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Analysis of $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ Retrial Queueing System with Priority Services, Working Breakdown, Non-Persistent Customers, Modified Bernoulli Vacation, Emergency Vacation and Repair

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Abstract

This paper considers $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general retrial queueing system with priority services. Two types of customers from different classes arrive at the system in different independent compound Poisson processes. The server follows the pre-emptive priority rule subject to working breakdown, non-persistent customers and modified Bernoulli vacation, emergency vacation with general (arbitrary) vacation periods. After completing the service, if there is no high priority customers present in the system, the server may go for a vacation or remains idle in the system. During service period of high priority customers the server may take an emergency vacation. The priority customers who find the server busy are queued in the system. If a low-priority customer finds the server busy, they are routed to orbit that attempts to get the service. The system may become defective at any point of time when it is in operation. However, when the system is defective, instead of stopping service completely, the service continues only to the high priority customers at a slower rate. The low priority customers are considered as a non-persistent customers. The non-persistent customer may gives up if the server is busy upon arrival and may give up after staying in the orbit. Using the supplementary variable technique, we give the joint distribution of the server state and the number of customers in the system. Finally, some performance measures and numerical examples are presented.

Keywords: Priority Queueing systems, Retrial, Modified Bernoulli Vacation, Working breakdown, Emergency vacation, Non-persistent customer.

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1. INTRODUCTION:

Retrial queues are studied by numerous researchers, where the customers are joined the orbit if the server is busy and retry for their service. The perfectly reliable servers are virtually nonexistent and overlooked by many researchers. In fact, the servers may well be subject to lengthy and unpredictable breakdowns while serving a customer. For example, in manufacturing systems the machine may breakdown. This results in a period of unavailable time until the machines(servers) are repaired. Such a system with repairable server has been studied as a queueing model and reliability model by many authors. Studying queueing models with server breakdown, it is generally assumed that the server stops service when the server breaks down. However, in most of the models considered so far of queueing systems with server breakdowns, the underlying assumption has been that a server breakdown disrupts the service completely in the system.

In computer, the presence of a virus in the system may slow down the performance of the computer system. The computer system may still be able to perform various tasks but at a considerably slower rate. Here the failure of the computer system does not stop the work completely. Motivated by this factor, we have therefore considered in this paper a class of queueing systems with the working breakdown policy with various parameters.

Kalidass et.al [1] introduced the working breakdown policy, in which the server works at a lower service rate rather than stopping service during the breakdown period. In the working breakdown, service can decrease complaints from the customers who should wait for the server to be repaired. Therefore, the working breakdown service is a more reasonable repair policy for unreliable queueing systems. Tao Li et.al [2] and Zaiming Liu et.al [3] describes more about working breakdowns. Kim B.K. et.al [4] studied the M/G/1 queueing system with disasters and working breakdowns. Cheng-Dar Liou [5] applied the matrix-geometric method to examine an infinite capacity Markovian queue with an unreliable server subject to working breakdowns and impatient customers. Recenty, Yang et.al [6] studied the analysis of a finite capacity system with working breakdown and retention of impatient customers.

If the server is busy at the time of arrival of a customer, then with probability π_1 it leaves the system without service and with probability $1-\pi_1$ it enters into an orbit. Similarly, if the server is occupied at the time of arrival of an orbital customer, with probability $1-\pi_2$ it leaves the system without service and with probability π_2 it goes back to the orbit. This type of customers are known as non-persistent customers. Ebenesar A.B et.al [7] deals with non-persistent customers with two types of heterogeneous service. Phung-Duc T [8] describes about Multi server retrial queue with two types of nonpersistent customers. Ayyappan et. al [9] studied about non-persistent customers with orbital search.

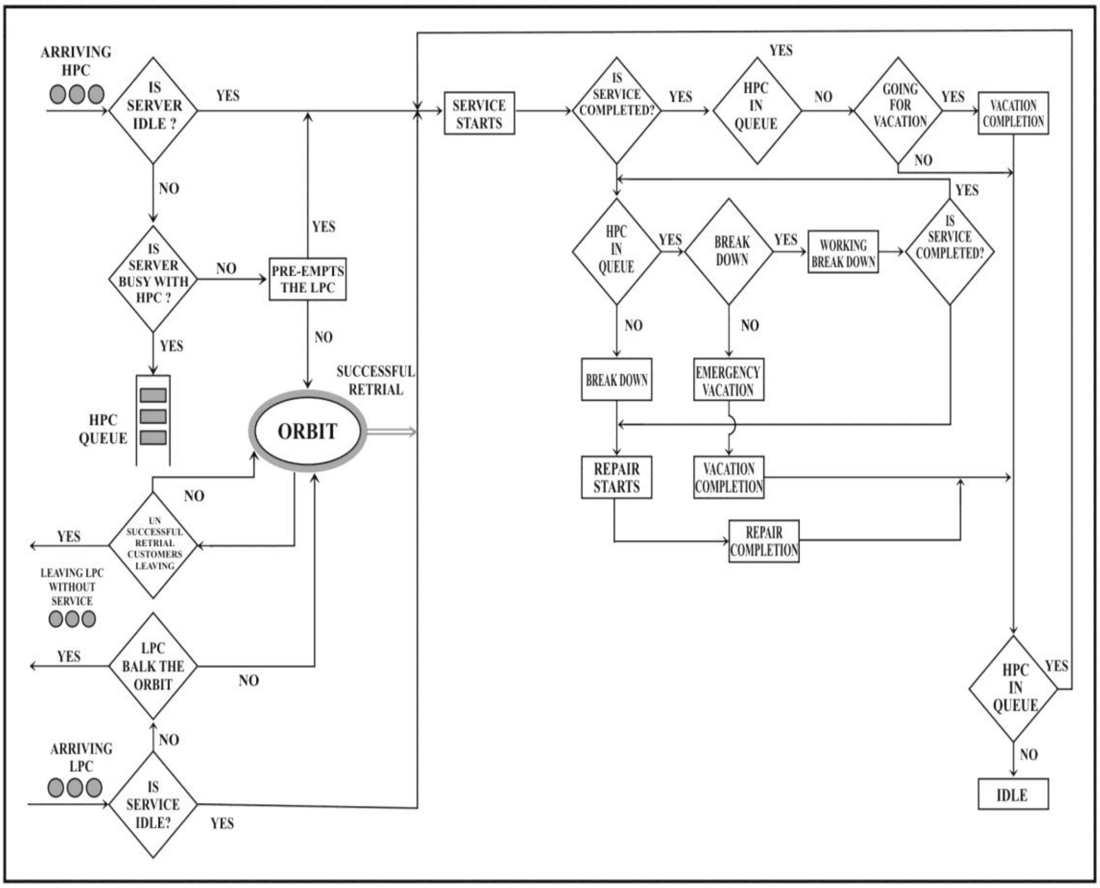
This paper considers $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general retrial queueing system with priority services. Two different sorts of customers arrive at the system in two independent compound Poisson processes. Under the pre-emptive priority rule, the server providing general service to the arriving customers. We propose a retrial queueing model with the additional characteristics of server's working breakdown, repair, non-persistent customers and emergency vacation. Models with this behaviour of a working breakdown can be used to analyse computer networks with virus affection and breakdowns due to a reset order. Low priority customers are considered as a non-persistent customer. The arriving low-priority customer on finding the server busy cannot be queued they leave the service area and join the orbit as a retrial customer. After completing service, if there is no high priority customer present in the system, the server may go for a vacation. During the service period of high priority customer the server may go for an emergency vacation. After completing vacation, service completion and repair if there is no high priority customer present in the system then the server becomes idle.

