Global Journal of Mathematical Sciences: Theory and Practical.
ISSN 0974-3200 Volume 9, Number 3 (2017), pp. 387-397

http://www.irphouse.com

On Implicative and Strong Implicative Filters of Lattice Wajsberg Algebras

A. Ibrahim* and D. Saravanan**

** P.G. and Research Department of Mathematics,*
H. H. The Rajah's College, Pudukkottai, Tamilnadu, India.

*** Department of Mathematics, Chikkanna Government Arts College,*
Tirupur, Tamilnadu, India.

Abstract

In this paper, we discuss properties of implicative filter. We introduce the notion of strong implicative filter and investigate some properties with interesting illustrations. Also, we introduce the dual of kernel denoted as $\overline{ker}$ and derive some properties of it. We obtain the relation between an implicative filter and strong implicative filter in lattice Wajsberg algebra.

Keywords: Wajsberg algebra, Lattice Wajsberg algebra, Implicative filter, Strong implicative filter, Implication homomorphism, Lattice implication homomorphism, dual of kernel ($\overline{ker}$).

Mathematical Subject Classification 2010: 03G10, 03E70.

1. INTRODUCTION

Mordchaj Wajsbreg [6] introduced the concept of Wajsberg algebras in 1935 and studied by Font, Rodriguez and Torrens in [4]. They [4] defined lattice structure of Wajsberg algebras. Also, they [4] introduced the notion of a implicative filter of lattice Wajsberg algebras and discussed some of their properties. Basheer Ahamed and Ibrahim [1, 2] introduced the definitions of fuzzy implicative filter and an anti fuzzy implicative filter of lattice Wajsberg algebras and obtained some properties with illustrations.

In the present paper, we discuss the properties of implicative filter. We introduce the notion of strong implicative filter of lattice Wajsberg algebra, and discuss some properties with examples. Further, we discuss some related properties of implicative and strong Implicative filters of lattice Wajsberg algebra with useful illustrations. Finally, we define dual of kernel ($\overline{ker}$) of a lattice implication homomorphism and investigate the properties of implicative and strong implicative filters related to $\overline{ker}$.

2. PRELIMINARIES

In this section, we review some basic definitions and properties which are helpful to develop our main results.

Definition 2.1[4] Let $(A, \rightarrow, *, 1)$ be an algebra with quasi complement " $*$" and a binary operation "$\rightarrow$" is called Wajsberg algebra if and only if it satisfies the following axioms for all $x, y, z \in A$.

(i) $1 \rightarrow x = x$

(ii) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$

(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(iv) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$

Proposition 2.2[4] The Wajsberg algebra $(A, \rightarrow, *, 1)$ satisfies the following equations and implications for all $x, y, z \in A$.

(i) $x \rightarrow x = 1$

(ii) If $x \rightarrow y = y \rightarrow x = 1$ then $x = y$

(iii) $x \rightarrow 1 = 1$

(iv) $x \rightarrow (y \rightarrow x) = 1$

(v) If $x \rightarrow y = y \rightarrow z = 1$ then $x \rightarrow z = 1$

(vi) $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$

(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

(viii) $x \rightarrow 0 = x \rightarrow 1^* = x^*$

(ix) $(x^*)^* = x$

(x) $x^* \rightarrow y^* = y \rightarrow x$

Proposition 2.3[4] TheWajsberg algebra $(A, \rightarrow, *, 1)$ satisfies the following equations and implications for all $x, y, z \in A$.

(i) If $x \leq y$ then $x \rightarrow z \geq y \rightarrow z$

(ii) If $x \leq y$ then $z \rightarrow x \leq z \rightarrow y$

(iii) $x \leq y \rightarrow z$ iff $y \leq x \rightarrow z$

(iv) $(x \vee y)^* = (x^* \wedge y^*)$

(v) $(x \wedge y)^* = (x^* \vee y^*)$

(vi) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

(vii) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$

(viii) $(x \rightarrow y) \vee (y \rightarrow x) = 1$

(ix) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$

(x) $(x \wedge y) \rightarrow z = (x \rightarrow y) \vee (x \rightarrow z)$

(xi) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

(xii) $(x \wedge y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$

Definition 2.4[4] The Wajsberg algebra A is called a lattice Wajsberg algebra if it satisfies the following conditions for all $x, y \in A$.

(i) A partial ordering " $\leq$ "on a lattice Wajsberg algebra A, such that $x \leq y$ if and only if $x \rightarrow y = 1$

(ii) $(x \vee y) = (x \rightarrow y) \rightarrow y$

(iii) $(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$. Thus, we have $(A, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.

