International Journal of Engineering Research and Technology. ISSN 0974-3154, Volume 12, Number 10 (2019), pp. 1626-1638

Laplace Transform Method for the Elastic Buckling Analysis of Moderately Thick Beams

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Abstract

The elastic buckling problem of moderately thick plates, presented as a classical problem of the mathematical theory of elasticity is solved in this work using the Laplace transform method. The governing equation solved was a fourth order ordinary differential equation (ODE) and solutions were obtained for various end support conditions, namely fixed-fixed ends, fixed-pinned ends, pinned-fixed ends and pinned-pinned ends. Application of the Laplace transformation to the governing domain equation simplified the ODE to an algebraic equation in the Laplace transform space. Inversion yielded the general solution in the physical domain space in terms of the initial values of the buckled deflection and its derivatives. Boundary conditions for the considered end support conditions were then used on the general solution, reducing the problem to algebraic eigenvalue problem represented by a system of homogeneous equations. The condition for nontrivial solution was used to obtain the characteristic buckling equation for each considered boundary condition. The characteristic buckling equations were solved using Mathematica and other mathematical and computational software tools to obtain the first four eigenvalues. The least eigenvalue for each case considered was used to obtain the critical elastic buckling loads for rectangular and circular cross-sections; which were presented for each considered case in Tables. It was found that for $t/l < 0.02$ and $d/l < 0.02$ for various end support conditions considered, the critical elastic buckling load coefficient obtained approximated the corresponding solutions for the Bernoulli-Euler beam. For $t/l > 0.02$ and $d/l > 0.02$, the critical elastic buckling load coefficients obtained for the various end support conditions were smaller than the corresponding values from the Bernoulli-Euler theory. The Bernoulli-Euler theory was thus found to overestimate the critical elastic buckling load capacities of moderately thick beams for the end support conditions considered; and this is due to the effect of shear deformation on the elastic buckling load capacities which were disregarded in the Bernoulli-Euler theory but considered in the present study.

Keywords-algebraic eigenvalue problem, characteristic buckling equation, critical elastic buckling load coefficient, Laplace transform method, moderately thick beam.

I. INTRODUCTION

Elastic buckling problems of thick and moderately thick beams and beam columns are basically problems of the mathematical theory of elasticity. Their governing equations are derived using the fundamental equations of the theory of elasticity – namely: the constitutive laws, the kinematics relations, and the differential equations of equilibrium [1 – 10].

Theories that have been presented for the buckling of beams include: (i) the classical Bernoulli-Euler beam theory, (ii) Timoshenko beam theory, (iii) Mindlin beam theory, (iv) shear deformation beam theories presented by Levinson [11], Krishna Murty [12], Heylinger and Reddy [13], Ghugal [14], Ghugal and Shimpi [15], Sayyad and Ghugal [16], Ghugal and Sharma [17], Soldatos [18], and (v) unified beam theory (UBT) presented by Sayyad [19] and Sayyad and Ghugal [20].

The Bernoulli-Euler beam theory (BEBT) was developed using the hypothesis that plane cross-sections initially orthogonal to the undeformed neutral axis remains plane and orthogonal to the neutral axis after deformation. The implication of the orthogonality hypothesis is that the effects of transverse shear deformation on the flexural, stability and dynamic behaviour of the beam are disregarded. Thus the BEBT is limited in scope of application to thin beams where the thickness, t, to span, l, ratios are less than 0.05 (1/20), ($t/l < 0.05$) and where transverse shear deformation effects make insignificant contributions to the stability, flexural and dynamic behaviours [21 – 23]. The BEBT thus does not produce realistic and good estimates of flexural displacements, frequencies and buckling load capacities of thick and moderately thick beams since shear deformation effects have remarkable contributions to their flexural, dynamic and buckling responses [24 – 27].

Other theories were developed primarily to improve on the BEBT and address its limitations especially, to include and consider the effect of transverse shear deformations.

The Timoshenko beam theory (TBT) a first order shear deformation theory (FSDT) for beams was developed using the relaxation or slight modification of the Bernoulli-Euler beam orthogonality hypothesis. In TBT, the hypothesis used is that plane cross-sections that are initially orthogonal to the undeformed neutral axis (middle axis) of the beam would remain plane but would not necessarily be orthogonal to the neutral axis after deformation. Timoshenko theory assumes that the transverse shear strain is constant through the beam thickness, thus leading to non vanishing of transverse shear stresses on the top and bottom surfaces of the beam, and violating the transverse shear stress free boundary conditions of some beam stability, dynamics and flexure problems.

Shear modification factors have been used in the TBT and other first order shear deformation beam theories to express the transverse shear forces and thus appropriately represent the strain-energy of deformation [19]. Another feature of the TBT and other FSDT of beams is that they are expressed using two unknown displacement functions.

The shortcomings of both the BEBT, TBT and other FSDT of beams inspired the research into the development of higher order shear deformation theories of moderately thick and thick beams. Levinson [11], Krishna Murty [12], Heylinger and Reddy [13] and others formulated parabolic shear deformation theories for moderately thick and thick beams for stability, vibration and flexural behaviours. Ghugal [14] formulated an extension of the parabolic shear deformation theory for thick and moderately thick beams to incorporate transverse, normal strains and transverse shear strain effects for the bending, vibration and stability analysis of isotropic beams. Ghugal and Shimpi [15] developed a trigonometric shear deformation theory incorporating transverse shear deformation effects and which could be useful for the flexural, vibration and stability analysis of moderately thick and thick beams.

Sayyad and Ghugal [16] developed a trigonometric shear and normal deformation theory which included the effects of transverse shear and normal deformation for the analysis of moderately thick and thick isotropic and laminated beams. Karama et al. [28] derived an exponential shear deformation theory for the analysis of moderately thick and thick beams subjected to flexural, vibrating and stability conditions and for various boundary conditions.

Soldatos [18] developed a hyperbolic shear deformation theory for moderately thick and thick beams under flexural, vibrating and stability cases and isotropic, homogenous beam materials for various boundary conditions. Sayyad and Ghugal [20] derived a unified beam theory (UBT) for the flexural, vibration and buckling analysis of moderately thick and thick beams. In the UBT, parabolic, hyperbolic, exponential and trigonometric functions are used in terms of the thickness coordinates to represent the effect of transverse shear deformation, thus assuring the applicability of the UBT to the moderately thick and thick beams.