The summary of the paper is as follows. Section 1 is an introduction to priority retrial queueing discipline and comprises literature review. Section 2 deals with model description, notations used, mathematical formulation and governing equations of the model. Section 3 elucidates the steady state solutions of the system. Section 4 demonstrates the performance measures of the model. In Section 5 the numerical results and graphs are computed following which the conclusion is given.

2. MODEL DESCRIPTION

We consider an unreliable single server retrial queueing model with two types of customers namely, high priority and low-priority customers. The basic operation of the model can be described as:

Arrival and retrial process: Two class of customers arrive at the system in two independent compound Poisson processes with arrival rate λ_1 and λ_2 respectively. Let $\lambda_1 c_{1,i}\, dt$ and $\lambda_2 c_{2,i}\, dt\; (i = 1,2,3,...)$ be the first order probability that a batch of 'i' customers arrives at the system during a short interval of time $(t, t+dt)$, where $0 \le c_{1,i} \le 1, \sum_{i=1}^{\infty} c_{1,i} = 1,\; 0 \le c_{2,i} \le 1, \sum_{i=1}^{\infty} c_{2,i} = 1$ and $\lambda_1 > 0, \lambda_2 > 0$. The high priority customer who find the server busy is queued and then is served. The arriving low-priority customer on finding the server busy, are routed to a retrial queue and they follow classical retrial policy that attempts to get the service. The retrial time is generally distributed with distribution function $I(s)$ and the density function $i(s)$.



Let $\eta(x)dx$ be the conditional probability of completion of retrial during the interval $(x, x+dx]$ given that the elapsed retrial time is x. The low priority customers are treated as a non-persistent customers, i.e., if the server is busy at the time of arrival of primary customer, then with probability $1-\pi_1$ it leaves the system without service and with probability π_1 it enters into an orbit. Similarly, if the server is occupied at the time of arrival of an orbital customer, with probability $1-\pi_2$ it leaves the system without service and with probability π_2 it goes back to the orbit.

Service process: If a high priority customer arrives in batch and finds a low priority customer in service, they pre-empt the low priority customer who is undergoing service; thus the service of the pre-empted low priority customer begins only after the completion of service of all high priority customers present in the system. The service times for the high priority and low priority customers are generally(arbitrary) distributed with distribution functions $B_i(s)$ and the density functions $b_i(s)$, i = 1,2 respectively. Let $\mu_i(x)dx$ be the conditional probability of completion of the high priority and low priority customers service during the interval $(x, x+dx]$, given that the elapsed service time is x.

Modified Bernoulli Vacation: After completing all high priority customers in the system the sever may go for a vacation with probability θ or remains idle in the system with probability $1-\theta$. Vacation time is generally distributed with distribution function $V(s)$ and the density function $v(s)$. Let $\beta_1(x)dx$ be the conditional probability of completion of vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x.

Emergency Vacation: During service period of high priority customer, if there is any emergency, the server may go for a vacation with probability β. The emergency vacation times follows an exponential distribution with rate β_2.

Working Breakdown state: The server may become inactive during busy period. At the time of breakdown the high priority customer who is in service will get service continuously by slower service rate μ_3 and it follows an exponential distribution. But, the low priority customer who is in service will send to the orbit.

Repair Process: The repair process for broken down server starts immediately so as to regain its functionality with exponential repair rate γ. In the repair process the server is in working breakdown, if any high priority customers present in the system.

Idle State: After completing the service, vacation or repair if there is any high priority customer waiting in the system the server starts doing the service. Otherwise, the server is simply present in the system for the customers to arrive.