Definition 2.5[4] Let A be a lattice Wajsberg algebra. A subset F of A is called an implicative filter of A if it satisfies the following axioms for all $x, y \in A$.

(i) $1 \in F$

(ii) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$.

Definition 2.6[3] Let A_1 and A_2 be lattice Wajsberg algebras, $f : A_1 \longrightarrow A_2$ be a mapping from $A_1$to A_2, if for any $x, y \in A_1$, $f(x \rightarrow y) = f(x) \rightarrow f(y)$ holds, then f is called an implication homomorphism from $A_1$to A_2.

Definition 2.7[3] Let A_1 and A_2 be lattice Wajsberg algebras, $f : A_1 \longrightarrow A_2$ be an implication homomorphism from A_1 to A_2, is called a lattice implication homomorphism from A_1 to A_2 if it satisfies the following axioms for all $x, y \in A_1$

(i) $f(x \wedge y) = f(x) \wedge f(y)$

(ii) $f(x \vee y) = f(x) \vee f(y)$

(iii) $f(x^*) = [f(x)]^*$

Definition 2.8[3] Let $f : A_1 \longrightarrow A_2$ be an implication homomorphism, the kernel of f written as $Ker(f)$ is defined as $Ker(f) = \{x \in A_1 \,/\, f(x) = 0\}$.

3. MAIN RESULTS

3.1. Properties of implicative filters

In this section, we discuss and investigate some properties of implicative filter of lattice Wajsberg algebra.

Proposition 3.1.1. Let $F \neq \emptyset$ be an implicative filter of lattice Wajsberg algebra A then $x \leq y$ and $x \in F$ imply $y \in F$ for all $x, y \in A$.

Proof. Let F be an implicative filter of A. By the definition 2.5 of an implicative filter $x \leq y$ if and only if $x \rightarrow y = 1 \in F$, $x \leq y$ and $x \in F$ imply $y \in F$. □

Proposition 3.1.2. Let F be a non-empty subset of A. Then F is an implicative filter of A if and only if it satisfies for all $x, y \in F$ and $z \in A$, $x \leq y \rightarrow z$ implies $z \in F$.

Proof. Let F be implicative filter of A. By the Proposition 3.1.1, $x \leq y$ and $x \in F$ imply $y \in F$. Let $z \in A$, we have $x \leq y \rightarrow z$.

Suppose $x \leq x \rightarrow 1$ for all $x \in F$. By the definition 2.5 of implicative filter, we have $1 \in F$. Let $x \rightarrow y \in F$ and $x \in F$, $x \rightarrow (y \rightarrow y) = 1$, $x \leq x \rightarrow y$ implies $y \in F$. Therefore, we get F is an implicative filter. □

Proposition 3.1.3. Let A be a lattice Wajsberg algebra, $F \subseteq A$. F is an implicative filter of A if and only if it satisfies the following:

(i) $1 \in F$

(ii) For any $x, y, z \in A$, $x \rightarrow y \in F$, $y \rightarrow z \in F$ imply $x \rightarrow z \in F$.

Proof. Suppose that F is an implicative filter of A, $x \to y \in F$, $y \to z \in F$. By $(x \to y) \to ((y \to z) \to (x \to z)) = 1 \in F$, it follows that $(x \to z) \in F$.

Conversely, if $F \subseteq A$ satisfies (i) and (ii) we have x and $x \to y \in F$.

It follows that $1 \to x \in F$, $x \to y \in F$, and hence, we get $y = 1 \to y \in F$. □

Proposition 3.1.4. Let F be a non-empty subset of lattice Wajsberg algebra A. F is an implicative filter of A if and only if it satisfies the following:

(i) $1 \in F$

(ii) For any $x, y, z \in A$, $(z \to y) \to x \in F$, $y \in F$ imply $z \to x \in F$.

Proof. Suppose that (i) and (ii) hold. If $x \to y \in F$ and $x \in F$, then $(1 \to x) \to y \in F$, which implies $y = 1 \to y \in F$ and F is an implicative filter.

Conversely, if F is an implicative filter, and $(z \to y) \to x \in F$, $y \in F$, then by $x \leq z \to x$ and $y \leq z \to y$. It follows that $(z \to y) \to x \leq (z \to y) \to (z \to x) \leq y \to (z \to x)$ and hence, we have $y \to (z \to x) \in F$, which implies $z \to x \in F$. □

Proposition 3.1.5. Let A_1 and A_2 be any two lattice Wajsberg algebras, f is a lattice implication homomorphism from A_1 to A_2. If $F_2 \subseteq A_2$ is an implicative filter of A_2, then $f^{-1}(F_2)$ is an implicative filter of A_1.

