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Some Fixed Point Results in Complete b-metric spaces

Vimlesh Kushwaha[1] and D.P. Shukla[2]

[1]Ph.D. Research Scholar in Mathematics, Govt. Model Science College, Rewa (M.P.), 486001, India.

[2]Dept. of Mathematics/Computer Science, Govt. Model Science College, Rewa (M.P.), 486001, India. (Affiliated to A.P.S. University, Rewa 486003, India)

Abstract

The purpose of this paper is to obtain unique fixed point theorems for in b-metric space which satisfies the earlier result of the Czerwik, s[6] Kir, Mehmet, Kiziltune, Hukmi [8] and several examples are discussed. In this paper we show that different contraction type of mapping exist in b-metric space.

Keywords: Unique fixed point, Complete b-metric spaces, Contractive mapping

2000 Mathematical Subject Classification: 47 H 10, 54 H 25

1. Introduction:

Fixed point theory is very good combination of functional analysis, real analysis, topology and geometry. Theory of fixed point is one of the most powerful tools in mathematics existence and properties of fixed point is known as fixed point theorems. Fixed point theory has been applied in such field as mathematics, mathematical sciences, physics, mathematical physics, chemistry, biological sciences economics, game theory, engineering, medical sciences, computer sciences.

The concept of b-metric space by Bakhtin [2] in 1989, Czerwik [6] extended the results of b-metric spaces in 1993. Using this idea by many mathematician presented generalization of the renowned Banach of fixed point theorems in the b-metric space.

Bota [4], Boriceanu [5], Czerwik, s[6], Mehmet Kir [7], Pacurar [8] and Swati et al [1] extended the fixed point theorems in b-metric space. Czerwik, [6] first presented first presented a generalization of Banach of fixed point theorem in b-metric spaces.

2. Preliminaries and Definitions:

Definition 2.1

Let X be a non-empty set and A function d: X x X → R is called a b-metric on X if and only if the following properties holds:

$[m_1]$: d(x,y) = 0 if and only if x = y ;

$[m_2]$: d(x, y) = d(y, x) for all x, y ∈ X ;

$[m_3]$: d(x,y) ≤ d(x, z) + d(z, y) for all x, y, z ∈ X.

then (X,d) is called a metric space on X.

Example 2.1

Let En (or R^n) = {x : $(x_1, x_2, \ldots, x_n)$, $x_i \in R$, R the set of real numbers} and Let d be defined as follows: if y = $(y_1, y_2,\ldots,y_n)$ then

$$d(x, y) = \left(\sum_i^n |x_i - y_i|^p\right)^{\frac{1}{p}} = dp(x, y)$$

where p is a fixed number in [1, ∞).

Definition 2.2

Let X be a non-empty set and s ≥ 1 be a given real number. A function d:X × X → R_+ is called a b-metric provided that for all x, y, z ∈ X

(1) d(x,y) = 0 if and only if x = y ,

(2) d(x, y) = d(y, x),

(3) d(x,y) ≤ s[d(x, z) + d(z, y)].

A pair (X,d) is called a b-metric space it is clear that definition of b-metric space is an extension of usual metric space.

Example 2.2 [5]

Let X = {0, 1, 2) and d(2, 0) = d(0, 2) = m ≥ 2, d(0,1) = d(1, 2) = d(1, 0) = d(2, 1) = 1

and d(0,0) = d(1,1) = d(2,2) = 0.

then $\quad d(x, y) \leq \frac{m}{2}[(d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

if $m > 2$ then the triangle inequality does not hold.

Example 2.3

$X = \{1, 2, 3\}$ and $d(1, 3) = d(3, 1) = m \geq 2$

$d(1, 2) = d(2, 3) = d(2, 1) = d(3, 2) = 1$

and $\quad d(1, 1) = d(2, 2) = d(3, 3) = 0$

Then $\quad d(x, y) \leq \frac{m}{2}[(d(x, z) + d(z, y)] \ \forall\ x, y, z \in X$.

if $m > 2$ then the triangle inequality does not hold.

