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Modified Ridge-Type Estimator with Prior Information

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Abstract

Literature has proved the inefficiency of ordinary least squares estimator when the linear model suffers multicollinearity problem. Several biased estimation techniques have been developed to tackle the problem of multicollinearity. In this study, we proposed a new estimator based on prior information and the modified ridge-type estimator by Lukman et al. (2019). It includes the modified ridge-type (MRT) estimator, ridge estimator (RRE) and the ordinary least square estimator (OLSE) as special cases. We established the superiority of this new estimator (MRTP) over others using the mean squared error criterion. Finally, the superiority of the MRTP estimator was confirmed through a simulation study and its application to real-life data.

Keywords: Linear model; Prior information; Multicollinearity; Modified ridge-type estimator

1. INTRODUCTION

The general linear regression model includes a n × 1 vector of the dependent variable labelled Y, a fixed n × p matrix of independent variables labelled as X, a p × 1 vector known as the regression coefficients which is commonly denoted by β, a n × 1 vector of disturbance denoted by ε which is assumed to follow a normal distribution, $N(0, \sigma^2 I)$. The model is generally written as:

$$Y = X\beta + \varepsilon \tag{1}$$

The ordinary least squares (OLS) estimator of β is

$$\hat{\beta}_{OLS} = (X'X)^{-1}X'Y \tag{2}$$

Gauss-Markov theorem proved that the OLS estimator is the best, linear and unbiased estimator possessing a relatively minimum variance in the class of all linear unbiased estimators. However, literature has proved the OLS estimator to provide misleading results when the model assumptions are not satisfied. One of the prominent violations is the problem of multicollinearity which occur when the independent variables are related (Hoerl and Kennerd, 1970; Lukman and Ayinde, 2017). Several biased estimators have been suggested in the literature to combat this problem. These include Liu estimator by Liu (1993), principal component estimator (Massy, 1965), ridge regression estimator (Hoerl and Kennard, 1970), modified ridge estimator (Swindel, 1976) and others. The ridge regression estimator (RRE) is defined as:

$$\hat{\beta}_{RRE}(k) = (X'X + kI)^{-1}X'Y = T_K\hat{\beta}_{OLS} \tag{3}$$

where $T_K = (X'X + kI)^{-1}X'X$ and $k > 0$. Swindel (1976) modified the ridge estimator by including a prior information. This is expressed mathematically as:

$$\hat{\beta}_{MRRE}(k) = (X'X + kI)^{-1}(X'Y + kb) \tag{4}$$

where b is the prior information on β and MRRE tends to b as k tends to infinity. Dorugade (2014) defined a ridge-type estimator as:

$$\hat{\beta}_D(k) = R_{kd}\hat{\beta} \tag{5}$$

where $R_{kd} = (X'X + kdI)^{-1}X'X$ with d introduced as additional biasing parameter. Lukman et al. (2019) proposed a modified ridge-type (MRT) estimator which is defined as:

$$\hat{\beta}_{MRT}(k,d) = (X'X + k(1+d)I)^{-1}X'Y = R_{kd}\hat{\beta} \tag{6}$$

where $R_{kd} = (X'X + k(1+d)I)^{-1}X'X, k > 0$ and $0 < d < 1$. This estimator includes OLS and RE as special cases.

This article focuses primarily on presenting an alternative method to combat the problem of multicollinearity in a linear regression model. The rest of the study is arranged as follows; a Modified Ridge-Type Estimator based on prior information (MRTP) is introduced in Section 2. This estimator was compared with the OLS, RRE, MRRE, D and MRT estimator through the mean squared error criterion in Section 3. Monte Carlo simulation study was carried out, and the new estimator alongside others was applied to a chemical data in Section 4 while Section 5 provides the concluding remarks.

2. THE NEW ESTIMATOR BASED ON A PRIOR INFORMATION

The MRRE in equation (4) can be re-expressed as

$$\hat{\beta}_{MRRE}(k) = (X'X + kI)^{-1}(X'Y + kb)$$
$$= (X'X + kI)^{-1}X'Y + k(X'X + kI)^{-1}b$$

$$= (X'X + kI)^{-1}X'X\hat{\beta}_{OLS} + k(X'X + kI)^{-1}b$$

$$= T_k\hat{\beta}_{OLS} + (I - T_k)b \quad (7)$$

where $T_k = (X'X + kI)^{-1}X'X = I - k(X'X + kI)^{-1}$. This implies that MRRE is a convex combination of the prior information, b and the OLS estimator. In the same manner, it follows from equation (6) that $R_{kd} = (X'X + k(1 + d)I)^{-1}X'X = I - k(1 + d)(X'X + k(1 + d)I)^{-1}$. Thus, a modified ridge type estimator based on a prior information can be defined as:

