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On the Entropy Numbers of a Bounded Linear Operator and Its Conjugate

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Abstract

We recall the problem of existence of some positive constant L such that the inequalities $e_n(T^*) \le Le_n(T)$ and $e_n(T) \le Le_n(T^*)$ hold. We show that if $\|I - A\| \prec 1$ then $e_n(T^*) \le 4\|T\|\|T^{-1}\|e_n(T)$ and $e_n(T) \le 4e_n\|T\|\|T^{-1}\|e_n(T^*)$.

INTRODUCTION

Recall that for T a bounded linear operator on the Banach space X to the Banach space Y, n a natural number, the nth entropy number of T is defined by $e_n(T) = \inf\{\varepsilon > 0 : \exists y_i, i = 1,...,2^{n-1}, T(B_X) \subset \cup\{y_i + \varepsilon B_Y : 1 \le i \le 2^{n-1}\}$. Here, B_X stands for the closed unit ball of X and analogously for B_Y. We consider Banach spaces X,Y and we write $T : X \to Y$ for T a (bounded linear) operator on X to Y. In this paper we obtain the inequality in the Abstract.

Preliminaries

The entropy numbers of T give information about the compactness of the operator. In fact, X being a Banach space it follows that T is compact if and only if it is totally bounded that is, if and only if $e_n(T) \to 0$.

Also recall that in the Banach space setting, the conjugate of $T : X \to Y$ is the operator $T^* : Y^* \to X^*$ defined by $\prec T^* y^*, x \succ = \prec y^*, Tx \succ, y^* \in Y^*, x \in X$. $\prec x^*, x \succ$ for the duality. A problem that has been considered for long is the connection between

$e_n(T)$ and $e_n(T^*)$ namely, if there exist constants C and L such that $e_m(T^*) \le Ce_n(T)$ and $e_n(T) \le Le_n(T^*)$. You can find in (2) that for T a compact operator on the Hilbert space H to H it holds that $e_n(T) = e_n(T^*)$.

We own to Kuratowski the concepts

The ball measure of non-compactness of the bounded subset A of the Banach space X is $\psi_X(A) = \inf\{\delta > 0 : \exists x_i, i = 1,...,n, A \subset \cup\{B(x_i,\delta) : 1 \le i \le n\}$. Clearly that A is compact if and only if $\psi_X(A) = 0$. For X and T Banach spaces and an operator $T : X \to Y$ we say that T is a k-ball contraction if and only if for every bounded subset B of X we have

(k(T))≡ $\psi_Y(T(B)) \le k\psi_X(B)$. We say that $\beta(T) = \inf\{k > 0 : k(T)\}$ is the ball measure of non compactness of T. Clearly that T is compact if and only if $\beta(T) = 0$.

Remark 1.1

For X and Y Banach spaces, T a bounded linear operator on X to Y it holds that lim $e_n(T) = \beta(T)$.

Proof: This follows easily from the definitions.

Remark 1.2

H being a Hilbert space and T a bounded linear operator on H to H it holds that $\beta(T) = \beta(T^*)$.

Following Edmunds and Evans in (1) for X a Banach space, X real with dimX=m and $I : X \to X$ the identity operator it holds that $1 \le 2^{(n-1)/m} e_n(I) \le 4$.

Following these authors, Gordon, König and Schütt (3) have proved, using work by Carl that for X and Y of type 2, there exist constants c and d depending only on X and Y such that for all $n = 1,2,...$,

$d^{-1}e_{[nc]}(T) \le e_n(T^*) \le de_{[nc]}(T)$, where [nc] stands for the integer part of nc.

Edmunds and Tylli have shown in (2) that for general Banach spaces and all natural numbers k and n,

$e_n(T^*) \le 2(1 + c(k,2^n))e_n(T) + \|T\|c(k,2^n)$ and

$e_n(T) \le 4(1 + c(k,2^n))e_n(T^*) + 2\|T\|c(k,2^n)$ where

$c(k,m) = \min\{1, 2(2^{(k-1)/m} - 1)^{-1}\}$.

Also recall in (5) that we say the sequence (x_n) in the real Banach space X is a Schauder basis (a basis) of X if and only if there exists a scalar sequence (λ_n) for each x in X such that, $x = \sum_n \lambda_n x_n$ in the sense that $\left\| x - \sum_{n=1}^{N} \lambda_n x_n \right\| \to_{N\to\infty} 0$.

Remark 1.3

The natural projections $P_N \sum_n \lambda_n x_n = \sum_{n=1}^{N} \lambda_n x_n$ are bounded with $\sup\{\|P_N\| : N = 1,2,...\} = \|P\|$.

The Results

In what follows we consider first the case where T is an invertible bounded linear operator on the Banach space X to itself.

Theorem 2.1

Letting X be a real N-dimensional Banach space and $T : X \to X$ be an invertible operator it holds that $e_n(T^*) \le 4\|T\|.\|T^{-1}\| e_n(T)$ and $e_n(T) \le 4\|T\|.\|T^{-1}\| e_n(T^*)$.

Proof: We have that T^* is invertible with $(T^*)^{-1} = (T^{-1})^*$ by (6) (p. 227). Using the above inequalities we find that $2^{(1-n)/m} e_n(T^{-1}T) \le 4\|T^{-1}\| e_m(T)$. Also letting I stand for the identity operator on X to X we have

$e_n(T^*) = e_n(T^* I^*) \le \|T^*\| e_n(I^*) \le \|T^*\| 4.2^{-(n-1)/m}$ thus

$e_n(T^*) \le 4\|T^*\|.\|T^{-1}\| e_n(T) = 4\|T^{-1}\|.\|T\| e_n(T)$ where we apply Theorem 8.2, p. 230 in (6). Analogously, we obtain that $e_n(T) \le 4\|T\|.\|T^{-1}\| e_n(T^*)$ and the theorem follows as wished.

Also recall in (6) that for an operator $T : X \to X$ such that $\|I - T\| < 1$, X a Banach space it holds that there exists the bounded linear inverse operator $T^{-1} : X \to X$.

Lemma 2.2

For T and S bounded linear operators on X to X it holds that $| \; e_n(T) - e_n(S) \; \} \le \|T - S\|$.

ProoF: This follows easily from the definitions.

X having a basis (x_n) and given an operator $T: X \to X$ it holds that $\|P_N Tx - Tx\| \geq \|T\|.\|(I - P_N)x\| \to_{N\to\infty} 0$ for each x in X. Also, $TP_N = P_N T$ and $\|T(I - P_N)\| \leq \|T\|.\|(I - P_N\| \to_{N\to\infty} 0$. This follows by (6), just after Theorem 1.2, pp. 190-1.

Theorem 2.3

For T a bounded linear operator on the real infinite dimensional Banach space X to X such that $\|I - T\| < 1$ it holds that

$e_n(T^*) \leq 4\|T\|.\|T^{-1}\|e_n(T)$ and $e_n(T) \leq 4\|T\|.\|T^{-1}\|e_n(T^*), n = 1,2,...$

Proof: This follows from theorem 2.1 and the above, since we find that $e_n(T^* : X \to X) = \lim_{N\to\infty} e_n(T : R(P_N) \to X) \leq \lim_{N\to\infty} 4\|T\|.\|T^{-1}\|e_n(T : X \to X)$ and analogously interchanging T and T^*.

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