Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 9, Number 3 (2017), pp. 357-366

http://www.irphouse.com

The Connected Total Monophonic Number of A Graph

[1]P. Arul Paul Sudhahar
Department of Mathematics,
Rani Anna Govt. College (W), Tirunelveli – 627 008, Tamilnadu, India

[2]V.K. Kalai Vani
Department of Mathematics,
Nanjil Catholic College of Arts and Science,
Kaliyakkavilai – 629 153, Kanyakumari District, Tamil Nadu, India.

[3]A. Vijayan
Department of Mathematics,
Nesamony Memorial Christian College, Marthandam – 629 165, Kanyakumari District, Tamil Nadu, India.

Abstract

A set M of vertices of a connected graph G is a monophonic set if every vertex of G lies on an x-y monophonic path for some elements x and y in M. The minimum cardinality of a monophonic set of G is the monophonic number of G, denoted by $m(G)$. A total monophonic set of a graph G is a monophonic set M such that the subgraph induced by M has no isolated vertices. The minimum cardinality of a total monophonic set of G is the total monophonic number denoted by $m_t(G)$. A connected total monophonic set of a graph G is a total monophonic set M such that the subgraph $\langle M \rangle$induced by M is connected The minimum cardinality of a connected total monophonic set of G is the connected total monophonic number of G and is denoted by $m_{ct}(G)$. It is proved that, for the integers a, b and c with $a < b < c$, there exists a connected graph G having the monophonic number a,the total monophonic number b, and the connected total monophonic number c.

Keywords : Monophonic set, monophonic number, total monophonic number, monophonic distance, connected total monophonic number. AMS Subject classification : 05C12.

1.INTRODUCTION

For any two vertices *x* and *y* in a connected graph *G*, the distance $d(x, y)$ is the length of a shortest *x-y* path in *G*. An *x-y* path of length $d(x, y)$ is called an *x-y* geodesic. A vertex *v* is said to lie on an *x-y* geodesic *P* if *v* is a vertex of *P* including the vertices *x* and *y*. *A* set *S* of vertices is a geodetic set if $I[S]$ = V, and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic set of cardinality $g(G)$ is called *a g*-set. The geodetic number of a graph was introduced in [2, 6] and further studied in [3, 4, 5]. A connected geodetic set of a graph G is a geodetic set S such that the subgraph G[S] induced by S is connected. The minimum cardinality of a connected geodetic set of G is the connected geodetic number of G and is denoted by $g_c(G)$. The connected geodetic number of a graph is introduced in [9] and further studied in [11,12] .

A chord of a path $u_1, u_2, \ldots, u_k$ in *G* is an edge $u_i u_j$ with $j \geq i + 2$. A *u-v* path *P* is called a monophonic path if it is a chordless path. A set *M* of vertices is a monophonic set if every vertex of *G* lies on a monophonic path joining some pair of vertices in *M*, and the minimum cardinality of a monophonic set is the monophonic number $m(G)$. The monophonic number of a graph *G* was studied in [10]. The eccentricity $e(v)$ of a vertex *v* in *G* is the maximum distance from *v* and *a* vertex farthest from *v*. The minimum eccentricity among the vertices of *G* is the radius, rad (*G*) or $r(G)$ and the maximum eccentricity is its diameter, diam *G* of *G*.

A total monophonic set of a graph *G* is a monophonic set *M* such that the subgraph induced by *M* has no isolated vertices. The minimum cardinality of a total monophonic set of *G* is the total monophonic number denoted by $m_t(G)$. The Total edge monophonic number of a graph was introduced and studied in [1]. A connected monophonic set of a graph *G* is a monophonic set *M* such that the subgraph $\langle M \rangle$ induced by *M* is connected . The minimum cardinality of a connected monophonic set of *G* is the connected monophonic number of *G* and is denoted by $m_c(G)$. The connected monophonic number of a graph was studied in [8].

The following Theorems are used in the sequel.

Theorem 1.1:[4] Each extreme vertex of a connected graph *G* belongs to every geodetic set of *G*.