$$\hat{\beta}_{MRTP}(k, d, b) = R_{kd}\hat{\beta}_{OLS} + (I - R_{kd})b \quad (8)$$

$$= (X'X + k(1 + d)I)^{-1}X'X\hat{\beta}_{OLS} + (I - (X'X + k(1 + d)I)^{-1}X'X)b$$

$$= (X'X + k(1 + d)I)^{-1}X'Y + (k(1 + d)(X'X + k(1 + d)I)^{-1})b$$

$$= (X'X + k(1 + d)I)^{-1}(X'Y + k(1 + d)b) \quad (9)$$

Equation (8) also presents MRTP has a convex combination of the prior information and OLS estimator. MRTP includes the special cases of OLS, RRE and MRT as follows:

$\hat{\beta}_{MRTP}(k, d, 0) = \hat{\beta}_{MRT}(k, d)$; Modified ridge type estimator (MRT)

$\hat{\beta}_{MRTP}(k, 0,0) = \hat{\beta}_{RRE}(k)$; Ridge regression estimator (RRE)

$\hat{\beta}_{MRTP}(0,0,0) = \hat{\beta}_{MRTP}(0, d, b) = \hat{\beta}_{MRTP}(0, d, 0) = \hat{\beta}_{MRTP}(0,0, b) = \hat{\beta}_{OLS}$; Ordinary least square estimator (OLSE)

In canonical form, model (1) is given written

$$Y = Z\alpha + \varepsilon \quad (10)$$

where $Z = XT, \alpha = T'\beta$ and T is the ortogonal matrix whose columns contains the eigenvectors of $X'X$. Then, $Z'Z = T'X'XT = \Lambda = diag(\lambda_1, \lambda_2, \dots, \lambda_p)$ where $\lambda_1, \lambda_2, \dots, \lambda_p > 0$ are the ordered eigenvalues of $X'X$. Thus, the corresponding OLS, RRE, MRRE, D, MRT and MRTP estimator for the canonical model is given as;

$$\hat{\alpha}_{OLS} = \Lambda^{-1}Z'Y \quad (11)$$

$$\hat{\alpha}_{RRE}(k) = (\Lambda + \mathrm{k}\mathrm{I})^{-1}Z'Y \quad (12)$$

$$\hat{\alpha}_{MRRE}(k, d) = (\Lambda + kI)^{-1}(Z'Y + kb) \quad (13)$$

$$\hat{\alpha}_{D}(k, d) = (\Lambda + kdI)^{-1}Z'Y \quad (14)$$

$$\hat{\alpha}_{MRT}(k, d) = (\Lambda + k(1 + d)I)^{-1}Z'Y \quad (15)$$

$$\hat{\alpha}_{MRTP}(k, d, b) = (\Lambda + k(1 + d)I)^{-1}(Z'Y + k(1 + d)b) \quad (16)$$

The properties of MRTP, that is, the expectation, bias vector, covariance matrix and mean square error matrix of MRTP are obtained as follows:

$$E(\hat{\alpha}_{MRTP}(k, d, b)) = E(R_{kd}\hat{\alpha} + (I - R_{kd})b)$$
$$= R_{kd}\hat{\alpha} + (R_{kd} - I)b \quad (17)$$

$$Bias(\hat{\alpha}_{MRTP}(k, d, b)) = Bias(R_{kd}\hat{\alpha} + (I - R_{kd})b)$$
$$= R_{kd}\hat{\alpha} + (I - R_{kd})b - \hat{\alpha}$$
$$= (R_{kd} - I)(\hat{\alpha} - b)(\hat{\alpha} - b)'(R_{kd} - I)' \quad (18)$$

$$Cov(\hat{\alpha}_{MRTP}(k, d, b))$$
$$= Cov(R_{kd}\hat{\alpha} + (I - R_{kd})b)$$
$$= R_{kd}Var(\hat{\alpha})R_{kd}'$$
$$= \sigma^2 R_{kd}\Lambda^{-1}R_{kd}' \quad (19)$$

Hence,

$$MSEM(\hat{\alpha}_{MRTP}(k, d, b))$$
$$= Var(R_{kd}\hat{\alpha} + (I - R_{kd})b) + Bias(R_{kd}\hat{\alpha} + (I - R_{kd})b)$$
$$= \sigma^2 R_{kd}\Lambda^{-1}R_{kd}' + (R_{kd} - I)(\hat{\alpha} - b)(\hat{\alpha} - b)'(R_{kd} - I)' \quad (20)$$

where $\hat{\alpha}$ is the OLS estimator, $R_{kd} = \Lambda(\Lambda + k(1 + d)I)^{-1}$, k > 0 and 0 < d < 1.