Proof. Let 1_1 and 1_2 be the greatest elements of A_1 and A_2, respectively.

Since $f(1_1) = f(1_1 \to 1_1) = 1_2 \in F_2$, implies $1_1 \in f^{-1}(F_2)$.
If x and $x \to y \in f^{-1}(F_2)$, then, we have $f(x) \in F_2$ and $f(x) \to f(y) = f(x \to y) \in F_2$.
It follows that $f(y) \in F_2$, and hence $y \in f^{-1}(F_2)$.

Therefore, $f^{-1}(F_2)$ is an implicative filter of A_1. □

Definition 3.1.6. Let A_1 and A_2 be lattice Wajsberg algebras, 1_1 and 1_2 be the greatest elements of A_1 and A_2, respectively. Let $f: A_1 \longrightarrow A_2$ be an implication homomorphism, then dual of kernel of f written as $\overline{ker(f)}$ is defined by $\overline{ker(f)} = \{x \,/\, x \in A_1, f(x) = 1_2\}$.

Proposition 3.1.7. Let A_1 and A_2 be any two lattice Wajsberg algebras, 1_1 and 1_2 be the greatest elements of A_1 and A_2 respectively, f is an implication homomorphism from A_1 to A_2, $\overline{ker(f)}$ is an implicative filter of A_1.

Proof. Given that f is an implication homomorphism, then $f(1_1) = 1_2$, and hence $1_1 \in \overline{ker(f)}$. If x and $x \to y \in \overline{ker(f)}$, then, we have $f(x) = 1_2$ and $f(x \to y) = 1_2$

Now, $f(y) = 1_2 \to f(y)$

$$= f(x) \to f(y)$$

$$= f(x \to y) = 1_2, \text{ since f is implication homomorphism}$$

Hence, we get $y \in \overline{ker(f)}$. Therefore, $\overline{ker(f)}$ is an implicative filter of A_1. □

Note. Let f be an implication epimorphism from A_1 to A_2. f *is* a lattice implication isomorphism if and only if $\overline{ker(f)} = \{1_1\}$, where 1_1 is the greatest element of A_1.

3.2. Strong Implicative Filters

In this section, we introduce strong implicative filters of lattice Wajsberg algebra and investigate some properties with illustrations.

Definition 3.2.1. Let A be a lattice Wajsberg algebra. A subset F of A is called a strong implicative filter of A if it satisfies the following axioms for all $x, y, z \in A$.

(i) $1 \in F$

(ii) $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$.

Example 3.2.2. Let $A = \{0, a, b, 1\}$ be a set with Figure (1) as partial ordering. Define a unary operation "$*$" and a binary operation "$\to$" on A as in the Table (1) and Table (2).

x	x^*
0	1
a	b
b	a
1	0

Table (1)

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Table (2)

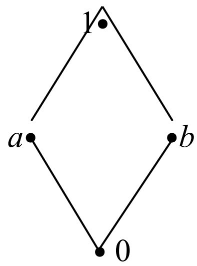

Figure (1)

Define $\vee$ and $\wedge$ operations on A as follow:

$(x \vee y) = x \to (y \to y)$ and $(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$. Then, we have A is a lattice Wajsberg algebra.

Consider the subset $F = \{a, b, 1\}$ of A, then F is a strong implicative filter of A. But, the subset $G = \{0, a, 1\}$ of A, then G is not a strong implicative filter of A. Since $1 \to (a \to 0) = b \notin$G.

Example 3.2.3. Let $A = \{0, a, b, c, 1\}$ with $0 < a < b < c < 1$, we define " $\wedge$ " and " $\vee$ " as $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ for all $x, y \in A$. Also, define a unary operation " $*$" and a binary operation "$\to$" on A as in the Table (3) and Table (4).

x	x^*
0	1
a	c
b	b
c	a
1	0

Table (3)

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	b	c	1	1
1	0	a	b	c	1

Table (4)

Then, we have A is a lattice Wajsberg algebra.

Consider the subset $F = \{b, c, 1\}$ of A, then F is a strong implicative filter of A. But, the subset $G = \{a, b, 1\}$ of A, then G is not a strong implicative filter of A. Since $b \to (1 \to a) = c \notin$ G.