Theorem 1.2: [7] For any non trivial tree *T* of order *p*, $g_c(T) = p$.

Theorem 1.3:[1] Each extreme vertex of *G* belongs to every total monophonic set of *G*.

Theorem 1.4:[7] The monophonic number of a tree *T* is the number of end vertices in *G*.

2. The Connected Total Monophonic Number Of a Graph

Definition 2.1: Let *G* be a connected graph with at least two vertices. A connected total monophonic set of a graph *G* is a total monophonic set *M* such that the subgraph $\langle M \rangle$induced by *M* is connected . The minimum cardinality of a connected total monophonic set of *G* is the connected total monophonic number of *G* and is denoted by $m_{ct}(G)$. A connected total monophonic set of cardinality $m_{ct}(G)$ is called a m_{ct} –set of *G* or a minimum connected total monophonic set of *G*.

Example 2.2: Consider the graph *G* of Fig. 2.1, $M = \{v_1, v_2, v_6\}$ is a minimum monophonic set of *G*. $M_1 = \{v_1, v_2, v_6, v_7,\}$ is a minimum total monophonic set of *G*, so that $m_t(G) = 4$. Here the induced subgraph $\langle M_1 \rangle$ is not connected , so that M_1 is not a connected total monophonic set of *G*. Now it is clear that $M_1 = \{v_1, v_2, v_3, v_6, v_7\}$ is a minimum connected total monophonic set of *G* and so $m_{ct}(G) = 5$.

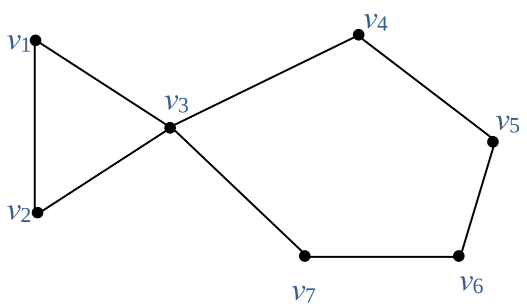


Figure 2.1

Observation 2.3: Every extreme vertex of a connected graph *G* belongs to every connected total monophonic set of *G*. In particular , every end vertex of *G* belongs to every connected total monophonic set of *G*.

Proof: Since every connected total monophonic set is also a total monophonic set, the result follows from Theorem 1.3.

Theorem 2.4: Let *G* be a connected graph with cut-vertices and let *M* be a connected total monophonic set of *G*. If *v* is a cut-vertex of *G*, then every component of *G-v* contains an element of *M*.

Proof: Suppose that there is a component *B* of *G* at a cut-vertex *v* such that *B* contains no vertex of *M*. Let $u \in V(B)$. Since *M* is a connected total monophonic set of *G*, there exists a pair of vertices *x* and *y* in *M* such that *u* lies on some *x-y* total monophonic path $P: x=u_0, u_1, u_2, \ldots, u_n= y$ in *G*. Since *v* is a cut-vertex of *G*, the *x-u* sub total monophonic path of *P* and the *u-y* total monophonic sub path of *P* both contain *v*, it follows that *P* is not a total monophonic path, contrary to the assumption. Therefore every component of *G-v* contains an element of *M*.

Theorem 2.5: Every cut-vertex of a connected graph *G* belongs to every connected total monophonic set of *G*.

Proof: Let *v* be any cut-vertex of *G* and let $G_1, G_2, \ldots, G_r (r \geq 2)$ be the components of *G-v*. Let *M* be any connected total monophonic set of *G*. Then by Theorem 2.4, *M* contains at least one element from each G_i $(1 \leq i \leq r)$. Since $\langle M \rangle$is connected, it follows that $v \in M$.

Corollary 2.6: For a connected graph *G* with *k* extreme vertices and *l* cut-vertices, $m_{ct}(G) \geq \max\{2, k+l\}$.