2.1 Definitions and notations

We define the following notations: The state of the server at time t is given by,

$$Y(t)=\begin{cases} 0, & \textit{if the server is in idle state;} \\ 1, & \textit{if the server is busy with high priority customer;} \\ 2, & \textit{if the server is busy with low priority customer;} \\ 3, & \textit{if the server is in modified Bernoulli vacation;} \\ 4, & \textit{if the server is in Emergency vacation;} \\ 5, & \textit{if the server is in working breakown state;} \\ 6, & \textit{if the server is in Repair process;} \end{cases}$$

2.2 Queue size distribution:

Since service time, vacation time and retrial time are not exponential, the process $\{Y(t), N_1(t), N_2(t)\}$ is non Markovian. In such case we introduce supplementary variables corresponding to elapsed times to make it Markovian (Cox[10]). Joint distributions of the server state and queue size are defined as,

$$\bar{I}_{0,n}(x,s,t)dx = Pr\{Y(t)=0, x < I^0(t) \le x+dx, N_1(t)=0, N_2(t)=n\}, n \ge 1$$

$$\overline{P}^{(1)}_{m,n}(x,s,t)dx = Pr\{Y(t)=1, x < B_1^0(t) \le x+dx, N_1(t)=m, N_2(t)=n\}, m \ge 0, n \ge 0$$

$$\overline{P}^{(2)}_{0,n}(x,s,t)dx = Pr\{Y(t)=2, x < B_2^0(t) \le x+dx, N_1(t)=0, N_2(t)=n\}, n \ge 0,$$

$$\overline{V}_{0,n}(x,s,t)dx = Pr\{Y(t)=3, x < V^0(t) \le x+dx, N_1(t)=0, N_2(t)=n\}, n \ge 0$$

$$\overline{E}_{m,n}(s,t) = Pr\{Y(t)=4, N_1(t)=m, N_2(t)=n\}, m \ge 1, n \ge 0$$

$$\overline{Q}_{m,n}(s,t) = Pr\{Y(t)=5, N_1(t)=m, N_2(t)=n\}, m \ge 1, n \ge 0$$

$$\overline{R}_{m,n}(s,t) = Pr\{Y(t)=6, N_1(t)=m, N_2(t)=n\}, m \ge 0, n \ge 1$$

2.3 Equations Governing the System:

The Kolmogorov forward equations which governs the model:

$$\frac{\partial}{\partial t}P^{(1)}_{m,n}(x,t) + \frac{\partial}{\partial x}P^{(1)}_{m,n}(x,t) = -(\lambda_1+\lambda_2+\alpha+\beta+\mu_1(x))P^{(1)}_{m,n}(x,t)$$
$$+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_{1,i}P^{(1)}_{m-i,n}(x,t) + (1-\delta_{0n})\pi_1\lambda_2\sum_{i=1}^{n}c_{2,i}P^{(1)}_{m,n-i}(x,t)$$
$$+\lambda_2(1-\pi_1)P^{(1)}_{m,n}(x,t) + \gamma Q_{m,n}(t) + \beta_2 E_{m,n}(t); m \ge 0, n \ge 0, \qquad (1)$$

$$\frac{\partial}{\partial t}V_{0,n}(x,t) + \frac{\partial}{\partial x}V_{0,n}(x,t) = -(\lambda_1+\lambda_2+\beta_1(x))V_{0,n}(x,t)$$
$$+(1-\delta_{0n})\lambda_2\pi_1\sum_{i=1}^{n}c_{2,i}V_{0,n-i}(x,t) + \lambda_2(1-\pi_1)V_{0,n}(x,t); m \ge 0, n \ge 0, \qquad (2)$$

$$\frac{\partial}{\partial t}P^{(2)}_{0,n}(x,t) + \frac{\partial}{\partial x}P^{(2)}_{0,n}(x,t) = -(\lambda_1+\lambda_2+\alpha+\mu_2(x))P^{(2)}_{0,n}(x,t)$$
$$+(1-\delta_{0n})\lambda_2\pi_1\sum_{i=1}^{n}c_{2,i}P^{(2)}_{0,n-i}(x,t) + \lambda_2(1-\pi_1)P^{(2)}_{0,n}(x,t); n \ge 0, \qquad (3)$$

$$\frac{d}{dt}Q_{m,n}(t) = -(\lambda_1+\lambda_2+\gamma+\mu_3)Q_{m,n}(t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_{1,i}Q_{m-i,n}(t)$$
$$+(1-\delta_{0n})\lambda_2\pi_1\sum_{i=1}^{n}c_{2,i}Q_{m,n-i}(t) + \lambda_2(1-\pi_1)Q_{m,n}(t) + \mu_3 Q_{m+1,n}(t)$$
$$+\alpha\int_0^\infty P^{(1)}_{m,n}(x,t)dx; m \ge 0, n \ge 0, \qquad (4)$$

$$\frac{d}{dt}R_{m,n}(t) = -(\lambda_1+\lambda_2+\gamma)R_{m,n}(t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_{1,i}R_{m-i,n}(t) + (1-\delta_{0n})\lambda_2\pi_1$$

$$\times\sum_{i=1}^{n}c_{2,i}R_{m,n-i}(t) + \lambda_2(1-\pi_1)R_{m,n}(t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_{1,i}R_{0,n}(t);$$

$$m\ge 0, n\ge 1, \quad (5)$$

$$\frac{d}{dt}R_{0,n}(t) = -(\lambda_1+\lambda_2+\gamma)R_{0,n}(t) + (1-\delta_{0n})\lambda_2\pi_1\sum_{i=1}^{n}c_{2,i}R_{0,n-i}(t) + \lambda_2(1-\pi_1)R_{0,n}(t)$$

$$+\alpha\int_0^\infty P_{0,n-1}^{(2)}(x,t)dx + \mu_3 Q_{0,n}(t); m\ge 0, n\ge 1, \quad (6)$$

$$\frac{d}{dt}E_{m,n}(t) = -(\lambda_1+\lambda_2+\beta_2)E_{m,n}(t) + (1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_{1,i}E_{m-i,n}(t)$$

$$+(1-\delta_{0n})\lambda_2\pi_1\sum_{i=1}^{n}c_{2,i}E_{m,n-i}(t) + \lambda_2(1-\pi_1)E_{m,n}(t)$$

$$+\beta\int_0^\infty P_{m,n}^{(1)}(x,t)dx; m\ge 0, n\ge 0, \quad (7)$$

$$\frac{d}{dt}I_{0,0}(t) = -(\lambda_1+\lambda_2)I_{0,0}(t) + (1-\theta)\int_0^\infty P_{0,0}^{(1)}(x,t)\mu_1(x)dx + (1-\theta)\int_0^\infty P_{0,0}^{(2)}(x,t)\mu_2(x)dx$$

$$+\int_0^\infty V_{0,0}(x,t)\beta_1(x)dx + \gamma R_{0,0}(t) \quad (8)$$

$$\frac{\partial}{\partial t}I_{0,n}(x,t) + \frac{\partial}{\partial x}I_{0,n}(x,t) = -(\lambda_1+\lambda_2+\eta(x))I_{0,n}(x,t); n\ge 1. \quad (9)$$