3. SUPERIORITY OF MRTP USING THE MSEM CRITERION

The following notations and lemmas are needful to prove the statistical property of $\hat{\beta}_{MRTP}(k, d, b)$.

Lemma 3.1. Let M be an n × n positive definite matrix, that is, M > 0, and α be some vector, then $M - \alpha\alpha' \geq 0$ if and only if $\alpha' M^{-1}\alpha \leq 1$ (Farebrother, 1976).

Lemma 3.2. Let $\hat{\beta}_i = A_i y, i = 1,2$ be two linear estimators of β. Suppose that $D = Cov(\hat{\beta}_1) - Cov(\hat{\beta}_2) > 0$, where $Cov(\hat{\beta}_i), i = 1,2$ denotes the covarince matrix of $\hat{\beta}_i$ and $b_i = Bias(\hat{\beta}_i) = (A_i X - I)\beta, i = 1,2$. Consequentlly,

$$\Delta(\hat{\beta}_1 - \hat{\beta}_2) = MSEM(\hat{\beta}_1) - MSEM(\hat{\beta}_2)$$
$$= \sigma^2 D + b_1 b_1' - b_2 b_2' > 0 \quad (21)$$

if and only if $b_2[\sigma^2 D + b_1 b_1']^{-1} b_2' < 1$, where $MSEM(\hat{\beta}_i) = Cov(\hat{\beta}_i) + b_i b_i'$ (Trenkler and Toutenburg, 1990).

3.1 Comparison between the MRTP and OLS using MSEM criterion.

From the canonical model, $\hat{\alpha}_{OLS} = \Lambda^{-1}Z'Y$, the MSEM of OLS is expressed as

$$Cov(\hat{\alpha}_{OLS}) = MSEM(\hat{\alpha}_{OLS}) = \sigma^2\Lambda^{-1} \quad (22)$$

Comparing (20) and (21),

$$MSEM(\hat{\alpha}_{OLS}) - MSEM(\hat{\alpha}_{MRTP}(k, d, b))$$
$$= \sigma^2\Lambda^{-1} - \sigma^2 R_{kd}\Lambda^{-1}R_{kd}' - (R_{kd} - I)(\hat{\alpha} - b)(\hat{\alpha} - b)'(R_{kd} - I)'$$
$$= \sigma^2(\Lambda^{-1} - R_{kd}\Lambda^{-1}R_{kd}') - (R_{kd} - I)(\alpha - b)(\alpha - b)'(R_{kd} - I)' \quad (23)$$

Let k > 0 and 0 < d < 1, the following theorem holds:

Theorem 3.1 Consider two biased competing homogeneous linear estimator $\hat{\alpha}_{OLS}$ and $\hat{\alpha}_{MRTP}(k,d,b)$. If k > 0 and 0 < d < 1, the estimator $\hat{\alpha}_{MRTP}(k,d,b)$ is superior to the estimator $\hat{\alpha}$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_{OLS}) - MSEM\big(\hat{\alpha}_{MRTP}(k,d,b)\big) > 0$ if and only if

$$(\alpha-b)'(R_{kd}-I)'\big[\sigma^2\big(\Lambda^{-1}-R_{kd}\Lambda^{-1}R_{kd}'\big)\big]^{-1}(R_{kd}-I)(\alpha-b)<1 \tag{24}$$

Proof: The difference between (19) and (22) was obtained as;

$$Cov(\hat{\alpha}_{OLS}) - Cov\big(\hat{\alpha}_{MRTP}(k,d,b)\big) = \sigma^2\big(\Lambda^{-1}-R_{kd}\Lambda^{-1}R_{kd}'\big) = \sigma^2 diag\left\{\frac{1}{\lambda_i}-\frac{\lambda_i}{(\lambda_i+k(1+d))^2}\right\}_{i=1}^{p} \tag{25}$$

$\Lambda^{-1}-R_{kd}\Lambda^{-1}R_{kd}'$ will be positive definite (pd) if and only if $2\lambda_i k(1+d)+k^2(1+d)^2>0$. Since k > 0 and 0< d < 1, we observed that $\Lambda^{-1}-R_{kd}\Lambda^{-1}R_{kd}'$ is pd. By Lemma 3.2, the proof is complete.