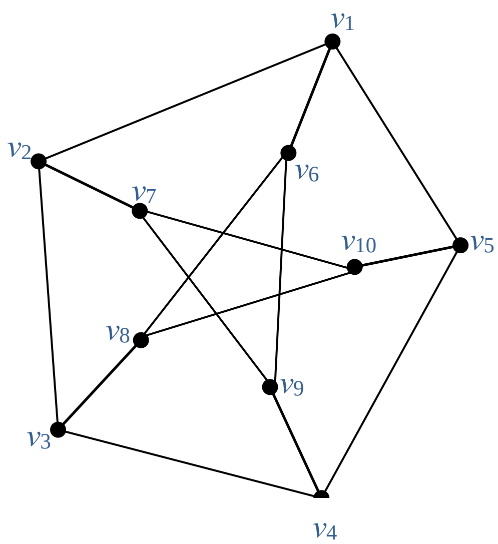
Proof: This follows from Observations 2.3 and Theorem 2.5.

In the following we determine the connected total monophonic number of some standard graphs.

Corollary 2.7: (i) For any non trivial tree *T* of order *p*, $m_{ct}(G) = p$.

(ii) For the complete graph K_p $(p \geq 2)$, $m_{ct}(K_p) = p$.

(iii) For the Petersen graph $K_{10,15}$ $m_{ct}(K_{10,15}) = 3$.



Theorem 2.8: For the cycle C_p $(p \geq 3)$, $m_{ct}(C_p) = 3$.

Proof: Let $v_1, v_2, \ldots, v_p, v_1$ be a cycle of length *p*. Let $x,y \in V(C_p)$ such that $d(x,y) = 2$. Then $M = \{x,y\}$ is a monophonic set of C_p. But $\langle M \rangle$is not connected . Let *u* be a vertex of C_p which is adjacent to both *x* and *y*. Then $M \cup \{u\}$ is connected total monophonic set , so that $m_{ct}(C_p) = 3$.

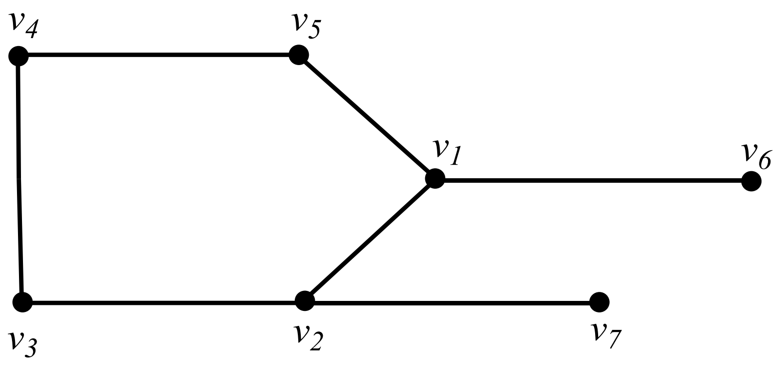
Theorem 2.9: For the complete bipartite graph $G = K_{m,n}$, $m_{ct}(G) = \begin{cases} 3 \text{ if } m = 2, n \geq 2 \\ 4 \text{ if } 3 \leq m \leq n \end{cases}$

Proof: Let $U = \{u_1, u_2, \ldots, u_m\}$ and $W = \{v_1, v_2, \ldots, v_n\}$ be the partite sets of G. First assume that $m = 2$, $n \geq 2$. Let $M \subseteq V(G)$. If $|M| = 2$, then either $\langle M \rangle$is disconnected or $\langle M \rangle$is an edge. It is clear that M is not a connected total monophonic set of G. However, $M = \{u_1, u_2, v_1\}$ is a connected total monophonic set of G, so that $m_{ct}(G) = 3$. Next assume that $3 \leq m \leq n$. Let $M \subseteq V(G)$. If $|M| = 2$, then it can be easily verified that M is not a connected total monophonic set of G. Let $|M| = 3$. If $M \subseteq U$ or $M \subseteq W$, then $\langle M \rangle$is not connected and so M is not a total monophonic set of G. If $M \subseteq U \cup W$, then it is easily seen that M is not a total monophonic set of G and so $m_{ct}(G) \geq 4$. Let $M = \{u_i, u_j, w_l, w_k\}$. It is easily verified that M is a total monophonic set of G. Since $\langle M \rangle$is connected, it follows that M is a connected total monophonic set of G and so $m_{ct}(G) = 4$.