The above set of equations are to be solved under the following boundary conditions at x=0.

$$I_{0,n}(0,t) = +(1-\theta)\int_0^\infty P_{0,n}^{(1)}(x,t)\mu_1(x)dx + (1-\theta)\int_0^\infty P_{0,n}^{(2)}(x,t)\mu_2(x)dx$$

$$+\int_0^\infty V_{0,n}(x,t)\beta_2(x)dx + \gamma R_{0,n}(t); n\ge 1. \quad (10)$$

$$P_{m,n}^{(1)}(0,t) = \lambda_1 c_{1,m+1}I_{0,n}(t) + (1-\theta)\int_0^\infty P_{m+1,n}^{(1)}(x,t)\mu_1(x)dx + (1-\delta_{0n})\lambda_1 c_{1,m+1}$$

$$\int_0^\infty P_{0,n-1}^{(2)}(x,t)dx + \int_0^\infty V_{m+1,n}(x,t)\beta_2(x)dx + \gamma R_{m+1,n}(t); m\ge 0, n\ge 1, \quad (11)$$

$$P_{0,n}^{(2)}(0,t)=\int_0^{\infty}I_{0,n+1}(x,t)\eta(x)dx+\lambda_2c_{2,n+1}I_{0,0}(t)+\lambda_2\pi_2\sum_{i=1}^{n}c_{2,i}\int_0^{\infty}I_{0,n+1-i}(x,t)dx$$

$$+(1-\pi_2)\lambda_2\sum_{i=1}^{n}c_{2,i}\int_0^{\infty}I_{0,n+1-i}(x,t)dx;\, n\geq 0, \tag{12}$$

$$V_{0,n}(0,t)=\theta\int_0^{\infty}P_{0,n}^{(1)}(x,t)\mu_1(x)dx+\theta\int_0^{\infty}P_{0,n}^{(2)}(x,t)\mu_2(x)dx;\, m=0, n\geq 0. \tag{13}$$

We assume that initially there are no customers in the system and the server is idle. Then the initial conditions are,

$$P_{m,n}^{(1)}(0)=P_{0,n}^{(2)}(0)=V_{0,n}(0)=E_{m,n}(0)=R_{m,n}(0)=Q_{m,n}(0)=0; m\geq 0, n\geq 0$$

$$I_{0,n}(0)=0 \text{ and } I_{0,0}(0)=1. \tag{14}$$

The Probability Generating Function(PGF) of this model:

$$I(x,z_2,t)=\sum_{n=1}^{\infty}z_2^nI_{0,n}(x,t), P_0^{(2)}(x,z_2,t)=\sum_{n=0}^{\infty}z_2^nP_{0,n}^{(2)}(x,t),$$

$$A(z_1,z_2,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}z_1^mz_2^nA_{m,n}(t), B(x,z_1,z_2,t)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}z_1^mz_2^nB_{m,n}(x,t)$$

where $A=Q,R,E$ and $B=P^{(1)},V$,

By taking Laplace transforms from (1) to (13) and solve the equations, we get,

$$\overline{I}_0(x,s,z_2)=\overline{I}_0(0,s,z_2)[1-\overline{I}(\varphi(a,s))]e^{-\varphi(a,s)x}, \tag{15}$$

$$\overline{P}^{(1)}(x,s,z_1,z_2)=\overline{P}^{(1)}(0,s,z_1,z_2)[1-\overline{B}_1(\varphi_1(s,z_1,z_2))]e^{-\varphi_1(s,z_1,z_2)x}, \tag{16}$$

$$\overline{P}^{(2)}(x,s,z_2)=\overline{P}^{(2)}(0,s,z_2)[1-\overline{B}_2(\varphi_2(s,z_2))]e^{-\varphi_2(s,z_2)x}, \tag{17}$$

$$\overline{V}(x,s,z_2)=\overline{V}(0,s,z_2)[1-\overline{V}(\varphi_3(s,z_2))]e^{-\varphi_3(s,z_2)x}, \tag{18}$$

$$\overline{Q}(s,z_1,z_2)=\frac{\alpha\overline{P}^{(1)}(0,s,z_1,z_2)[1-\overline{B}_1(\varphi_1(s,z_1,z_2))]e^{-\varphi_1(s,z_1,z_2)x}}{\varphi_4(s,z_1,z_2)}, \tag{19}$$

$$\overline{E}(s,z_1,z_2)=\frac{\alpha\overline{P}^{(1)}(0,s,z_1,z_2[1-\overline{B}_1(\varphi_1(s,z_1,z_2))]e^{-\varphi_1(s,z_1,z_2)x}}{\varphi_6(s,z_1,z_2)}, \tag{20}$$

$$\overline{R}(s,z_1,z_2)=\frac{\mu_3\overline{Q}(s,z_2)+\alpha\overline{P}_0^{(2)}(x,s,z_2)}{\varphi_5(s,z_1,z_2)}. \tag{21}$$

where,

$$\varphi(a,s) = s+\lambda_1+\lambda_2,$$

$$\varphi_1(s,z_1,z_2) = s+\lambda_1[1-C_1(z_1)]+\lambda_2\pi_1[1-C_2(z_2)]+\alpha[1-\frac{\gamma}{\varphi_4(s,z_1,z_2)}]+\beta[1-\frac{\beta_2}{\varphi_6(s,z_1,z_2)}],$$

$$\varphi_2(s,z_1,z_2) = s+\lambda_1+\alpha+\lambda_2\pi_1[1-C_2(z_2)],$$

$$\varphi_3(s,z_1,z_2) = s+\lambda_1[1-C_1(z_1)]+\lambda_2\pi_1[1-C_2(z_2)],$$

$$\varphi_4(s,z_1,z_2) = s+\lambda_1[1-C_1(z_1)]+\lambda_2\pi_1[1-C_2(z_2)]+\mu_3[1-\frac{1}{z_1}]+\gamma,$$

$$\varphi_5(s,z_1,z_2) = s+\lambda_1[1-C_1(z_1)]+\lambda_2\pi_1[1-C_2(z_2)]+\gamma,$$

$$\varphi_6(s,z_1,z_2) = s+\lambda_1[1-C_1(z_1)]+\lambda_2\pi_1[1-C_2(z_2)]+\beta_2,$$

