l \neq m$,
3. $G(l, l, m) \leqslant G(l, m, n)$, for all $l, m, n \in M$, with $n \neq m$,

4. $G(l,m,n) = G(l,n,m) = G(m,n,l)$ = ...(symmetry in all three variables),

5. $G(l,m,n) \leq G(l,a,a)+G(a,m,n)$, for all $l,m,n,a \in M$, (rectangular inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on M, and the pair (M,G) is called a *G-metric space*.

Definition 2.3. [1] A Relation R among sets P and Q is a subset of the Cartesian product of the sets P and $Q, R \subseteq P \times Q$.

Definition 2.4. [1] Let M be a nonempty set. A subset R of M^2 is called a binary relation on M. Notice that for each pair $l,m \in M$, one of the following conditions holds:

$(l,m) \in R$; which amounts to saying that "l is R-related to m" or "l relates to m under R". Sometimes, we write lRm instead of $(l,m) \in R$;
$(l,m) \notin R$; which means that "l is not R-related to m" or "l does not relate to m under R". Let R be a binary relation defined on a nonempty set M and $l,m \in M$. We say that l and m are R -comparative if either $(l,m) \in R$ or $(m,l) \in R$. We denote it by $[l,m] \in R$.

Proposition 2.5. *If (M,G) is a G-metric space, t is a ternary relation on M, Y is a self-mapping on M, Y is a self-mapping on M and $\alpha \in [0,1)$, then the following contractivity conditions are equivalent:*

(I) $G(Yl,Ym,Yn) \leq \alpha G(l,m,n) \forall,\ l,m,n \in M$ *with* $(l,m,n) \in t$.
(II) $G(Yl,Ym,Yn) \leq \alpha G(l,m,n) \forall,\ l,m,n \in M$ *with* $[l,m,n] \in t$.

Proof. The implication (II) $\Longrightarrow$ (I) is trivial. Conversely suppose that (I) holds. Take $l,m,n \in M$ and $[l,m,n] \in t$. If $(l,m,n) \in t$, then directly follows (I). Otherwise, if $(m,n,l) \in t$ using symmetry of G and (I), we obtain
$G(Yl,Ym,Yn) = G(Ym,Yn,Yl) \leq \alpha G(m,n,l) = \alpha G(l,m,n)$.
This shows that (I) $\Longrightarrow$ (II). □

Definition 2.6. [1] Let M be a nonempty set and Y a self-mapping on M. A ternary relation t defined on M is called Y-closed if for any $l,m,n \in M$,

$$(l,m,n) \in t \Rightarrow (Yl,Ym,Yn) \in t.$$

Proposition 2.7. *Let M, Y and t be the same as in Proposition 2.5. If t is Y-closed, then t^s is also Y-closed.*

Definition 2.8. [4, 6, 7, 9, 11] A binary relation R defined on a nonempty set M is called

1. *reflexive* if $(l, l) \in R$ for all $l \in M$,
2. *irreflexive* if $(l, l) \notin R$ for all $l \in M$,
3. *symmetric* if $(l, m) \in R$ implies $(m, l) \in R$,
4. *antisymmetric* if $(l, m) \in R$ implies $(m, l) \notin R$,
5. *transitive* if $(l, m) \in R$ and $(m, n) \in R$ implies $(l, n) \in R$,
6. *complete, connected or dichotomous* if $[l, n] \in R$ for all $l, m \in M$,
7. *weakly complete, weakly connected or trichotomous* if $[l, m] \in R$ or $l = m$ for all $l, m \in M$.
8. *strict order or sharp order* if R is irreflexive and transitive,
9. *near-order* if R is antisymmetric and transitive,
10. *pseudo-order* if R is reflexive and antisymmetric,
11. *quasi-order or preorder* if R is reflexive and transitive,
12. *partial order* if R is reflexive, antisymmetric and transitive,
13. *simple order* if R if weakly complete strict order,
14. *weak order* if R is complete preorder,
15. *total order, linear order or chain* if R is complete partial order,
16. *tolerance* if R is reflexive and symmetric,
17. *equivalence* if R is reflexive, symmetric and transitive.

Definition 2.9. [10] Let M be a set and $t \subseteq M \times M \times M$. Then t is said to be a *ternary relation* on M. A ternary relation t defined on a non empty set G is called