Theorem 2.10: For a connected graph G of order p, $2 \leq m(G) \leq m_{ct}(G) \leq g_{ct}(G) \leq p$.

Proof: Any monophonic set needs atleast two vertices and so $m(G) \geq 2$. Since every connected total monophonic set is also a total monophonic set, it follows that $m(G) \leq m_{ct}(G)$. Since every connected total geodetic set is also a connected total monophonic set, it follows that $m_{ct}(G) \leq g_{ct}(G)$. Also, since $\langle V \rangle$induces a connected total geodetic set of G, it is clear that $g_{ct}(G) \leq p$.

Remark 2.11: The bounds in Theorem 2.10 are sharp. For any non-trivial path P, $m(P) = 2$. For the complete graph K_p, $m_{ct}(K_p)$. For $G = K_{m,n}(4 \leq m \leq n)$. By Theorem 2.9, $m_{ct}(G) = 4$ and also it is easily verified that $g_{ct}(G) = 4$ so that $m_{ct}(G) = g_{ct}(G)$. By Theorem 1.2, For any non trivial tree T, $g_c(G) = p$, $g_{ct}(G) = p$. Also, all the inequalities in the theorem are strict. For the graph G given in Figure 2.2, $m(G) = 3$, $m_{ct}(G) = 5$, $g_{ct}(G) = 6$ and $p = 7$ so that $2 < m(G) < m_{ct}(G) < g_{ct}(G) < p$.



Theorem 2.12: Let G be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $m_{ct}(G) = 2$.

Proof: If $G = K_2$, then $m_{ct}(G) = 2$. Conversely, let $m_{ct}(G) = 2$. Let $M = \{u, v\}$ be a minimum connected total monophonic set of G. Then uv is an edge. If $G \neq K_2$, then there exists a vertex w different from u and v that lies on a path between u and v. Since uv is a chord, u-v is not a total monophonic path , so that M is not a m_{ct} –set, which is a contradiction. Thus $G = K_2$.

Theorem 2.13: Let G be a connected graph. Then every vertex of G is either a cut-vertex or an extreme vertex if and only if $m_{ct}(G) = p$.

Proof: Let G be a connected graph with every vertex of G is either a cut-vertex or an extreme vertex. Then the result follows from Observation 2.3 and Theorem 2.5.

Conversely, suppose $m_{ct}(G) = p$. Suppose that there is a vertex x in G which is neither a cut-vertex nor an extreme vertex. Since x is an extreme vertex , $N(x)$ does not induce a complete subgraph and hence there exist u and v in $N(x)$ such that $d_m(u, v) = 2$. Clearly, x lies on a u-v monophonic path in G. Also, since x is not a cut-vertex of G, G-x is connected. Thus V-$\{x\}$ is a connected total monophonic set of G and so $m_{ct}(G) \leq | V - \{x\} | \leq p-1$, which is a contradiction. Therefore every vertex of G is either a cut-vertex or an extreme vertex.

Theorem 2.14: If G is a non complete connected graphs such that it has a minimum cutset, then $m_{ct}(G) \leq p$-$K(G)$+1.