In this work, the elastic buckling problem of an isotropic homogeneous beam with a prismatic cross-section is derived from first principles from a simultaneous application of the differential equations of equilibrium, kinematics relations and the material constitutive laws, and shown to be a boundary value problem (BVP) of the mathematical theory of elasticity. The BVP is represented by a fourth order ordinary differential equation. The BVP is solved using the method of Laplace transformation to simplify the problem to an algebraic eigenvalue problem.

II.THEORETICAL FORMULATION

Moderately thick beam considered

The beam as shown in Figure 1, is defined using three dimensional (3D) Cartesian coordinates:

$$0 \le x \le l, -b/2 \le y \le b/2, -t/2 \le z \le t/2$$

where x, y, z are the 3D Cartesian coordinates, l is the beam length, b is the width, and t is the beam thickness. The cross-section lies on the yz plane while the x-axis is the longitudinal axis of the beam. The beam is assumed to be subject to various boundary conditions.

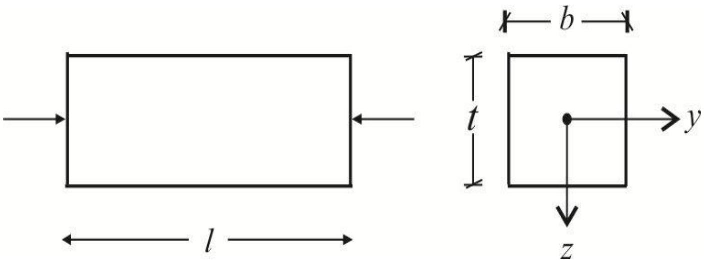

Figure 1: Elastic buckling of moderately thick beam under compressive loads

Fundamental assumptions

The assumptions of the formulation include:

(i) the inplane displacement is made up of a pure bending displacement component, and a component due to shear deformation.

(ii) the transverse displacement component in the z coordinate direction depends only on the position in the longitudinal coordinate axis.

(iii) stress-strain equations are one dimensional.

(iv) transverse displacements are so small as compared to the beam thickness that transverse strains are assumed to be insignificant.

(v) body forces are neglected.

(vi) transverse displacement field $w(x, z)$ has two components namely: flexural or bending component $w_b(x)$ and shear compoment $w_s(x)$.

(vii) the beam is made of homogeneous, linear elastic, isotropic material.

Displacement field

The x, y, and z Cartesian components of the displacement field are given by:

$$u(x,z) = u(x) + z\phi(x) \tag{1}$$

$$v(x,z) = 0 \tag{2}$$

$$w(x,z) = w_b(x) + w_s(x) = w(x) \tag{3}$$

where $\phi(x)$ is the rotation of the beam's cross-section at the neutral axis, $u(x, z)$, $v(x, z)$ and $w(x, z)$ are the x, y, and z components of the displacement $u(x)$ is the axial displacement corresponding to pure bending and $z\phi(x)$ is the displacement due to the shear deformation.

The rotation $\phi(x)$ is

$$\phi(x) = -\frac{dw_b(x)}{dx} \tag{4}$$

Strain fields from the kinematic relations

From the small displacement assumptions of elasticity theory, the strain fields are obtained using the kinematic relations as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w_b}{\partial x^2} \tag{5}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \tag{6}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \tag{7}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \tag{8}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \tag{9}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial w_s(x)}{\partial x} \tag{10}$$

where ε_{xx}, ε_{yy}, ε_{zz} are normal strains, while γ_{xy}, γ_{yz}, γ_{xz} are shear strains.

Stress fields

The stress fields are obtained from the stress-strain relations for one dimensional problems as:

$$\sigma_{xx} = E\varepsilon_{xx} = E\frac{du(x)}{dx} - Ez\frac{d^2 w_b(x)}{dx^2} \qquad 11)$$

$$\tau_{xz} = G\gamma_{xz} = G\frac{dw_s(x)}{dx} \tag{12}$$

$$\sigma_{yy} = \sigma_{zz} = \tau_{xy} = \tau_{xz} = 0 \tag{13}$$

where G is the shear modulus, and E is the Young's modulus of elasticity, σ_{xx}, σ_{yy} and σ_{zz} are normal stresses, while $\tau_{xy}, \tau_{xz}, \tau_{yz}$ are shear stresses.

Stress resultants

The bending moment $M_{xx}(x)$ is:

$$M_{xx}(x) = \int_{-t/2}^{t/2}\int_{-b/2}^{b/2} \sigma_{xx}\, z\, dy\, dz \tag{14}$$

For buckling problems,

$$M_{xx}(x) = \int_{-t/2}^{t/2}\int_{-b/2}^{b/2} -Ez^2\frac{d^2 w_b(x)}{dx^2}\, dy\, dz = -EI\frac{d^2 w_b(x)}{dx^2} \tag{15}$$

where $$I = \int_{-t/2}^{t/2}\int_{-b/2}^{b/2} z^2\, dy\, dz \tag{16}$$

where I is the moment of inertia.

Introduction of a shear modification factor k_s leads to the expression of the shear force Q_x as:

$$Q_x = k_s\int_{-t/2}^{t/2}\int_{-b/2}^{b/2} \tau_{xz}\, dy\, dz = k_s GA\frac{dw_s(x)}{dx} \tag{17}$$

where A is the beam cross-sectional area.

Differential equation of equilibrium

When body forces are neglected, the equilibrium equation is:

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{18}$$

By integration,

$$\iint_{R^2}\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}\right) z\, dy\, dz = 0 \tag{19}$$

$$\frac{dM_{xx}}{dx} - Q_x = 0 \tag{20}$$

$$-\frac{d}{dx}\left(EI\,\frac{d^2 w_b(x)}{dx^2} \right) = k_s AG\,\frac{dw_s(x)}{dx} \tag{21}$$

Hence, by integration,

$$w_s(x) = -\frac{EI}{k_s AG}\frac{d^2 w_b(x)}{dx^2} = -\frac{2(1+\mu)I}{k_s A}\frac{d^2 w_b(x)}{dx^2} \tag{22}$$

where μ is the Poisson's ratio.