Similarly for the boundary conditions we can get,

$$z_1\overline{P}^{(1)}(0,s,z_1,z_2) = \lambda_1 C_1(z_1)\overline{I}_0(x,s,z_2)+\int_0^\infty \overline{P}^{(1)}(x,s,z_1,z_2)\mu_1(x)dx+\lambda_1 C_1(z_1)z_2$$
$$\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)dx-\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx+\gamma\overline{R}(s,z_1,z_2)-\gamma\overline{R}_0(s,z_2)$$
$$+\int_0^\infty \overline{V}(x,s,z_1,z_2)\beta_1(x)dx-\int_0^\infty \overline{V}_0(x,s,z_2)\beta_1(x)dx. \quad (22)$$

$$\overline{I}_0(0,s,z_2) = 1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s)+\int_0^\infty \overline{V}_0(x,s,z_2)\beta_1(x)dx+(1-\theta)$$
$$\int_0^\infty \overline{P}_0^{(1)}(x,s,z_2)\mu_1(x)dx++(1-\theta)\int_0^\infty \overline{P}_0^{(2)}(x,s,z_2)\mu_2(x)dx+\gamma\overline{R}_0(s,z_2) \quad (23)$$

$$z_2\overline{P}_0^{(2)}(0,s,z_2) = \overline{I}_0(0,s,z_2)[\overline{I}(f(a,s))+((1-\pi_2)\lambda_2+\lambda_2\pi_2 C_2(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]]$$
$$+\lambda_2 C_2(z_2)\overline{I}_{0,0}(s). \quad (24)$$

By applying Rouche's theorem, we get,

$$\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1(\varphi_1(s,z_2)) = \frac{\lambda_1 C_1(g(z_2))\overline{I}_0(0,s,z_2)[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]+\overline{P}_0^{(2)}(0,s,z_2)\overline{\zeta}_3(s,z_2)}{\overline{\zeta}_4(s,z_2)}. \quad (25)$$

By substituting the above in required equations we get,

$$\overline{P}_0^{(1)}(0,s,z_1,z_2)=\frac{\left[\overline{I}_0(x,s,z_2)\{\lambda_1 C_1(z_1)\overline{\zeta}_4(s,z_2)-\lambda_1 C_1(g(z_2))\overline{\zeta}_1(s,z_1,z_2)\} + \overline{P}_0^{(2)}(0,s,z_2)\{\overline{\zeta}_2(s,z_1,z_2)\overline{\zeta}_4(s,z_2)-\overline{\zeta}_1(s,z_1,z_2)\overline{\zeta}_3(s,z_2)\}\right]}{\left[\overline{\zeta}_4(s,z_2)\{z_1-\overline{B_1}(f_1(s,z_1,z_2))\}\right]},$$

$$\overline{P}_0^{(2)}(0,s,z_2)=\frac{\left[(1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s))\overline{\zeta}_4(s,z_2)\overline{\zeta}_6(s,z_2) + \lambda_2\pi_2 C_2(z_2)\overline{I}_{0,0}(s)\{\overline{\zeta}_4(s,z_2)-\lambda_1 C_1(g(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\overline{\zeta}_5(s,z_2)\}\right]}{\left[z_2\{\overline{\zeta}_4(s,z_2)-\lambda_1 C_1(g(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\overline{\zeta}_5(s,z_2)\} - \overline{\zeta}_6(s,z_2)\overline{\zeta}_7(s,z_2)\right]}$$

$$\overline{I}_0(0,s,z_2)=\frac{\left[z_2(1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s))\overline{\zeta}_4(s,z_2) + \lambda_2\pi_2 C_2(z_2)\overline{I}_{0,0}(s)\overline{\zeta}_7(s,z_2)\right]}{\left[z_2\{\overline{\zeta}_4(s,z_2)-\lambda_1 C_1(g(z_2))[\frac{1-\overline{I}(f(a,s))}{f(a,s)}]\overline{\zeta}_5(s,z_2)\} - \overline{\zeta}_6(s,z_2)\overline{\zeta}_7(s,z_2)\right]}$$

$$\overline{V}(0,s,z_2)=\theta\overline{P}^{(1)}(0,s,z_2)\overline{B_1}(f_1(s,z_2))+\theta\overline{P}_0^{(2)}(0,s,z_2)\overline{B_2}(f_2(s,z_2)), \tag{26}$$

$$\overline{\zeta}_1(s,z_1,z_2)=\{1-\theta\overline{V}(\varphi_3(s,z_1,z_2))+\theta\overline{V}(\varphi_3(s,z_2))\}\overline{B_1}(\varphi_1(s,z_2)) + \frac{\alpha\gamma\mu_3(1-\overline{B_1}(\varphi_1(s,z_2)))}{\varphi_1(s,z_2)\varphi_4(s,z_2)}\{\frac{1}{\varphi_5(s,z_2))-\frac{1}{\varphi_5(s,z_1,z_2))}}\} \tag{27}$$

$$\overline{\zeta}_2(s,z_1,z_2)=\lambda_1 C_1(z_1)z_2\frac{1-\overline{B_2}(\varphi_2(s,z_2)))}{\varphi_2(s,z_2))} + \theta\overline{B_2}(\varphi_2(s,z_2)))\{\overline{V}(\varphi_3(s,z_1,z_2))-\overline{V}(\varphi_3(s,z_2))\}$$