1. *reflexive* if $(l, l, l) \in t$ for all $l \in M$,
2. *symmetric* if and only if $(l, m, n) \in t \Longrightarrow (m, n, l) \in t$ for any $l, m, n \in G$,
3. *transitive* if and only if $(l, m, n) \in t, (m, n, u) \in t$ imply $(l, n, u) \in t$ for any l, m, n, u in G,
4. *asymmetric* if and only if $(l, n, m) \in t$ implies $(m, n, l) \notin t$ for any l, m, n in G,
5. *irreflexive* if $(l, m, n) \in t \Longrightarrow (m, l, p) \notin t$ for any $p \in G$,
6. *irreversible* if $(l, m, n) \in t \Longrightarrow (m, l, p) \notin t$ for any $p \in G$,
7. *feebly regular* if $(l, m, p) \in t \Longrightarrow (m, n, q) \in t \Longrightarrow (l, m, n) \in t$ for any $p \in G$,
8. *regular* if t is feebly regular and $(l, m, p) \in t, (l, n, q) \in t \Longrightarrow (l, m, n) \in t$,
9. *feebly translative* if $(l, m, n) \in t, (m, p, q) \in t \Longrightarrow \exists r \in G : (l, p, r) \in t$,
10. *translative* if t is feebly translative and $(l, m, n) \in t \Longrightarrow \exists r \in t : (l, n, r) \in t$ or $(m, n, r) \in t$,
11. *cyclic* if and only if $(l, m, n) \in t \Longrightarrow (m, n, l) \in t$ for any l, m, n in G.
12. *complete* if $[l, m, n] \in t \forall l, m, n \in M$,
13. *weakly complete* if $[l, m, n] \in t$ or $l = m = n \forall l, m, n \in t$

Proposition 2.10. *For a ternary relation* t *defined on a nonempty set* M*,*

$$(l, m, n) \in t^s \Leftrightarrow [l, m, n] \in t$$

Proof. This can be solved as

$$(l, m, n) \in t^s \Leftrightarrow (l, m, n) \in t \bigcup t^{-1}$$

$$\Leftrightarrow (l, m, n) \in t \text{ or } (l, m, n) \in t^{-1}$$

$$\Leftrightarrow (l, m, n) \in t \text{ or } (n, m, l), (l, n, m), (m, n, l), (m, l, n), (n, l, m) \in t$$

$$\Leftrightarrow [l, m, n] \in t$$

□

3. FIXED POINT THEOREM

Now, we state and prove our main result, which is as follows.

Theorem 3.1. *Let (M, G) be a complete G-metric space, t a ternary relation on M and Y a self-mapping on M. Suppose that the following conditions hold:*
(i) $M(Y; t)$ is nonempty,
(ii) t is Y - closed,
(iii) either Y is continuous or t is G-self-closed,
(iv) there exists $\alpha \in [0,1)$ such that $G(Yl, Ym, Yn) \leq \alpha G(l, m, n) \forall l, m, n \in M$ with $(l, m) \in t$.
Then Y has a fixed point.
Moreover, if
(v) $\Upsilon(l, m, n, t^s)$ is nonempty, for each $l, m, n \in M$,
then Y has a unique fixed point.

Proof. Let l_0 be an arbitrary element of $M(Y, t)$. Define the sequence l_u of Picard iterates, i.e., $l_u = Y^p(l_0)$ for all $p \in N_0$. As $(l_0, Yl_0, Yl_0) \in t$, using assumption (ii), we obtain

$(Yl_0, Y^2l_0, Y^2l_0), (Y^2l_0, Y^3l_0, Y^3l_0,), ..., (Y^pl_0, Y^{p+1}l_0, Y^{p+1}l_0), ... \in t$
so that

$$(l_p, l_{p+1}, l_{p+1}) \in t \forall p \in N_0 \tag{3.1}$$

Thus the sequence l_p is t-preserving. Applying the contractivity condition (iv) to (3.1), we deduce, for all $p \in N_0$, that

$$G(l_{p+1}, l_{p+2}, l_{p+2}) \leq \alpha G(l_p, l_{p+1}, l_{p+1}),$$

which by induction gives that

$$G(l_{p+1}, l_{p+2}, l_{p+2}) \leq \alpha^{p+1} G(l_0, Yl_0, Yl_0) \forall p \in N_0. \tag{3.2}$$

Using (3.2) and rectangular inequality, for all $p \in N_0, q \in N, q \geq 2$, we have

$$G(l_{p+1}, l_{p+q}, l_{p+q}) \leq G(l_{p+1}, l_{p+2}, l_{p+2}) + G(l_{p+2}, l_{p+3}, l_{p+3}) + ... + G(l_{p+q-1}, l_{p+q}, l_{p+q})$$

$$\leq (\alpha^{p+1} + \alpha^{p+2} + ... + \alpha^{p+q-1}) G(l_0, Yl_1, Yl_0)$$

$$= \alpha^p G(l_0, Yl_0, Yl_0) \textstyle\sum_{j=1}^{q-1} \alpha^j \to 0 \text{ as } p \to \infty,$$

which implies that the sequence l_p is G-Cauchy in M. As (M, G) is complete, there exists $l \in M$ such that

$$l_p \underset{\rightarrow}{G} l.$$

Now, in lieu of (iii), assume that Y is continuous, we have

$$l_{p+1} = Yl_p \underset{\rightarrow}{G} Y(l)$$

Owing to the uniqueness of limit, we obtain $Y(l) = l$, i.e., l is a fixed point of Y. Alternately, let us assume that t is G-self-closed. As l_p is an t-preserving sequence and

$$l_p \underset{\rightarrow}{G} l,$$

there exists a subsequence l_{pk} of l_p with

$$[l_{pk}, l, l] \in t \text{ for all } k \in N_0.$$

Using (iv), Proposition (2.5), $[l_{pk}, l, l] \in t$ and $l_{pk} \underset{\rightarrow}{G} l$, we obtain

$$G(l_{p_k+1}, Yl, Yl) = G(Yl_{pk}, Yl, Yl) \leq \alpha G(l_{pk}, l, l) \to 0 \text{ as } k \to \infty$$

so that $l_{p_k+1} \underset{\rightarrow}{G} Y(l)$. Again, owing to the uniqueness of limit, we obtain $Y(l) = l$ so that l is a fixed point of Y.