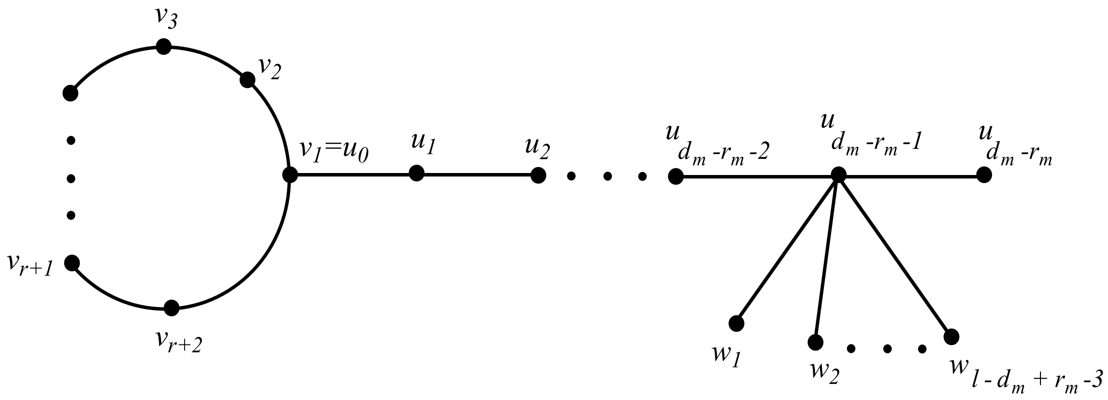
Proof: Since G is non complete , it is clear that $1 \leq K(G) \leq p-2$. Let $U = \{ u_1, u_2, . . . ,u_k\}$ be a minimum cutset of G. Let $G_1,G_2, . . . ,G_r$ $(r \geq 2)$ be the components of G-U and let $M = V(G) – U$. Then every vertex $u_i (1 \leq i \leq k)$ is adjacent to at least one vertex of Gj, for every $j(1\leq j \leq r)$. It is clear that M is a monophonic set of G and $\langle M\rangle$is not connected. Also , it is clear that $\langle M \cup \{x\}\rangle$is a connected total monophonic set for any vertex x in U so that $m_{ct}(G) \leq p – K(G) + 1$.

Remark 2.15: The bound in Theorem 2.14 is sharp. For the cycle $G = C_4$, $m_{ct}(G) = 3$. Also, K $(G) = 2$, $p – \mathrm{K}(G) + 1 = 3$. Thus $m_{ct}(G) = p – \mathrm{K}(G) + 1$.

3 REALIZATION RESULTS:

Theorem 3.1: For positive integers r_m, d_m and $l > d_m - r_m + 3$ with $r_m < d_m \leq 2r_m$, there exists a connected graph G with $\mathrm{rad}_m(G) = r_m$, $\mathrm{diam}_m(G) = d_m$ and $m_{ct}(G) = l$.

Proof: When $r_m = 1$, we let $G = K_{1,l-1}$. Then the result follows from Corollary 2.7 (i). Let $r_m \geq 2$, let C_{r+2}: $v_1, v_2, . . . , v_{r+2}, v_1$ be a cycle of length $r+2$ and let $P_{d_m - r_m + 1}$: $u_0, u_1, u_2, . . . ,u_{d_m - r_m}$ be a path of length $d_m - r_m + 1$. Let H be a graph obtained from C_{r_m+2} and $P_{d_m-r_m+1}$ by identifying v_1 in C_{r_m+2} and u_0 in $P_{d_m-r_m+1}$. Now add $l - d_m + r_m - 3$ new vertices $w_1, w_2, . . . , w_{l-d_m+r_m-3}$ to H and join each w_i $(1 \leq i < l - d_m + r_m - 3)$ to the vertex $u_{d_m-r_m-1}$ and obtain the graph G as shown in Figure 3.1.



Then $rad_m(G) = r_m$ and diam $m(G) = d_m$. Let $M = \{ u_0, u_1, \ldots, u_{d_m - r_m}, w_1, w_2, \ldots, w_{l-d_m+r_m-3}\}$ be the set of all cut vertices and end vertices of G. By Theorem 2.5, M is a subset of every connected total monophonic set of G. It is clear that M is not a connected total monophonic set of G . Also $M \cup \{x\}$, where $x \notin M$ is not a connected total monophonic set of G. Hence $M \cup \{ v_2, v_3\}$ is a connected total monophonic set of G, so that $m_{ct}(G) = l$.

Theorem 3.2: For any positive integers a, b, c with $a < b \leq c$, there exists a connected graph G such that $m(G) = a$, $m_t(G) = b$, $m_{ct}(G) = c$.

Proof: Case 1 : $a < b = c$. Let G be any tree. Then by Theorem 1.4 , $m(G) = a$, and by corollary 2.7 (i), $m_t(G) = m_{ct}(G) = b$.