$$\overline{\zeta}_3(s,z_2)=\lambda_1 C_1(g(z_2))z_2\frac{1-\overline{B_2}(\varphi_2(s,z_2)))}{\varphi_2(s,z_2))}+\theta\overline{B_2}(\varphi_2(s,z_2)))\{\overline{V}(\varphi_3(s,g(z_2))) - \overline{V}(\varphi_3(s,z_2))\} \tag{28}$$

$$\overline{\zeta}_4(s,z_2)=\{1-\theta\overline{V}(\varphi_3(s,g(z_2)))+\theta\overline{V}(\varphi_3(s,z_2))\}\overline{B_1}(\varphi_1(s,z_2))+\frac{\alpha\gamma\mu_3(1-\overline{B_1}(\varphi_1(s,z_2)))}{\varphi_1(s,z_2)\varphi_4(s,z_2)}$$

$$\{\frac{1}{\varphi_5(s,z_2)}-\frac{1}{\varphi_5(s,g(z_2))}\}, \tag{29}$$

$$\overline{\zeta}_5(s,z_2)=\{1-\theta+\theta\overline{V}(\varphi_3(s,z_2))\}\overline{B_1}(\varphi_1(s,z_2))+\frac{\alpha\gamma\mu_3(1-\overline{B_1}(\varphi_1(s,z_2)))}{\varphi_1(s,z_2)\varphi_4(s,z_2)\varphi_5(s,z_2)} \tag{30}$$

$$\overline{\zeta}_6(s,z_2)=\overline{I}_0(0,s,z_2)+\overline{I}_0(0,s,z_2)[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}\{(1-\pi_2)\lambda_2+\pi_2\lambda_2C_2(z_2)\}, \tag{31}$$

$$\overline{\zeta}_7(s,z_2)=\{1-\theta+\theta\overline{V}(\varphi_3(s,z_2))\}\overline{B_2}(\varphi_1(s,z_2))\overline{\zeta}_4(s,z_2)+\overline{\zeta}_3(s,z_2)\overline{\zeta}_5(s,z_2).$$

2.3 Theorem: The inequality

$$P^{(1)}(1,1)+P^{(2)}(0,1)+Q(1,1)=\rho<1,$$

is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server's state, queue size and orbit size distributions are given by,

$$\overline{I}_{(0)}(s,z_2)=\overline{I}_0(0,s,z_2)[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}], \tag{32}$$

$$\overline{P}^{(1)}(s,z_1,z_2)=\overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B_1}(\varphi_1(s,z_1,z_2))}{\varphi_1(s,z_1,z_2)}], \tag{33}$$

$$\overline{V}(s,z_1,z_2)=\{\theta\overline{P}^{(1)}(0,s,z_1,z_2)\overline{B_1}(\varphi_1(s,z_1,z_2))+\theta\overline{P}_0^{(2)}(0,s,z_2)\overline{B_2}(\varphi_1(s,z_2))\}$$

$$\times[\frac{1-\overline{V}(\varphi_3(s,z_1,z_2))}{\varphi_3(s,z_1,z_2)}], \tag{34}$$

$$\overline{P}_0^{(2)}(s,z_2)=\overline{P}_0^{(2)}(0,s,z_2)[\frac{1-\overline{B_2}(\varphi_2(s,z_2))}{\varphi_2(s,z_2)}], \tag{35}$$

$$\overline{Q}(s,z_1,z_2)=\frac{\alpha\overline{P}^{(1)}(0,s,z_1,z_2)[1-\overline{B}_1(\varphi_1(s,z_1,z_2))]}{\varphi_1(s,z_1,z_2)\varphi_4(s,z_1,z_2)}, \tag{36}$$

$$\overline{R}(s,z_1,z_2)=\frac{\mu_3\overline{Q}_0(s,z_2)}{\varphi_5(s,z_1,z_2)}, \tag{37}$$

$$\overline{E}(s,z_1,z_2)=\frac{\beta\overline{P}^{(1)}(0,s,z_1,z_2)[1-\overline{B}_1(\varphi_1(s,z_1,z_2))]}{\varphi_1(s,z_1,z_2)\varphi_6(s,z_1,z_2)}. \tag{38}$$

3. STEADY STATE ANALYSIS: LIMITING BEHAVIOUR

By applying the well-known Tauberian property,

$$\lim_{s\to 0} s\overline{f}(s)=\lim_{t\to\infty} f(t),$$

to the above equations, and adding we obtain the steady- state solutions of this model. In order to determine $I_{0,0}$, we use the normalizing condition

$$P^{(1)}(1,1)+V(1,1)+P^{(2)}(1)+Q(1,1)+R(1,1)+E(1,1)+I_0(0,1)+I_{0,0}=1.$$

For this, let $P_q(z_1,z_2)$ be the probability generating function of the queue size irrespective of the state of the system. Then adding equations from (50) to (57) , we obtain,

$$P_q(z_1,z_2)=P^{(1)}(z_1,z_2)+V(z_1,z_2)+P_0^{(2)}(z_2)+E(z_1,z_2)+Q(z_1,z_2)+R(z_1,z_2), \tag{39}$$

$$P_q(z_1,z_2)=\frac{N_1(z_1,z_2)}{D_1(z_1,z_2)}+\frac{N_2(z_1,z_2)}{D_2(z_1,z_2)}+\frac{N_3(z_1,z_2)}{D_3(z_1,z_2)},$$

where

$$N_1(z_1,z_2)=I_0(0,z_2)[\frac{1-\overline{I}(\varphi(a))}{\varphi(a)}]\{\zeta_4(z_2)\varphi_1(z_2)\varphi_4(z_2)\varphi_5(z_1,z_2)+\lambda_1C_1(g(z_2))\{\theta\overline{B}_1(\varphi_1(s,z_2))$$
$$\overline{V}(\varphi_3(s,z_2))\varphi_1(s,z_2)\varphi_4(s,z_2)\varphi_5(s,z_1,z_2)+\alpha\mu_3(1-\overline{B}_1(\varphi_1(s,z_2)))\}\},$$