From the differential equation of equilibrium in the *z* direction,

$$\frac{dQ_x}{dx} + q = 0 \tag{23}$$

where *q* is the applied transverse load.

Stability equation

The stability equation is given by:

$$N_x\,\frac{dw(x)}{dx} + \frac{dM_{xx}}{dx} - Q_x = 0 \tag{24}$$

where N_x is the axial load.

$$\frac{dQ_x}{dx} = 0 \tag{25}$$

if $q = 0$, there is no applied transverse load on the beam.

Substitution and simplification gives:

$$N_x\left(\frac{d^2 w_b(x)}{dx^2} + \frac{d^2 w_s(x)}{dx^2} \right) - EI\,\frac{d^4 w_b(x)}{dx^4} = -q(x) \tag{26}$$

Substitution for $w_s(x)$ and simplification gives: for compressive load P_x

where $P_x = -N_x$

$$\left(EI - \frac{P_x t^2(1+\mu)}{6k_s} \right)\frac{d^4 w_b(x)}{dx^4} + P_x\,\frac{d^2 w_b(x)}{dx^2} = q(x) \tag{27}$$

for rectangular cross-sections, and

$$\left(EI - \frac{P_x d^2(1+\mu)}{8k_s} \right)\frac{d^4 w_b(x)}{dx^4} + P_x\,\frac{d^2 w_b(x)}{dx^2} = q(x) \tag{28}$$

for circular cross-sections. When transverse loads $q(x)$ are absent, $q(x) = 0$, and the governing equations become:

$$\left(EI - \frac{P_x t^2(1+\mu)}{6k_s} \right)\frac{d^4 w_b(x)}{dx^4} + P_x\,\frac{d^2 w_b(x)}{dx^2} = 0 \tag{29}$$

$$\left(EI - \frac{P_x d^2(1+\mu)}{8k_s} \right)\frac{d^4 w_b(x)}{dx^4} + P_x\,\frac{d^2 w_b(x)}{dx^2} = 0 \tag{30}$$

For rectangular cross-sections, k_s = 5/6 while for circular cross-sections,

$$k_s = \frac{6(1+\mu)}{7+6\mu} \tag{31}$$

$$k_s(\mu = 0.25) = 0.8824 \tag{32}$$

$$k_s(\mu = 0.30) = 0.8864 \tag{33}$$

Hence, for circular cross-sections, $k_s \simeq 0.90$ (34)

Then, the governing equations for elastic buckling of moderately thick beams are:

$$\left(EI - \frac{P_x t^2(1+\mu)}{5} \right)\frac{d^4 w_b(x)}{dx^4} + P_x\,\frac{d^2 w_b(x)}{dx^2} = 0 \tag{35}$$

for rectangular cross-sections, and

$$\left(EI - \frac{P_x d^2(1+\mu)}{7.2} \right)\frac{d^4 w_b(x)}{dx^4} + P_x\,\frac{d^2 w_b(x)}{dx^2} = 0 \tag{36}$$

for circular cross-sections.

III. METHODOLOGY

The governing equations for elastic buckling problems are presented generally as:

$$\frac{d^4 w_b(x)}{dx^4} + \beta^2\,\frac{d^2 w_b(x)}{dx^2} = 0 \tag{37}$$

where
$$\beta^2 = \frac{P_x}{EI - \dfrac{P_x t^2(1+\mu)}{5}} \tag{38}$$

for rectangular cross-sections, and

$$\beta^2 = \frac{P_x}{EI - \dfrac{P_x d^2(1+\mu)}{7.2}} \tag{39}$$

for circular cross-sections.

By application of the Laplace transformation, the governing equation is transformed to the Laplace integral.

$$\int_0^\infty e^{-sx}\left(\frac{d^4 w_b(x)}{dx^4} + \beta^2\,\frac{d^2 w_b(x)}{dx^2} \right) dx = 0 \tag{40}$$

By the linearity property of the Laplace transform, it is obtained that:

$$\int_0^\infty e^{-sx}\frac{d^4w_b(x)}{dx^4}dx+\int_0^\infty e^{-sx}\beta^2\frac{d^2w_b(x)}{dx^2}dx=0 \qquad (41)$$

Hence:

$$\int_0^\infty e^{-sx}\frac{d^4w_b(x)}{dx^4}dx+\beta^2\int_0^\infty e^{-sx}\frac{d^2w_b(x)}{dx^2}dx=0 \qquad (42)$$

Simplifying,

$$s^4W(s)-s^3w(0)-s^2w'(0)-sw''(0)-$$
$$w'''(0)+\beta^2\left(s^2W(s)-sw(0)-w'(0)\right)=0 \qquad (43)$$

where

$$W(s)=\int_0^\infty e^{-sx}w_b(x)dx=\mathcal{L}\, w_b(x) \qquad (44)$$

$W(s)$ is the Laplace transform of $w_b(x)$

Simplifying Equation (4), we have:

$$W(s)=\frac{w(0)}{s}+\frac{w'(0)}{s^2}+\frac{w''(0)}{s(s^2+\beta^2)}+\frac{w'''(0)}{s^2(s^2+\beta^2)} \qquad (45)$$

Hence, by inversion,

$$w_b(x)=\mathcal{L}^{-1}W(s)=w(0)\mathcal{L}^{-1}\frac{1}{s}+w'(0)\mathcal{L}^{-1}\frac{1}{s^2}+$$
$$w''(0)\mathcal{L}^{-1}\frac{1}{s(s^2+\beta^2)}+w'''(0)\mathcal{L}^{-1}\frac{1}{s^2(s^2+\beta^2)} \qquad (46)$$

$$w_b(x)=w_b(0)+xw_b'(0)+$$
$$w_b''(0)\left(\frac{1-\cos\beta x}{\beta^2}\right)+w'''(0)\left(\frac{x}{\beta^2}-\frac{\sin\beta x}{\beta^3}\right) \qquad (47)$$

IV. RESULTS

Results for moderately thick beams with fixed-fixed ends

The boundary conditions for moderately thick beams with fixed (clamped) ends at x = 0, and $x = l$, as shown in Fig 2, are:

$$w_b(0)=w_b'(0)=0 \qquad (48)$$

$$w_b(l)=w_b'(l)=0 \qquad (49)$$

Figure 2: Moderately thick beam with fixed (clamped) ends at x = 0, and $x = l$ under compressive load