Example 2.4 [5]

The space $L_p[0, 1)$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0,1]$ such that $\int_0^1 |x(t)|^p \, dt < \infty$, is a b-metric space if we take

$$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p \, dt\right)^{\frac{1}{p}}, \text{ for each } x, y \in L_p[0, 1].$$

Definition 2.3

Let (X, d) be a b-metric space then a sequence $\{x_n\}$ in X is called convergent sequence if and only if there exists $x \in X$ such that for all there exists $n(\in) \in N$ such that for all $n \geq n(\in)$ we have $d(x_n, x) < \in$. In this case we write $\lim_{n\to\infty} x_n = x$.

Definition 2.4

Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

Definition 2.5

The b-metric space is complete if every Cauchy sequence convergent.

3 MAIN RESULTS

Theorem 3. 1.

Let (X, d) be a complete b-metric space. Let T be a self mapping on X such that $d(Tx,Ty) \leq \alpha\{d(x,Tx) + d(y,Ty)\} + \beta\{d(x, Ty) + d(y, Tx)\} + \gamma \{d(y, Ty) + d(y, Tx)\} + \delta\{d(x, Ty) + d(y, Tx)\}$ (1)

where $\alpha, \beta, \gamma, \delta > 0$ such that

$\beta + \gamma + 2s\delta < 1, \forall$ x, y $\in$X and

$s \geq 1$ then T has a unique fixed point.

Proof:

Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in x defined by the recursion.

$x_n = Tx_{n-1} = T^n x_0 \ \ \forall \ n \in N$

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \alpha\{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\} + \beta\{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_{n-1})\} \\
&\quad + \gamma\{d(x_n, Tx_n) + d(x_n, Tx_{n-1})\} + \delta\{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})\} \\
&\quad \alpha d(x_{n-1}, Tx_{n-1}) + \alpha d(x_n, Tx_n) + \beta d(x_{n-1}, Tx_{n-1}) + \beta d(x_n, Tx_{n-1}) \\
&\quad + \gamma d(x_n, Tx_n) + \gamma d(x_n, Tx_{n-1}) + \delta d(x_{n-1}, Tx_n) + \delta d(x_n, Tx_{n-1}) \\
&\quad \alpha d(x_{n-1}, x_n) + \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \beta d(x_n, x_n) \\
&\quad + \gamma d(x_n, x_{n+1}) + \gamma d(x_n, x_n) + \delta d(x_{n-1}, x_{n-1}) + \delta d(x_n, x_n) \\
&\quad \alpha d(x_{n-1}, x_n) + \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) + \delta d(x_{n-1}, x_{n-1})
\end{aligned}
$$

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) + \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) + \delta d(x_{n-1}, x_{n+1})$$

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \alpha d(x_{n-1}, x_n) + \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) \\
&\quad + \alpha\delta s\{d(x_{n-1}, x_n) + d(x_n, x_{n+1})\}
\end{aligned}
$$

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq \alpha d(x_{n-1}, x_n) + \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) \\
&\quad + \delta s d(x_{n-1}, x_n) + \delta s d(x_n, x_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
d(x_n, x_{n-1}) &\leq \alpha d(x_{n-1}, x_n) + \alpha d(x_n, x_{n+1}) \\
&\quad + \beta d(x_{n-1}, x_n) + \gamma d(x_n, x_{n+1}) \\
&\quad + \delta s d(x_{n-1}, x_n) + s\delta d(x_n, x_{n+1})
\end{aligned}
$$

$d(x_n, x_{n+1}) \quad - \alpha d(x_n, x_{n+1}) - \gamma d(x_n, x_{n+1}) - \delta s d(x_n, x_{n+1})$

$\leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \delta s d(x_{n-1}, x_n)$