$$N_2(z_1,z_2)=P_0^{(2)}(0,z_2)\{\zeta_3(z_2)\varphi_2(s,z_2)\{\theta\overline{B}_1(\varphi_1(s,z_2))\overline{V}(\varphi_3(s,z_2))\varphi_1(s,z_2)\varphi_4(s,z_2)$$
$$\varphi_5(s,z_1,z_2)+\alpha\mu_3(1-\overline{B}_1(\varphi_1(s,z_2)))\}+\zeta_4(z_2)\varphi_1(z_2)\varphi_4(z_2)\varphi_5(z_1,z_2)$$
$$[1-\overline{B}_2(\varphi_2(s,z_2))+\theta\overline{B}_2(\varphi_2(s,z_2))\overline{V}(\varphi_3(s,z_2))\varphi_2(s,z_2)]\}$$

$$N_3(z_1,z_2)=P^{(1)}(0,z_1,z_2)\{(1-\overline{B}_1(\varphi_1(z_1,z_2)))[\alpha\varphi_6(z_1,z_2))+\beta\varphi_4(z_1,z_2))$$
$$+\varphi_4(z_1,z_2))\varphi_6(z_1,z_2))\},$$
$$D_1(z_1,z_2)=\zeta_4(z_2)\varphi_1(z_2)\varphi_4(z_2)\varphi_5(z_1,z_2),$$

$$D_2(z_1,z_2) = \varphi_2(z_2)\zeta_4(z_2)\varphi_1(z_2)\varphi_4(z_2)\varphi_5(z_1,z_2)$$

$$D_3(z_1,z_2) = \varphi_1(z_1,z_2)\varphi_4(z_1,z_2)\varphi_6(z_1,z_2).$$

In order to obtain the probability of idle time $I_{0,0}$, we use the normalizing condition,

$$P_q^{(1)}(1,1) + I_{0,0} = 1.$$

From which we can have,

$$I_{0,0} = \frac{\varphi_2(1)\zeta_4(1)\varphi_1(1)\varphi_4(1)\varphi_4(1,1)\varphi_5(1,1)\varphi_6(1,1)}{Dr},$$

$$Dr = \varphi_2(z_2)\zeta_4(1)\varphi_1(1)\varphi_4(1)\varphi_4(1,1)\varphi_5(1,1)\varphi_6(1,1) + I_0(0,1)[\frac{1-\bar{I}(\varphi(a))}{\varphi(a)}]\{\zeta_4(1)\varphi_1(1)\varphi_4(1)$$

$$\varphi_5(1,1) + \lambda_1\{\theta\bar{B}_1(\varphi_1(1))\bar{V}(\varphi_3(1))\varphi_1(1)\varphi_4(1)\varphi_5(1,1) + \alpha\mu_3(1-\bar{B}_1(\varphi_1(1)))\}\}$$

$$+ P_0^{(2)}(0,1)\{\zeta_3(1)\varphi_2(1)\{\theta\bar{B}_1(\varphi_1(1))\bar{V}(\varphi_3(1))\varphi_1(1)\varphi_4(1)\varphi_5(1,1) + \alpha\mu_3(1-\bar{B}_1(\varphi_1(1)))\}$$

$$+ \zeta_4(1)\varphi_1(1)\varphi_4(1)\varphi_5(1,1)[1-\bar{B}_2(\varphi_2(1)) + \theta\bar{B}_2(\varphi_2(1))\bar{V}(\varphi_3(1))\varphi_2(1)]\}$$

$$+ P^{(1)}(0,1,1)\{(1-\bar{B}_1(\varphi_1(1,1)))[\alpha\varphi_6(1,1)) + \beta\varphi_4(1,1)) + \varphi_4(1,1))\varphi_6(1,1))\}.$$

4 . THE AVERAGE QUEUE LENGTH:

The Mean number of customers in the queue and in the orbit under the steady state condition is,

$$L_{q_1} = \frac{d}{dz_1} P_{q_1}(z_1,1)|_{z_1=1}, \quad L_{q_2} = \frac{d}{dz_2} P_{q_2}(1,z_2)|_{z_2=1}. \tag{40}$$

then,

$$L_{q_1} = \frac{D_1'(1,1)N_1''(1,1) - D_1''(1,1)N_1'(1,1)}{2(D_1'(1,1))^2} + \frac{D_2'(1,1)N_2''(1,1) - D_2''(1,1)N_2'(1,1)}{2(D_2'(1,1))^2}$$

$$+ \frac{D_3''(1,1)N_3'''(1,1) - D_3'''(1,1)N_3''(1,1)}{3(D_3''(1,1))^2},$$

$$L_{q_2} = \frac{d_1'(1,1)n_1''(1,1) - d_1''(1,1)n_1'(1,1)}{2(d_1'(1,1))^2} + \frac{d_2'(1,1)n_2''(1,1) - d_2''(1,1)n_2'(1,1)}{2(d_2'(1,1))^2}$$

$$+\frac{d_3''(1,1)n_3'''(1,1)-d_3'''(1,1)n_3''(1,1)}{3(d_3''(1,1))^2},$$

4.1 The Average Waiting Time in the Queue and Orbit:

Average waiting time of a customer in the high priority queue is

$$W_{q_1}=\frac{L_{q_1}}{\lambda_1}, \tag{41}$$

Average waiting time of a customer in the low priority orbit is

$$W_{q_2}=\frac{L_{q_2}}{\lambda_2}, \tag{42}$$

where L_{q_1} and L_{q_2} have been found in the abouve equations.