From Equation (48), the buckled deflection function is

$$w_b(x)=w_b''(0)\left(\frac{1-\cos\beta x}{\beta^2}\right)+w_b'''(0)\left(\frac{x}{\beta^2}-\frac{\sin\beta x}{\beta^3}\right) \qquad (50)$$

Using the boundary conditions, Equation (49), the problem reduces to the algebraic eigenvalue problem given by the system of homogeneous equations:

$$\begin{pmatrix}\left(\frac{1-\cos\beta l}{\beta^2}\right) & \frac{1}{\beta^2}\left(l-\frac{\sin\beta l}{\beta}\right)\\ \frac{\sin\beta l}{\beta} & \left(\frac{1-\cos\beta l}{\beta^2}\right)\end{pmatrix}\begin{pmatrix}w_b''(0)\\ w_b'''(0)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \qquad (51)$$

For nontrivial solutions, the characteristic bucking equation is given by:

$$\begin{vmatrix}\left(\frac{1-\cos\beta l}{\beta^2}\right) & \frac{1}{\beta^2}\left(l-\frac{\sin\beta l}{\beta}\right)\\ \frac{\sin\beta l}{\beta} & \left(\frac{1-\cos\beta l}{\beta^2}\right)\end{vmatrix}=0 \qquad (52)$$

Expansion of the determinant and simplification, yields the characteristic buckling equation as the transcendental equation

$$\beta l\sin\beta l+2\cos\beta l-2=0 \qquad (53)$$

The characteristic buckling equation has an infinite number of roots each corresponding to the buckling modes. The first four nontrivial roots of the transcendental buckling equation obtained using Newton-Raphson's iteration, Mathematica and other mathematical and computational software tools are given by:

$$\beta_1 l=\pm 6.28318530717959 \qquad (54)$$

$$\beta_2 l=\pm 8.98681891581813 \qquad (55)$$

$$\beta_3 l=\pm 12.5663706143592 \qquad (56)$$

$$\beta_4 l=\pm 15.4505036738754 \qquad (57)$$

The least eigenvalue β_1 is used to determine the critical elastic buckling load P_{xcr}. For rectangular cross-sections:

$$\beta_1^2=\left(\pm\frac{6.283185}{l}\right)^2=\left(\frac{P_{x_{cr}}}{EI-\frac{P_{x_{cr}}t^2(1+\mu)}{5}}\right) \qquad (58)$$

Simplifying,

$$P_{x_{cr}}=\frac{39.478414}{1+7.89568(1+\mu)\left(\frac{t}{l}\right)^2}\frac{EI}{l^2}=K_{cr_1}\times\frac{EI}{l^2} \qquad (59)$$

For μ = 0.25,

$$P_{x_{cr}}(\mu=0.25)=\frac{39.478414}{1+9.8696\frac{t^2}{l^2}}\frac{EI}{l^2} = K_{cr_1}(\mu=0.25)\frac{EI}{l^2} \quad (60)$$

For μ = 0.30,

$$P_{x_{cr}}(\mu=0.30)=\frac{39.478414}{1+10.264384\left(t/l\right)^2}\frac{EI}{l^2} = K_{cr_1}(\mu=0.30)\frac{EI}{l^2} \quad (61)$$

where K_{cr1} is the critical elastic buckling load coefficient for moderately thick beam with fixed-fixed ends and rectangular cross-section.

For circular cross-sections,

$$\beta_1^2=\left(\pm\frac{6.283185}{l}\right)^2=\frac{P_{x_{cr}}}{EI-\frac{P_{x_{cr}}(1+\mu)d^2}{7.2}} \quad (62)$$

Hence,

$$P_{x_{cr}}=\frac{39.478414}{1+5.483111(1+\mu)\left(d/l\right)^2}\frac{EI}{l^2}=K_{cr_2}\frac{EI}{l^2} \quad (63)$$

$$P_{x_{cr}}(\mu=0.25)=\frac{39.478414}{1+6.85389\left(d/l\right)^2}\frac{EI}{l^2} = K_{cr_2}(\mu=0.25)\frac{EI}{l^2} \quad (64)$$

$$P_{x_{cr}}(\mu=0.30)=\frac{39.478414}{1+7.12804\left(d/l\right)^2}\frac{EI}{l^2} = K_{cr_2}(\mu=0.30)\frac{EI}{l^2} \quad (65)$$

where K_{cr2} is the critical elastic buckling load coefficient for moderately thick beam with fixed-fixed ends and circular cross-section.

Equations (60) and (61) are used to determine K_{cr1}, the critical elastic buckling load coefficients for moderately thick beams of rectangular cross-section for various values of t/l and for values of Poisson's ratio, μ = 0.25, and μ = 0.30, and the values presented as Table 1.

Equations (64) and (65) are used to determine K_{cr2}, the critical elastic buckling load coefficients for moderately thick beams of circular cross-section for various values of d/l and for Poisson's ratio μ = 0.25 and μ = 0.30 which are presented in Table 2.

Table 1: K_{cr1}, the critical elastic buckling load coefficients for moderately thick beams with rectangular cross-section with fixed-fixed ends for μ = 0.25 and μ = 0.30 for various values of t/l

t/l	μ = 0.25 $P_{x_{cr}}(\mu=0.25) = K_{cr_1}(\mu=0.25)EI/l^2$	μ = 0.30 $P_{x_{cr}}(\mu=0.30) = K_{cr_1}(\mu=0.30)EI/l^2$
0.01	39.4395	39.4379
0.02	39.3232	39.3170
0.05	38.3278	38.4907
0.10	35.9321	35.8034
0.15	32.3046	32.0715
0.20	28.3043	27.9875
0.25	24.4169	24.0500
0.30	20.9073	20.5211
0.35	17.8714	17.4885
0.40	15.3068	14.9409
0.45	13.1656	12.8238
0.50	11.3856	11.0705