Case 2: $a < b < c$. Let P_{c-b+4}: $z_1, z_2, \ldots, z_{c-b+4}$ be a path of length $c-b+4$. Let Q: x_i, y_i $(1 \leq i \leq a)$ be a path of length 1. Let H be a graph obtained from P_{c-b+4} by adding $a-2$ new vertices $\{u_1, u_2, \ldots, u_{a-2}\}$ to P_{c-b+4} and join $u_1, u_2, \ldots, u_{a-2}$ to z_2. Subdivide the edge z_2u_i, where $1 \leq i \leq b-a-2$, calling the new vertices $v_1, v_2, \ldots, v_{b-a-2}$, where u_i is adjacent to v_i and v_i is adjacent to z_2, for all $i \in \{1, 2, \ldots, b-a-2\}$. Let G be a graph shown in Figure 3.2 obtained from H by joining each x_i to z_2 and each y_i to z_4 ($1 \leq i \leq a$).

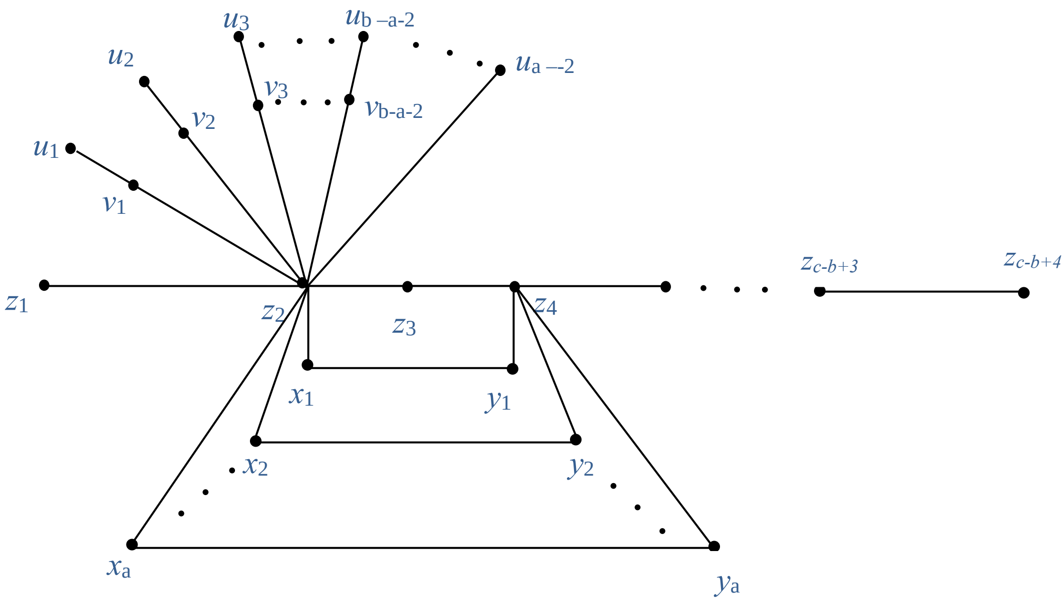
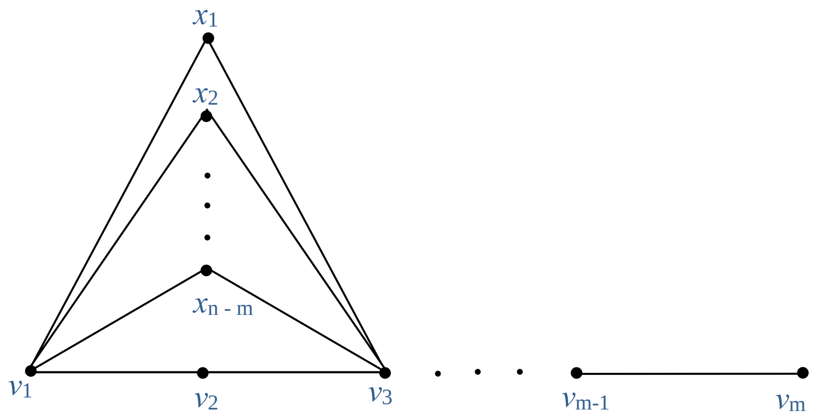