Example 3.2.4. Let $A = \{0, a, b, c, d, 1\}$ be a set with Figure (2) as partial ordering. Define unary a operation " $*$" and a binary operation "$\to$" on A as in the Table (5) and Table (6).

x	x^*
0	1
a	c
b	d
c	a
d	b
1	0

Table (5)

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Table (6)

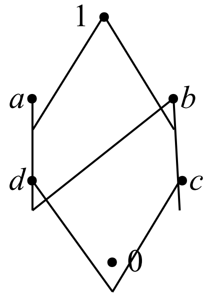

Figure (2)

Define $\vee$ and $\wedge$ operations on A as follow:

$(x \vee y) = x \to (y \to y)$ and $(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$. Then, we have A is a lattice Wajsberg algebra.

Consider the subset $F = \{b, c, 1\}$ of A, then F is a strong implicative filter of A. But the subset $G = \{c, d, 1\}$ of A, then G is not a strong implicative filter of A. Since $c \to (1 \to d) = a \notin G$.

Proposition 3.2.5. Every strong implicative filter F of A is an implicative filter.

Proof. Let F be strong implicative filter of A and $x \to y \in F$ and $x \in F$ for all $x, y \in A$. Replace z by y in the definition 3.2.1.

Then, we have $x \to (y \to z) = x \to (y \to y)$

$$= x \to 1$$

$$= x \in F.$$

Also, $x \to y \in F$. Then, we get $x \to z \in F$.

Thus, every strong implicative filter is an implicative filter. □

Note. The converse may not be true. In Example 3.2.3, $\{1\}$ is an implicative filter but not a strong implicative filter since $a \to (b \to 0) = a \to b = 1 \in \{1\}$, and $a \to 0 = c \notin \{1\}$.

Proposition 3.2.6. Let F be an implicative filter of A such that $x \to (y \to (y \to z)) \in F$ and $x \in F$ imply $x \to z$ for all $x, y, z \in A$. Then F is a strong implicative filter of A.

Proof. Let $x \to (y \to (y \to z)) \in F$ and $x \to y \in F$ for all $x, y, z \in A$.

From (vii) of definition 2.2 and (ii) of definition 2.1, we have

$$x \to (y \to z) = y \to (x \to z) \leq (x \to y) \to (x \to (x \to z)).$$

By the Proposition 3.1.1 we get, $(x \to y) \to (x \to (x \to z)) \in F$.

Since $x \to y$, we have $x \to z \in F$.Therefore, F is a strong implicative filter. □

Next, we show the definition of the closed interval $[a, 1]$.

Definition 3.2.7. Let A be a lattice Wajsberg algebra, $a \in A$. The interval $[a, 1]$ defined as $[a, 1] = \{x / x \in A, a \leq x\}$ of A denoted as $I(a)$.

Proposition 3.2.8. Let A be a lattice Wajsberg algebra, $a \in A$, then $\{1\}$ is a strong implicative filter of A if and only if $I(a)$ is an implicative filter of A for any $a \in A$.

Proof. Suppose that $\{1\}$ is a strong implicative filter of A. For any $a \in A$, $1 \in A$ is trivial. If $x \in I(a)$ and $x \to y \in I(a)$, then $a \leq x$, $a \leq x \to y$, That is, $a \to x = 1 \in \{1\}$ and $a \to (x \to y) = 1 \in \{1\}$. It follows that $a \to y \in \{1\}$, $a \leq y$, and hence $y \in I(a)$. Thus, $I(a)$ is an implicative filter of A.

Conversely, assume that $I(a)$ is an implicative filter of A for any $a \in A$. For any $x, y, z \in A$, if $x \to (y \to z) \in \{1\}$ and $x \to y \in \{1\}$, then $x \leq y \to z$, $x \leq y$, it follows that $x \leq z$ because $I(x)$is an implicative filter, hence $x \to z = 1 \in \{1\}$. Therefore, we have $\{1\}$ is a strong implicative filter of A. □

Proposition 3.2.9. Let A be a lattice Wajsberg algebra, $F \subseteq A$. The following statements are equivalent

(i) F is a strong implicative filter

(ii) F is an implicative filter and for any $x, y \in A$, $x \to (x \to y) \in F$ implies $x \to y \in F$

(iii) F is an implicative filter and for any $x, y, z \in A$, $x \to (y \to z) \in F$ implies $(x \to y) \to (x \to z) \in F$

(iv) $1 \in F$ and for any $x, y, z \in A$, $z \to (x \to (x \to y)) \in F$ and $z \in F$ imply $x \to y \in F$.

Proof.

(i) ⇒ (ii).

For any $x, y \in A$, if $x \to (x \to y) \in$ F, since $x \to x = 1 \in F$, from (ii) of definition 3.2.1 we have $x \to y \in F$.