Table 2: K_{cr2}, the critical elastic buckling load coefficients for moderately thick beams (circular cross-section) with fixed-fixed ends for μ = 0.25, and μ = 0.30 for various values of d/l

d/l	μ = 0.25 $P_{x_{cr}}(\mu=0.25) = K_{cr_2}(\mu=0.25)EI/l^2$	μ = 0.30 $P_{x_{cr}}(\mu=0.30) = K_{cr_2}(\mu=0.30)EI/l^2$
0.01	39.4514	39.4503
0.02	39.3705	39.3662
0.05	38.8134	38.7872
0.10	36.9462	36.8516
0.15	34.2038	34.0219
0.20	30.9840	30.7196
0.25	27.6388	27.3112
0.30	24.4169	24.0499
0.35	21.4603	21.0756
0.40	18.8295	18.4437
0.45	16.5326	16.1570
0.50	14.5490	14.1906

Results for moderately thick beams with fixed-pinned ends

The boundary conditions for moderately thick beams of length l fixed at $x = 0$, and pinned at $x = l$, as shown in Figure 3 are:

$$w_b(0)=w_b'(0)=0 \quad (66)$$

$$w_b(l) = w_b''(l) = 0 \tag{67}$$

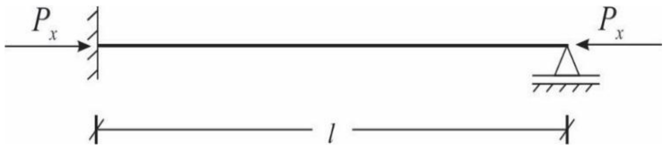

Figure 3: Elastic buckling of moderately thick beam with fixed end at $x = 0$, and pinned at $x = l$ due to load P_x

The buckled deflection function is obtained by using the boundary conditions at $x = 0$, as:

$$w_b(x) = w_b''(0)\left(\frac{1-\cos\beta x}{\beta^2}\right) + w_b'''(0)\left(\frac{x}{\beta^2} - \frac{\sin\beta x}{\beta^3}\right) \tag{68}$$

Enforcement of the boundary conditions at $x = l$, Equation (67), leads to the homogeneous eigenvalue problem given by the system of homogeneous equations:

$$\begin{pmatrix} \left(\dfrac{1-\cos\beta l}{\beta^2}\right) & \dfrac{1}{\beta^2}\left(l - \dfrac{\sin\beta l}{\beta}\right) \\ \cos\beta l & \dfrac{\sin\beta l}{\beta} \end{pmatrix} \begin{pmatrix} w_b''(0) \\ w_b'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{69}$$

For nontrivial solutions, the characteristic buckling equation is obtained as:

$$\begin{vmatrix} \left(\dfrac{1-\cos\beta l}{\beta^2}\right) & \dfrac{1}{\beta^2}\left(l - \dfrac{\sin\beta l}{\beta}\right) \\ \cos\beta l & \dfrac{\sin\beta l}{\beta} \end{vmatrix} = 0 \tag{70}$$

Expansion and simplification yields the characteristic buckling equation as the transcendental equation:

$$\tan\beta l = \beta l \tag{71}$$

The methods for solution of the transcendental equation are: Newton-Raphson's iteration, Regula falsi, simple iteration or use of Mathematica software or other symbolic algebra software. The transcendental equation has an infinite number of nontrivial roots (zeros) which are the eigenvalues of the characteristic buckling equation. The first four roots (zeros) or eigenvalues of the transcendental equation are:

$$\beta_1 l = \pm 4.49341 \tag{72}$$

$$\beta_2 l = \pm 7.72525184 \tag{73}$$

$$\beta_3 l = \pm 10.90412166 \tag{74}$$

$$\beta_4 l = \pm 14.0661939 \tag{75}$$

The critical buckling load P_{xcr} for this boundary condition is obtained using the least eigenvalue. Thus, for rectangular cross-sections,

$$\beta_1^2 = \left(\frac{\pm 4.49341}{l}\right)^2 = \left(\frac{P_{x_{cr}}}{EI - \dfrac{P_{x_{cr}}(1+\mu)t^2}{5}}\right) \tag{76}$$

Hence,

$$P_{x_{cr}} = \frac{20.19073}{1 + 4.038146(1+\mu)\dfrac{t^2}{l^2}}\frac{EI}{l^2} = K_{cr_3}\frac{EI}{l^2} \tag{77}$$

$$\begin{aligned} P_{x_{cr}}(\mu = 0.25) &= \frac{20.19073}{1 + 5.04768\left(t/l\right)^2}\frac{EI}{l^2} \\ &= K_{cr_3}(\mu = 0.25)\frac{EI}{l^2} \end{aligned} \tag{78}$$

$$\begin{aligned} P_{x_{cr}}(\mu = 0.30) &= \frac{20.19073}{1 + 5.24959\left(t/l\right)^2}\frac{EI}{l^2} \\ &= K_{cr_3}(\mu = 0.30)\frac{EI}{l^2} \end{aligned} \tag{79}$$

where K_{cr3} is the critical elastic buckling load coefficient for moderately thick beam with fixed-pinned ends and rectangular cross-section.

For circular cross-sections,

$$\beta_1^2 = \left(\frac{\pm 4.49341}{l}\right)^2 = \frac{P_{x_{cr}}}{EI - \dfrac{P_{x_{cr}} d^2(1+\mu)}{7.2}} \tag{80}$$

Hence,

$$P_{x_{cr}} = \frac{20.19073}{1 + 2.80427(1+\mu)\left(d/l\right)^2}\frac{EI}{l^2} = K_{cr_4}\frac{EI}{l^2} \tag{81}$$

$$\begin{aligned} P_{x_{cr}}(\mu = 0.25) &= \frac{20.19073}{1 + 3.50534\left(d/l\right)^2}\frac{EI}{l^2} \\ &= K_{cr_4}(\mu = 0.25)\frac{EI}{l^2} \end{aligned} \tag{82}$$

$$\begin{aligned} P_{x_{cr}}(\mu = 0.30) &= \frac{20.19073}{1 + 3.64555\left(d/l\right)^2}\frac{EI}{l^2} \\ &= K_{cr_4}(\mu = 0.30)\frac{EI}{l^2} \end{aligned} \tag{83}$$

where K_{cr4} is the critical elastic buckling load coefficient for moderately thick beam with fixed-pinned ends and circular cross-section.

K_{cr3}, the critical elastic load buckling coefficients are calculated for this boundary condition for rectangular cross-sections and presented in Table 3. Similarly, K_{cr4}, the critical elastic load buckling coefficients for fixed-pinned ends and circular cross section are presented in Table 4.