$(1 - \alpha - \gamma - \delta s)\, d(x_n, x_{n+1}) \leq (\alpha + \beta + \delta s)\, d(x_{n-1}, x_n)$

$(1 - \alpha - \gamma - \delta s)\, d(x_n, x_{n+1}) \leq (\alpha + \beta + \delta s)\, d(x_{n-1}, x_n)$

$d(x_n, x_{n+1}) \leq \left(\frac{\alpha + \beta + \delta s}{1 - \alpha - \gamma - + \delta s}\right) d(x_{n-1}, x_n)$

$d(x_n, x_{n+1}) \leq kd\,(x_{n-1}, x_n)$

where $k = \frac{1 - \beta + \delta s}{1 - \alpha - \gamma - \delta s} < 1$

$\Rightarrow d(x_n, x_{n+1}) \leq kd\,(x_{n-1}, x_n)$

$\Rightarrow d(x_n, x_{n+1}) \leq k^2 d\,(x_{n-2}, x_{n-1})$

continuing this process, we have

$\Rightarrow d(x_n, x_{n+1}) \leq k^n d\,(x_0, x_1)$ ———————————————— (A)

Thus T is a contractive mapping

Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X

Let m, n $\in$ N, m > n

$d(x_n, x_m) \leq s\{d(x_n, x_{n+1}) + d(x_{n+1}, x_m)\}$

$d(x_n, x_m) \leq sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m)$

$\leq sd(x_n, x_{n+1}) + s^2\{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)\}$

$\leq sd(x_n, x_{n+1}) + s^2\{d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m)$

$\leq sk^n d(x_0, x_1) + s^2 k^{n+1} d(x_0, x_1) + s^3\, k^{n+3}\, d(x_0, x_1) + \ldots.$

$\leq sk^n d(x_0, x_1)\, [1 + sk + (sk)^2 + (sk)^3 + \ldots.]$

$\leq \frac{sk^n}{1-sk} d(x_0, x_1)$

$d(x_n, x_m) \leq \frac{sk^n}{1-sk} d(x_0, x_1)$

Then $\lim_{n\to\infty} d(x_n, x_m) = 0$ as n, m $\to \infty$

Since k < 1

$\lim_{n\to\infty} \frac{sk^n}{1-sk} d(x_0, x_1) = 0$ as n, m $\to \infty$ ———————————————— (B)

Hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X

Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence

$\{x_n\}$ converges to $x^* \in X$

Now, we show that x* is the fixed point of T

$d(x^*, Tx^*) \leq s\{d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)\}$

$d(x^*, Tx^*) \leq sd(x^*, x_{n+1}) + sd(x_{n+1}, Tx^*)$

$d(x^*, Tx^*) \leq sd(x^*, x_{n+1}) + sd(Tx_n, Tx^*)$

$d(x^*, Tx^*) \leq sd(x^*, x_{n+1}) + sd(Tx_n, Tx^*)$

$d(x^*, Tx^*) \leq sd(x^*, x_{n+1}) + s\alpha d(x_n, Tx_n) + s\alpha d(x^*, Tx^*)$

$+ s\beta d(x_n, Tx_n) + s\beta d(x^*, Tx_n) + s\gamma d(x^*, Tx^*) + s\gamma d(x^*, Tx_n)$

$+ s\delta d(x_n, Tx^*) + s\delta d(x^*, Tx_n)$

$(x^*, Tx^*) \leq sd(x^*, x_{n+1}) + s\alpha d(x_n, x_{n+1}) + s\alpha d(x^*, Tx^*)$

$+ s\beta d(x_n, x_{n+1}) + s\beta d(x^*, x_{n+1}) + s\gamma d(x^*, Tx^*)$

$+ s\gamma d(x^*, x_{n+1}) + s\delta d(x_n, Tx^*) + s\delta d(x^*, x_{n+1})$

$d(x^*, Tx^*) \leq sd(x^*, x_{n+1}) + s^2\alpha d(x_n, x^*) + s^2\alpha d(x^*, x_{n+1}) + s\alpha d(x^*, Tx^*)$