(ii) ⇒ (iii).

Assume that (ii) holds. For any $x, y, z \in A$, suppose $x \to (y \to z) \in F$, from (vii) of proposition 2.2, (i), (ii) of proposition 2.3 and (vi) of proposition 2.2, we have $x \to (y \to z) \leq x \to \big((x \to y) \to (x \to z)\big)$. Therefore, from proposition 3.1.1 and (vii) of proposition 2.2, we get $x \to (x \to ((x \to y) \to z)) = x \to ((x \to y) \to (x \to z)) \in F$.

From (ii) and (vii) of proposition 2.2, we have

$$x \to ((x \to y) \to z) = (x \to y) \to (x \to z) \in F.$$

(iii) ⇒ (iv).

Now assume that (iii) holds and we prove (iv). In fact, $1 \in F$ is trivial. For any $x, y, z \in A$, suppose $z \to (x \to (x \to y)) \in F$ and $z \in F$, then $x \to (y \to z) \in F$. From (ii) of definition 2.5, we have $x \to y = 1 \to (x \to y) = (x \to x) \to (x \to y) \in F$.

(iv) ⇒ (i).

Suppose $x \in F$ and $x \to y \in F$, then we get $x \to (1 \to (1 \to y)) = x \to y \in F$, It follows that $y = 1 \to y \in F$ and hence F is an implicative filter. For any $x, y, z \in A$, $x \to (y \to z) \in F$ and $x \to y \in F$,

Now, $(x\to (y\to z)) \to ((x\to y) \to (x \to (x\to z)))$

$$= (x\to y) \to ((x\to (y \to z)) \to (x \to (x\to z)))$$

$$= (x\to y) \to ((y\to (x \to z)) \to (x \to (x\to z)))$$

$$= (x\to y) \to (x\to (y\vee (x\to z)))$$

$$= (x\to y) \to ((x\to y) \vee (x\to (x\to z))) = 1 \in F.$$

It follows that $(x \to y) \to (x \to (x \to z) \in F$ and hence $x \to z \in F$ by $x \to y \in F$ and (iv).□

Proposition 3.2.10. Let A be a lattice Wajsberg algebra, F_1 and F_2 are any two implicative filters of A, $F_1 \subseteq F_2$. If F_1 is a strong implicative filter, so is F_2.

Proof. Suppose $x \to (x\to y) \in F_2$, we only to prove $x \to y \in F_2$.
Now $x\to (x\to ((x \to (x \to y))\to y)) = (x \to (x \to y)) \to (x \to (x \to y)) = 1 \in F_1$.

It follows that $x \to ((x \to (x \to y))\to y) \in F_1 \subseteq F_2$.
That is, $(x \to (x \to y)) \to (x \to y) \in F_2$ and hence $x \to y \in F_2$. Therefore, we have F_2 is a strong implicative filter. □

4. CONCLUSION

In this paper, we have introduced the notion of strong implicative filter of lattice Wajsberg algebra, and discussed some of their properties with examples. Further, we have shown that some related properties of implicative and strong implicative filters of lattice Wajsberg algebra with useful illustrations. Finally, we have defined $\overline{ker}$ of a lattice implication homomorphism and investigate the properties of implicative and strong implicative filters related to $\overline{ker}$.

REFERENCES

[1] BasheerAhamed. M and Ibrahim. A, *Fuzzy implicative filters of lattice Wajsberg Algebras,* Advances in Fuzzy Mathematics, Volume 6, Number 2 (2011), 235-243.

[2] BasheerAhamed. M and Ibrahim. A, *Anti fuzzy implicative filters in lattice W-algebras,* International Journal of Computational Science and Mathematics, Volume 4, Number 1 (2012), 49-56.

[3] Ibrahim. A, and C. Shajitha Begam, *On WI-ideals of Lattice Wajsberg algebras,* Global Journal of Pure and Applied Mathematics, Volume13, Number 10 (2017), 7237-7256.

[4] Font. J. M. Rodriguez. A. J and Torrens. A,*Wajsberg algebras,*STOCHASTICA Volume VIII, Number 1 (1984), 5-31

[5] Jun. Y. B, *Implicative filters of lattice implication algebras,*Bull. Korean Math. Soc. 34, Number 2 (1997), 193-198.

[6] Wajsberg. M, *BeitragezumMetaaussagenkalkul*1, Monat. Mat. phys. 42 (1935) page 240.

[7] Yang Xu, *Lattice implication algebras,* J. of Southwest Jiaotong Univ. 28 Number 1(1993), 20-27.