To prove uniqueness, take $l, m, n \in F(Y)$, i.e.,

$$Y(l) = l, Y(m) = m \tag{3.3}$$

By assumption (v), there exists a path (say $z_0, z_1, z_2, ..., z_k$) of some finite length k in t^s from l to m and n so that

$$z_0 = l, z_k = m, [z_i, z_{i+1}, z_{i+1}] \in t \text{ for each } i(0 \leq i \leq k-1).$$

As t is Y-closed, by using Proposition (2.7), we have

$$[Y^p z_i, Y^p z_{i+1}] \in t \text{ for each } i(0 \leq i \leq k-1) \text{ and for each } p \in N_0$$

Making use of (3.3), (3.4), (3.5), rectangular inequality, assumption (iv) and Proposition 2.5, we obtain

$$G(l,m,n)=G(Y^pz_0,Y^pz_k,Y^pz_k)\le\sum_{i=0}^{k-1}G(Y^pz_i,Y^pz_{i+1},Y^pz_{i+1})$$

$$\le\alpha\sum_{i=0}^{k-1}G(Y^{p-1}z_i,Y^{p-1}z_{i+1},Y^{p-1}z_{i+1})$$

$$\le\alpha^2\sum_{i=0}^{k-1}G(Y^{p-2}z_i,Y^{p-2}z_{i+1},Y^{p-2}z_{i+1})$$

$$\le$$

$$\vdots$$

$$\le\alpha^p\sum_{i=0}^{k-1}G(z_i,z_{i+1},z_{i+1})$$

$$\to 0 \text{ as } p\to\infty$$

so that $l=m$. Hence Y has a unique fixed point. □

Example 3.2. Let $M=R$ and $G=max\{|l-m|,|m-n|,|n-l|\}$, then (M,G) is a complete G-metric space. Define a ternary relation $t=\{(l,m,n)\in R^3: l-m, m-n, n-l\ge 0, l\in Q\}$ on M. Consider a mapping $Y:M\to M$ defined by

$$Y(l)=6+\tfrac{1}{3}l.$$

Clearly, t is Y-closed and Y is continuous. Now, for $l,m,n\in M$ with $(l,m,n)\in t$, we have
$G(Yl,Ym,Yn)=max\{(|6+\frac{1}{3}l|-|6+\frac{1}{3}m|),(|6+\frac{1}{3}m|-|6+\frac{1}{3}n|),(|6+\frac{1}{3}n|-|6+\frac{1}{3}l|)\}$
$=\frac{1}{3}max\{|l-m|,|m-n|,|n-l|\}$
$=\frac{1}{3}G(l,m,n)<\frac{3}{5}G(l,m,n)$
i.e., Y satisfies assumption (iv) of Theorem 3.1 for $\alpha=\frac{3}{5}$. Thus all the conditions (i)-(iv) of Theorem 3.1 are satisfied and Y has a fixed point in M. Moreover, here assumption (v) of Theorem 3.1 also holds and therefore, Y has a unique fixed point $l=9$.

Example 3.3. Consider $M=[0,2]$ equipped with the G-metric $G=max\{|l-m|,|m-n|,|n-l|\}$ so that (M,G) is a complete G-metric space. Define a ternary relation

$$t=\{(0,0,0),(0,1,1),(1,0,0),(1,1,1),(0,2,2)\}$$

on M and the mapping Y-closed but Y is not continuous. Take an t-preserving sequence $\{l_u\}$ such that

$$l_p \to^G l$$

so that $(l_p, l_{p+1}, l_{p+1}) \in t$ for all $p \in N_0$. Here one may notice that

$$(l_p, l_{p+1}, l_{p+1}) \notin \{(0, 2, 2)\}$$

so that

$$(l_p, l_{p+1}, l_{p+1}) \in \{(0, 0, 0), (0, 1, 1), (1, 0, 0), (1, 1, 1)\} \forall p \in N_0,$$

which gives rise to $l_p \subset \{0, 1\}$. As $\{0, 1\}$ is closed, we have $[l_p, l, l] \in t$. Therefore, t is G-self-closed. By a routine calculation, one can verify assumption (iv) of Theorem 3.1 with $\alpha = 1/3$. Thus all the conditions (i)-(iv) of Theorem 3.1 are satisfied and Y has a fixed point in M (namely, l=0).

Notice that in Example 3.3, the binary relation t is not one of the earlier known standard ternary relations such as reflexive, irreflexive, symmetric, asymmetric, transitive, complete and weakly complete.

Note:

By using the concept of universal relation(i.e. $t = M^3$), Theorem 3.1 reduces to the result proved by Mustafa and Sims. Clearly, under this relation, (i), (ii), (iii) and (v) hold trivially.

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