$+ s^2\beta d(x_n, x^*) + s^2\beta d(x^*, x_{n+1}) + s\beta d(x^*, x_{n+1}) + s\gamma d(x^*, Tx^*)$

$+ s\gamma d(x^*, x_{n+1}) + s^2\delta d(x_n, x^*) + s^2\delta d(x^*, Tx^*) + s\delta(x^*, x_{n+1})$

$d(x^*, Tx^*) - s\alpha d(x^*, Tx^*) - s\gamma d(x^*, Tx^*) - s^2\delta d(x^*, Tx^*)$

$\leq sd(x^*, x_{n+1}) + s^2\alpha d(x^*, x_{n+1}) + s^2\beta d(x^*, x_{n+1})$

$+ s\beta d(x^*, x_{n+1}) + s\gamma d(x^*, x_{n+1}) + s\delta(x^*, x_{n+1}) + s^2\alpha d(x_n, x^*)$

$+ s^2\beta d(x_n, x^*) + s^2\delta d(x_n, x^*)$

$(1 - s\alpha - s\gamma - s^2\delta)\, d(x^*, Tx^*)$

$\leq (s + s^2\alpha + s^2\beta + s\beta + s\gamma + s\delta)\, d(x^*, x_{n+1})$

$+ (s^2\alpha + s^2\beta + s^2\delta) d(x_n, x^*)$

$$d(x^*, Tx^*) \leq \frac{s+s^2\alpha+s^2\beta+s\beta+s\gamma+s\delta}{(1-s\alpha-s\gamma-s^2\delta)} d(x^*, x_{n+1}) + \frac{s^2\alpha + s^2\beta + s^2\delta}{1-s\alpha-s\gamma-s^2\delta} d(x_n, x^*)$$

Taking $\lim_{n\to\infty}$, we get

$\lim_{n\to\infty} d(x^*, Tx^*) = 0$

$\Rightarrow \quad Tx^* = x^*$

$\Rightarrow$ x^* is the fixed point of T

Uniqueness of fixed point

We have to show that x^* is unique fixed point of T

Let us suppose that x' is another fixed point of T Then Tx' = x'

and $(x^*, x') = d(Tx^*, Tx')$

$$(x^*, x') = (Tx^*, Tx')$$

$$\leq \alpha d(x^*, Tx^*) + \alpha d(x', Tx') + \beta d(x^*, Tx^*)$$

$$+ \beta d(x', Tx^*) + \gamma d(x', Tx^1) + \gamma d(x', Tx^*)$$

$$+ \delta d(x^*, Tx') + \delta d(x', Tx^*)$$

$$d(x^*, x') \leq \alpha d(x^*, x^*) + \alpha d(x', x') + \beta d(x^*, x^*)$$

$$+ \beta d(x', x^*) + \gamma d(x', x') + \gamma d(x', x^*)$$

$$+ \delta d(x^*, x') + \delta d(x', x^*)$$

$$\leq \beta d(x', x^*) + \gamma d(x', x^*) + \delta d(x^*, x') + \delta d(x', x^*)$$

$$d(x^*, x') \leq (\beta + \gamma + 2s\delta)\, d(x^*, x')$$

which is contradiction. Therefore $x^* = x'$.

Theorem 3.2

Let (X, d) be a complete b-metric space. Let T be a self mapping on X such that $d(Tx.Ty) \leq \alpha\{d(x. Tx) + d(y, Ty)\} + \beta\{d(x, Ty) + d(y, Tx)\} + \gamma\{d(x, y) + d(Tx, Ty)\}$

where $\alpha, \beta, \gamma > 0$ such that

$\beta + 2s\gamma < 1, \forall\, x, y \in X$ and $s \geq 1$ then T has a unique fixed point of T.

Proof: Same as Theorem 3.1

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