4.2 Particular Cases:

Case: 1 $M^X/G/1$ Queueing model:

If there are no priority arrival, no vacation, no working breakdown, no balking, no retrial, no closedown/startup time and batch arrival. The model under study becomes classical $M^X/G/1$ queueing system. In this case, the PGF of the busy state is given as,

$$P(z)=\frac{(1-\overline{B}(\lambda-\lambda C(z)))I_{0,0}}{\overline{B}(\lambda-\lambda C(z))-z}. \tag{43}$$

Case: 2 $M/G/1$ Queueing model:

If there are no priority arrival, no vacation, no working breakdown, no balking, no retrial, no closedown/startup time and single arrival. The model under study becomes classical $M/G/1$ queueing system. In this case, the PGF of the busy state is given as,

$$P(z)=\frac{(1-\overline{B}(\lambda-\lambda z))I_{0,0}}{\overline{B}(\lambda-\lambda z)-z}. \tag{44}$$

The above two results are coincide with the results of Gross.D and Harris.M [11].

5 NUMERICAL RESULTS

In this section, we present some numerical examples to study the effect of various parameters. For the purpose of a numerical illustration, we assume that all distribution function like retrial, regular service for customers, vacation are exponentially distributed. All the parameters values are selected with the aim of satisfying the steady state condition.

Table 1: $(\lambda_2, \mu, \mu_3, \eta, \theta, \alpha, \gamma, \beta, \beta_1, \beta_2, \pi, \pi) = (0.6,5,3,6,0.5,0.8,8,0.4,3,0.2,0.5,0.7)$

Table 1: Effect of High priority arrival rate

λ_1	$I_{0,0}$	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
1.4	0.0528	0.9472	2.1748	0.2704	1.5534	0.4507
1.5	0.0505	0.9495	2.4237	0.3049	1.6158	0.5082
1.6	0.0479	0.9521	2.6700	0.4030	1.6687	0.6717
1.7	0.0452	0.9548	2.9133	0.5643	1.7137	0.9405
1.8	0.0425	0.9575	3.1536	0.7881	1.7520	1.3135
1.9	0.0398	0.9602	3.3908	1.0738	1.7846	1.7897
2.0	0.0373	0.9627	3.6250	1.4207	1.8125	2.3679
2.1	0.0349	0.9651	3.8564	1.8285	1.8364	3.0476
2.2	0.0327	0.9673	4.0852	2.2970	1.8569	3.8283
2.3	0.0306	0.9694	4.3115	2.8260	1.8746	4.7100

Table 2:
$(\lambda_1, \lambda_2, \mu_3, \eta, \theta, \alpha, \gamma, \beta, \beta_1, \beta_2, \pi_1, \pi_2) = (0.1,0.2,0.1,3,0.1,1,2,0.3,2,2,0.2,0.5)$

Table 2: Effect of Service rate

μ	$I_{0,0}$	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
1.5	0.5992	0.4008	0.0178	0.2528	0.1782	1.2641
1.6	0.6006	0.3994	0.0162	0.2338	0.1615	1.1690
1.7	0.6021	0.3979	0.0146	0.2125	0.1464	1.0623
1.8	0.6037	0.3963	0.0133	0.1892	0.1326	0.9459
1.9	0.6052	0.3948	0.0120	0.1643	0.1198	0.8217
1.0	0.6069	0.3931	0.0108	0.1382	0.1080	0.6911
1.1	0.6085	0.3915	0.0097	0.1111	0.0970	0.5556
1.2	0.6102	0.3898	0.0087	0.0833	0.0868	0.4163
1.3	0.6119	0.3881	0.0077	0.0549	0.0772	0.2745
1.4	0.6137	0.3863	0.0068	0.0262	0.0681	0.1311

Table 1 clearly shows that as long as the arrival rate of high priority customers increases the servers idle time decreases. Simultaneously the utilisation factor, average queue length for both high priority and low priority customers are increases. Table 2 clearly shows that as long as the service rate increases the server's idle time increases and the utilisation factor, average queue length for high priority and low priority customers are decreases.

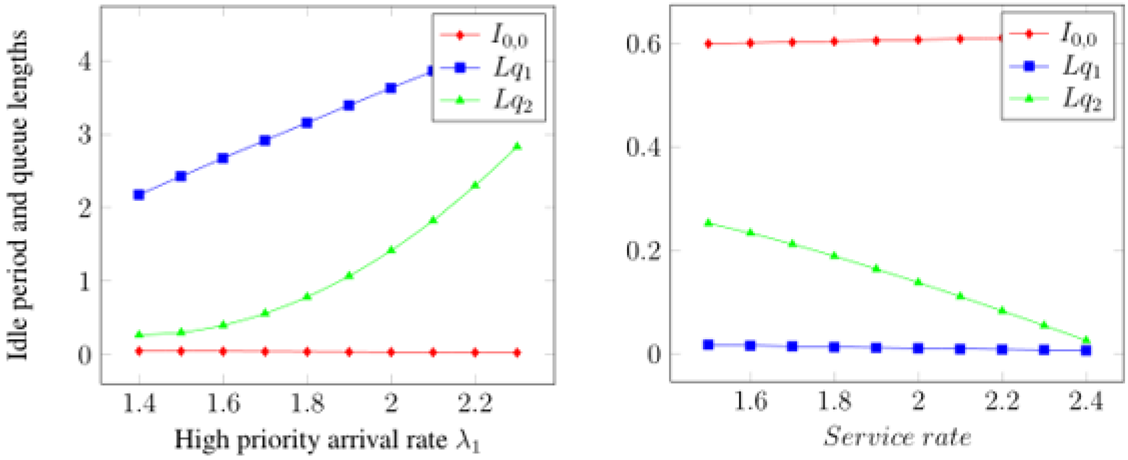


Figure 1: Graphical study

6 . CONCLUSION

In this paper we have analysed a $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queue with priority service under Bernoulli vacation subject to the server working breakdown along with it the modified Bernoulli vacation and emergency vacation are also investigated. In addition, the effect of non persistent behaviour of the low priority customer on a service system is studied. The joint distribution of the number of customers in the queue and the number of customers in the orbit are derived. Numerical examples have been carried out to observe the trend of the mean number of customers in the system for varying parametric values.

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