Table 3: K_{cr3}, the critical elastic buckling load coefficients for moderately thick beams (rectangular cross-sections) (a) fixed at $x = 0$, and pinned at $x = l$ for $\mu = 0.25$, and $\mu = 0.30$; (b) pinned at $x = 0$, and fixed at $x = l$ for $\mu = 0.25$, and $\mu = 0.30$ for various values of t/l

t/l	$P_{x_{cr}}(\mu = 0.25) = K_{cr_3}(\mu = 0.25) EI/l^2$	$P_{x_{cr}}(\mu = 0.30) = K_{cr_3}(\mu = 0.30) EI/l^2$
0.01	20.1805	20.1801
0.02	20.1500	20.1484
0.05	19.9391	19.9292
0.10	19.2205	19.1837
0.15	18.1315	18.0578
0.20	16.7989	16.6868
0.25	15.3486	15.2027
0.30	13.8836	13.7122
0.35	12.4762	12.2884
0.40	11.1697	10.9736
0.45	9.9848	9.7869
0.50	8.9264	8.7315

Table 4: K_{cr4}, the critical elastic buckling load coefficients for moderately thick beams (circular cross-sections) (a) fixed at $x = 0$, and pinned at $x = l$ for $\mu = 0.25$, and $\mu = 0.30$; (b) pinned at $x = 0$, and fixed at $x = l$ for $\mu = 0.25$, and $\mu = 0.30$ for various values of d/l

d/l	$P_{x_{cr}}(\mu = 0.25) = K_{cr_4}(\mu = 0.25) EI/l^2$	$P_{x_{cr}}(\mu = 0.30) = K_{cr_4}(\mu = 0.30) EI/l^2$
0.01	20.1837	20.1834
0.02	20.1625	20.1613
0.05	20.0153	20.0084
0.10	19.5069	19.4806
0.15	18.7147	18.6601
0.20	17.7078	17.6212
0.25	16.5622	16.4440
0.30	15.3486	15.2027
0.35	14.1253	13.9576
0.40	12.9357	12.7524
0.45	11.8086	11.6157
0.50	10.7607	10.5634

Results for moderately thick beams pinned at $x = 0$ and fixed at $x = l$

The boundary conditions for moderately thick beams of length l pinned at $x = 0$, and fixed at $x = l$, as shown in figure 4 are:

$$w_b(0) = w_b''(0) = 0 \tag{84}$$

$$w_b(l) = w_b'(l) = 0 \tag{85}$$

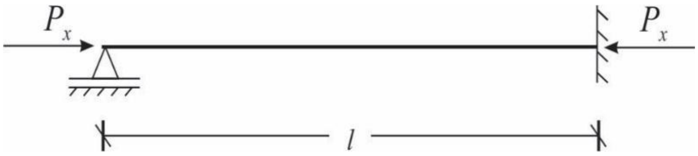

Figure 4: Moderately thick beam under compressive load – case of pinned end at $x = 0$, and fixed end at $x = l$.

The buckled deflection function for this case is:

$$w_b(x) = w_b'(0)x + w_b'''(0)\left(\frac{x}{\beta^2} - \frac{\sin\beta x}{\beta^3}\right) \tag{86}$$

Enforcement of boundary conditions at $x = l$ yields the homogeneous equations:

$$\begin{pmatrix} l & \left(\dfrac{l}{\beta^2} - \dfrac{\sin\beta l}{\beta^3}\right) \\ 1 & \left(\dfrac{1}{\beta^2} - \dfrac{\cos\beta l}{\beta^2}\right) \end{pmatrix} \begin{pmatrix} w_b'(0) \\ w_b'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{87}$$

For nontrivial solutions, the characteristic buckling equation is obtained as:

$$\begin{vmatrix} l & \left(\dfrac{l}{\beta^2} - \dfrac{\sin\beta l}{\beta^3}\right) \\ 1 & \left(\dfrac{1}{\beta^2} - \dfrac{\cos\beta l}{\beta^2}\right) \end{vmatrix} = 0 \tag{88}$$

Expansion and simplification yields Equation (71) which is the same solution as the moderately thick beam clamped at $x = 0$, and pinned at $x = l$. The critical elastic buckling load expressions for this case is the same as the case for the beam fixed at $x = 0$ and pinned at $x = l$ and is presented in Tables 3 and 4 for the rectangular and circular cross-sections respectively.

Results for moderately thick beams with pinned ends at $x = 0$, and $x = l$

The boundary conditions for moderately thick beams with ends at $x = 0$, $x = l$ pinned as shown in Figure 5 are:

$$w_b(0) = w_b''(0) = 0 \tag{89}$$

$$w_b(l) = w_b''(l) = 0 \tag{90}$$

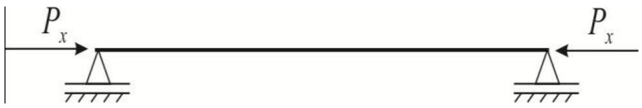

Figure 5: Moderately thick beam under compressive load with pinned ends $x = 0$, $x = l$

The buckling deflection function $w_b(x)$ is:

$$w_b(x) = w_b'(0)x + w_b'''(0)\left(\frac{x}{\beta^2} - \frac{\sin\beta x}{\beta^3}\right) \tag{91}$$

Enforcement of the boundary conditions at $x = l$ yields:

$$\begin{pmatrix} l & \left(\dfrac{l}{\beta^2} - \dfrac{\sin\beta l}{\beta^3}\right) \\ 0 & \dfrac{\sin\beta l}{\beta} \end{pmatrix}\begin{pmatrix} w_b'(0) \\ w_b'''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{92}$$

For nontrivial solutions, the characteristic buckling equation is given as:

$$\begin{vmatrix} l & \left(\dfrac{l}{\beta^2} - \dfrac{\sin\beta l}{\beta^3}\right) \\ 0 & \dfrac{\sin\beta l}{\beta} \end{vmatrix} = 0 \tag{93}$$

Expansion of the determinant and simplification yields the characteristic buckling equation as:

$$\sin\beta l = 0 \tag{94}$$

The eigenvalues (zeros) are:

$$\beta_n l = n\pi \tag{95}$$

$n = 1, 2, 3, 4, \ldots$

The eigenvalues β_n is used to obtain the buckling loads as follows:

For rectangular cross-sections:

$$\beta_n^2 = \left(\frac{n\pi}{l}\right)^2 = \frac{P_x}{EI - \dfrac{P_x(1+\mu)t^2}{5}} \tag{96}$$

$$P_x = \frac{(n\pi)^2}{1 + \dfrac{(n\pi)^2(1+\mu)}{5}\dfrac{t^2}{l^2}}\frac{EI}{l^2} \tag{97}$$

The critical buckling load is obtained using the least eigenvalue as:

$$P_{x_{cr}} = P_x(n=1) = \frac{\pi^2}{1 + \dfrac{(1+\mu)\pi^2}{5}\dfrac{t^2}{l^2}}\frac{EI}{l^2} = K_{cr_5}\frac{EI}{l^2} \tag{98}$$

$$P_{x_{cr}}(\mu = 0.25) = \frac{\pi^2}{1 + 0.25\pi^2\dfrac{t^2}{l^2}}\frac{EI}{l^2} = K_{cr_5}(\mu = 0.25)\frac{EI}{l^2} \tag{99}$$

$$P_{x_{cr}}(\mu = 0.30) = \frac{\pi^2}{1 + 0.26\pi^2\left(\dfrac{t}{l}\right)^2}\frac{EI}{l^2} = K_{cr_5}(\mu = 0.30)\frac{EI}{l^2} \tag{100}$$

K_{cr5} is the critical elastic buckling load coefficient for moderately thick beam with pinned-pinned ends and rectangular cross-section.

For circular cross-sections,

$$\beta_n^2 = \left(\frac{n\pi}{l}\right)^2 = \frac{P_x}{EI - \dfrac{P_x d^2(1+\mu)}{7.2}} \tag{101}$$

$$P_x = \frac{(n\pi)^2}{1 + \dfrac{(n\pi)^2(1+\mu)}{7.2}\dfrac{d^2}{l^2}}\frac{EI}{l^2} \tag{102}$$

$$P_{x_{cr}} = P_x(n=1) = \frac{\pi^2}{1 + \dfrac{(1+\mu)\pi^2}{7.2}\left(\dfrac{d}{l}\right)^2}\frac{EI}{l^2} = K_{cr_6}\frac{EI}{l^2} \tag{103}$$

$$P_{x_{cr}}(\mu = 0.25) = \frac{\pi^2}{1 + 1.713475\left(\frac{d}{l}\right)^2}\frac{EI}{l^2} \qquad (104)$$

$$= K_{cr_6}(\mu = 0.25)\frac{EI}{l^2}$$

$$P_{x_{cr}}(\mu = 0.30) = \frac{\pi^2}{1 + 1.782014\left(\frac{d}{l}\right)^2}\frac{EI}{l^2} \qquad (105)$$

$$= K_{cr_6}(\mu = 0.30)\frac{EI}{l^2}$$

K_{cr6} is the critical elastic buckling load coefficient for moderately thick beam with pinned-pinned ends and circular cross-section.

Equations (99) and (100) are used to calculate the critical elastic buckling load coefficients for moderately thick beams with pinned ends and rectangular cross-sections as presented in Table 5. Similarly, Equations (104) and (105) are used to calculate the critical elastic buckling loads for moderately thick beams with pinned ends and circular cross-sections as presented in Table 6.

Table 5: K_{cr5}, critical elastic buckling load coefficients for moderately thick beams with pinned ends and rectangular cross-sections for values of t/l and for $\mu = 0.25$ and $\mu = 0.30$.

t/l	$P_{x_{cr}}(\mu = 0.25) = K_{cr_5}(\mu = 0.25) EI/l^2$	$P_{x_{cr}}(\mu = 0.30) = K_{cr_5}(\mu = 0.30) EI/l^2$	Pakhare et al [25] $P_{x_{cr}} = K_{cr}\, EI/l^2$
0.01	9.86717	9.8671	9.8671
0.02	9.85987	9.8595	9.8595
0.05	9.8091	9.8067	9.8067
0.10	9.63195	9.6227	9.6227
0.15	9.35050	9.3309	9.3309
0.20	8.98302	8.9509	8.9509
0.25	8.55094	8.5055	8.5055
0.30	8.07616	8.0179	8.0179
0.35	7.57885	7.5091	7.5091
0.40	7.07608	6.9969	6.9969
0.45	6.58128	6.49472	
0.50	6.10422	6.01246	

Table 6: K_{cr6}, critical elastic buckling load coefficients for moderately thick beams with pinned ends and circular cross-sections for values of d/l and for $\mu = 0.25$ and $\mu = 0.30$.

d/l	$P_{x_{cr}}(\mu = 0.25) = K_{cr_6}(\mu = 0.25) EI/l^2$	$P_{x_{cr}}(\mu = 0.30) = K_{cr_6}(\mu = 0.30) EI/l^2$
0.01	$0.999828\pi^2$	$0.999822\pi^2$
0.02	$0.999315\pi^2$	$0.999288\pi^2$
0.05	$0.99573\pi^2$	$0.995565\pi^2$
0.10	$0.983154\pi^2$	$0.982492\pi^2$
0.15	$0.962878\pi^2$	$0.96145\pi^2$
0.20	$0.935857\pi^2$	$0.933462\pi^2$
0.25	$0.903267\pi^2$	$0.899786\pi^2$
0.30	$0.866391\pi^2$	$0.861786\pi^2$
0.35	$0.826514\pi^2$	$0.820818\pi^2$
0.40	$0.784833\pi^2$	$0.778136\pi^2$
0.45	$0.74240\pi^2$	$0.73483\pi^2$
0.50	$0.700099\pi^2$	$0.69180\pi^2$

V. DISCUSSION

The elastic buckling problems of thick and moderately thick beams have been presented in this work as boundary value problems of the mathematical theory of elasticity. The formulation considered thick and moderately thick beams with cross-sectional plane on the *yz* plane, and longitudinal coordinate axis on the *x*-axis. The considered beam was assumed isotropic, homogeneous, linear elastic, and subject to small deformations.

The displacement field components were considered as Equations (1) – (3), where the rotation of the cross-section at the vertical axis is given by Equation (4). Simultaneous considerations of the fundamental equations of elasticity were used to obtain the governing equations as Equation (35) for rectangular cross-sections and Equation (36) for circular cross-sections when transverse loads are absent.