Figure 3.2

Let $M = \{z_1, u_1, u_2, \ldots, u_{a-2}, z_{c-b+4}\}$ be the set of all end vertices of G. Now it is easily seen that M is a monophonic set of G, so that $m(G) = a$. Let $M_1 = \{v_1, v_2, \ldots, v_{b-a-2}, z_2, z_{c-b+3}\}$. By Theorem 1.4, every total monophonic set contains M. Clearly $M_2 = M \cup M_1$ is a total monophonic set of G, $m_t(G) = b$. Clearly $\langle M_2 \rangle$is not connected. However $M_2 \cup \{z_3, z_4, \ldots, z_{c-b+3}\}$ is a connected total monophonic set of G , so that $m_{ct}(G) = c$.

Theorem 3.3: For every pair m, n of positive integers with $3 \leq m \leq n$, there exists a connected graph G of order n such that $m_{ct}(G) = m$.

Proof: Let P_m: $v_1, v_2, \ldots, v_m$ be a path of m vertices. Add n-m new vertices $x_1, x_2, \ldots, x_{n-m}$ and join each x_i ($1 \leq i \leq$n-m) to both v_1 and v_3 , we get the connected graph G as shown in Figure 3.3. Its order is $(n-m) + m = n$.



Clearly $M_1 = \{v_1, v_m\}$ is a monophonic set of G and $M_2 = M_1 \cup \{v_2, v_{m-1}\}$ is the toal monophonic set of G. But M_2 is not connected . Now $M_3 = M_2 \cup \{v_3, v_4, . . . , v_{m-2}\}$ is a connected total monophonic set of G. Now $|M_3| = 4 + m - 4 = m$.

REFERENCES

[1] P.Arul Paul Sudhahar, V.K.Kalai Vani, The Total Edge Monophonic Number Of a Graph, International Journal of Applied Mathematics, ISSN 1819-4966 Volume 11,Number 2 (2016) pp.159-166.

[2] F.Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, Redwood City, CA, (1990).

[3] F. Buckley, F. Harary, and L. V. *Quintas*, Extremal Results on the Geodetic Number of a Graph, Scientia A2 (1988) 17-26.

[4] G. Chartrand, F. Harary, and P. Zhang, On the Geodetic Number of a Graph, *Networks*. 39 (2002) 1-6.

[5] G. Chartrand, G.L. John, and P. Zhang, On the Detour Number and Geodetic Number of a Graph, *ArsCombinatoria*l 72 (2004) 3-15.

[6] F. Harary, E. Loukakis, and C. Tsouros, The Geodetic Number of a Graph, *Math. Comput.Modeling* 17 (11) (1993) 87-95.

[7] J. John, S. Panchali, The Upper Monophonic Number of a Graph, International J. Math, Combin. Vol. 4 (2010), 46-52.

[8] J. John, P. Arul Paul Sudhahar, A. Vijayan, The connected monophonic number of a graph, IJCGTA,Vol 5, No 1, (January-June 2012) , pp. 41-48.

[9] D.A.Mojdeh, and N.J. Rad, Connected Geodomination in Graphs, Journal of Discrete Mathematical Sciences Cryptography, 9(1), (2006), 177-186.

[10] A.P. Santhakumaran, P. Titus, K. Ganesamoorthy, On the Monophonic Number of a Graph, *J. Appl. Math. & Informatics* Vol. 32 (2014), No. 1 – 2, pp. 255 – 266.

[11] A.P. Santhakumaran, P. Titus, J. John, On the connected Geodetic number of a Graph, Journal of Com.Math. Com. Comp, 69, (2009), 219-229.

[12] A.P. Santhakumaran, P. Titus, J. John, The Upper connected Geodetic number and Forcing Connected Geodetic Number of a Graph, Discrete Applied Mathematics, 157,(2009), 1571-1580.