The method of Laplace transform was applied to solve the homogeneous fourth order ODE governing the elastic stability of moderately thick beams. Application of the Laplace transformation to the governing ODE give the Laplace integral equation, Equation (40). Use of the linearity property of the Laplace transformation, and simplification gave the solution in the Laplace transform space $W(s)$ as Equation (45). Inversion of $W(s)$ yielded the solution in the physical domain space as Equation (47). The solution in the physical domain space was obtained in terms of the initial values (parameters) of w_b, w_b', w_b'' and w_b''' at $x = 0$, the origin.

Several end support conditions were considered, namely: fixed-fixed ends, fixed-pinned ends, pinned-fixed ends and simply supported ends. For moderately thick beams with fixed-fixed ends, the buckled deflection function was found as Equation (50). The use of the boundary conditions Equation (49) simplified the BVP to a system of homogeneous algebraic equations presented in matrix form – Equation (50). The characteristic buckling equation was obtained for nontrivial solutions as Equation (52), which upon expansion and simplification yielded Equation (53), a transcendental equation with infinite number of roots (eigenvalues). The first four eigenvalues obtained using Mathematica computational software tools are given as Equations (54 – 57). The least eigenvalue was used to obtain the critical elastic buckling load for rectangular cross-sections as Equation (59) or (60) for $\mu = 0.25$, and Equation (61) for $\mu = 0.30$. The critical elastic buckling load for the case of circular cross-section was similarly obtained as Equations (63), (64) for $\mu = 0.25$, and Equation (65) for $\mu = 0.30$. The critical elastic buckling load coefficients for moderately thick beams with fixed ends for rectangular and circular cross-sections are presented in Tables 1 and 2 respectively for various values of t/l, d/l and for $\mu = 0.25$, and $\mu = 0.30$. For the considered case of fixed-pinned ends, the buckled deflection function was obtained as Equation (68). The use of boundary conditions at $x = l$ yielded the eigenvalue problem – Equation (69). The conditions for nontrivial solutions of Equation (69) yielded the characteristic buckling equation as Equation (70), which upon expansion and simplification gave Equation (71), a transcendental equation with an infinite number of roots (eigenvalues). The solution of the transcendental characteristic equation using Mathematica computational software tool give the first four eigenvalues as Equations (72 – 75). The least eigenvalue was used to obtain the critical elastic buckling load for rectangular cross-sections as Equation (77), and Equation (78) for $\mu = 0.25$, Equation (79) for $\mu = 0.30$, and for circular cross-sections as Equation (81) and Equation (82) for $\mu = 0.25$, and Equation (83) for $\mu = 0.30$. The critical elastic buckling load coefficients for this case are presented in Tables 3 and 4 respectively for rectangular cross-sections and circular cross-sections.

For moderately thick beams with pinned-fixed ends, the buckled deflection was found as Equation (86). Enforcement of boundary conditions at $x = l$ gave the homogeneous equation in matrix form as Equation (87). The condition for nontrivial solutions was used to obtain the characteristic buckling equation for this case as Equation (88) which yielded upon expansion and simplification the same characteristic buckling equation for the case of fixed-pinned ends, and hence the same elastic buckling loads and the same critical elastic buckling load.

For moderately thick beams with pinned-pinned ends, the buckling deflection function is obtained as Equation (91). The use of boundary conditions at $x = l$ gave the homogeneous algebraic problem – Equation (92). The characteristic buckling equation for nontrivial solutions is obtained as Equation (93) which upon expansion and simplification yields Equation (94). The eigenvalues are obtained as Equation (95), and the buckling loads are found as Equations (97) for rectangular cross-section, and Equation (102) for circular cross-section. The least eigenvalue is used to obtain the critical buckling loads as Equation (98), and Equation (99) for $\mu = 0.25$, and Equation (100) for $\mu = 0.30$ for rectangular cross-sections and Equation (103), Equation (104) for $\mu = 0.25$, and Equation (105) for $\mu = 0.30$ for circular cross-sections. The critical elastic buckling load coefficients for moderately thick beams with pinned-pinned ends were presented in Tables 5 and 6 for rectangular cross-sections and circular cross-sections respectively.

VI. CONCLUSION

In conclusion,

(i) the elastic buckling problem of moderately thick beams formulated using first order shear deformation theory is a boundary value problem (BVP) of the mathematical theory of elasticity.

(ii) the BVP is represented in general as a fourth order inhomogeneous ordinary differential equation when distributed transverse loads act together with axial compressive loads, and a fourth order homogeneous ODE when transverse distributed loads are absent.

(iii) the method of Laplace transformation simplified the BVP to an algebraic equation in the Laplace transform space, which was solved to obtain an initial value (parameter) presentation of the solution in the Laplace transform space. Inversion of the solution in the Laplace transform space yielded the general solution in the physical domain space in terms of the initial values at the origin.

(iv) boundary conditions for the considered cases of end supports were used to present the problem as algebraic eigenvalue problems and find the characteristic stability equations for nontrivial solutions.

(v) the characteristic stability equations were obtained as transcendental equations which were solved using computational software tools and other numerical algorithms for solving nonlinear and transcendental solutions to obtain the eigenvalues (zeros or roots).

(vi) the least eigenvalue for each considered end support conditions were used to obtain the critical elastic buckling loads corresponding to the end support condition.

(vii) the n eigenvalues for each considered end support conditions were used to obtain the n elastic buckling loads corresponding to the n eigenvalues.

(viii) for $t/l < 0.02$, and $d/l < 0.02$, the critical buckling loads coefficients found for the various end support conditions are approximately equal to the critical elastic buckling load coefficients for the Bernoulli-Euler beam with the corresponding end support conditions.

(ix) for $t/l > 0.05$ and $d/l > 0.05$, for each considered end support conditions, the critical elastic buckling load coefficient is much smaller than the critical elastic buckling load coefficient obtained using the Bernoulli-Euler beam theory.

(x) for $t/l > 0.05$, $d/l > 0.05$, the Bernoulli-Euler beam theory is found to significantly overestimate the critical elastic buckling load capacity of beams for each considered end support condition.

(xi) shear deformation effects should be accounted for to give better estimates of the critical elastic buckling load capacities of moderately thick beams for safe design.

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