3.2 Comparison between the MRTP and RRE using MSEM criterion.

From (12), the bias, covariance and MSEM of RRE is given as follows:

$$Bias\big(\hat{\alpha}_{RRE}(k)\big) = -kB_K\hat{\alpha} \tag{26}$$

$$Cov(\hat{\alpha}_{RRE}(k)) = \sigma^2 B_K\Lambda^{-1}B_K' \tag{27}$$

Thus,

$$MSEM\big(\hat{\alpha}_{RRE}(k)\big) = \sigma^2 B_K\Lambda^{-1}B_K' + k^2B_K\hat{\alpha}\hat{\alpha}'B_K' \tag{28}$$

where $B_K = (\Lambda+kI)^{-1}$. The difference between the MSEM of the RRE and MRTP is given as follows:

$$MSEM\big(\hat{\alpha}_{RRE}(k)\big) - MSEM\big(\hat{\alpha}_{MRTP}(k,d,b)\big) = \sigma^2(B_K\Lambda^{-1}B_K' - R_{kd}\Lambda^{-1}R_{kd}') + k^2B_K\hat{\alpha}\hat{\alpha}'B_K' - (R_{kd}-I)(\hat{\alpha}-b)(\hat{\alpha}-b)'(R_{kd}-I)' \tag{29}$$

Let k > 0 and 0 < d < 1, the following theorem holds:

Theorem 3.2 Consider two biased competing homogeneous linear estimator $\hat{\alpha}_{RRE}(k)$ and $\hat{\alpha}_{MRTP}(k,d,b)$. If k > 0 and 0 < d < 1, the estimator $\hat{\alpha}_{MRTP}(k,d,b)$ is superior to estimator $\hat{\alpha}$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_{RRE}(k)) - MSEM\big(\hat{\alpha}_{MRTP}(k,d,b)\big) > 0$ if and only if

$$(\hat{\alpha}-b)'(R_{kd}-I)'\big[\sigma^2(B_K\Lambda B_K' - R_{kd}\Lambda^{-1}R_{kd}') + k^2B_K\hat{\alpha}\hat{\alpha}'B_K'\big]^{-1}(R_{kd}-I)(\hat{\alpha}-b)<1 \tag{30}$$

Proof: The difference between (19) and (27) was obtained as;

$$\sigma^2(B_K\Lambda B_K' - R_{kd}\Lambda^{-1}R_{kd}') = \sigma^2 diag\left\{\frac{\lambda_i}{(\lambda_i+k)^2} - \frac{\lambda_i}{(\lambda_i+k(1+d))^2}\right\}_{i=1}^{p} \tag{31}$$

From (31), $2\lambda_i{}^2kd + kd(d+2) > 0$, which implies $B_K\Lambda B_K' - \sigma^2R_{kd}\Lambda^{-1}R_{kd}' > 0$ with k > 0 and 0< d < 1. By Lemma 3.2, the proof is complete.

3.3 Comparison between the MRTP and MRRE using MSEM criterion.

From (13), the bias, covariance and MSEM of MRRE is given as follows:

$$Bias\big(\hat{\alpha}_{MRRE}(k,b)\big) = (B_K-I)(\alpha-b) \tag{32}$$

$$Cov\big(\hat{\alpha}_{MRRE}(k,b)\big) = \sigma^2B_K\Lambda^{-1}B_K' \tag{33}$$

Thus,

$$MSEM\big(\hat{\alpha}_{MRRE}(k,b)\big) = \sigma^2B_K\Lambda^{-1}B_K' + (B_K-I)(\alpha-b)(\alpha-b)'(B_K-I)' \tag{34}$$

where $B_K = (\Lambda+kI)^{-1}$. The difference between the MSEM of the MRRE and MRTP is given as follows:

$$MSEM\big(\hat{\alpha}_{MRRE}(k)\big) - MSEM\big(\hat{\alpha}_{MRTP}(k,d,b)\big) = \sigma^2(B_K\Lambda B_K' - R_{kd}\Lambda^{-1}R_{kd}') + (B_K-I)(\hat{\alpha}-b)(\hat{\alpha}-b)'(B_K-I)' - (R_{kd}-I)(\alpha-b)(\alpha-b)'(R_{kd}-I)' \tag{35}$$

Let k > 0 and 0 < d < 1, the following theorem holds:

Theorem 3.3 Consider two biased competing homogeneous linear estimator $\hat{\alpha}_{MRRE}(k,b)$ and $\hat{\alpha}_{MRTP}(k,d,b)$. If k > 0 and 0 < d < 1, the estimator $\hat{\alpha}_{MRTP}(k,d,b)$ is superior to estimator $\hat{\alpha}$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_{MRRE}(k,b)) - MSEM\big(\hat{\alpha}_{MRTP}(k,d,b)\big) > 0$ if and only if

$$(\hat{\alpha}-b)'(R_{kd}-I)'[\sigma^2(B_K\Lambda B_K' - R_{kd}\Lambda^{-1}R_{kd}') + (B_K-I)(\hat{\alpha}-b)(\hat{\alpha}-b)'(B_K-I)']^{-1}(R_{kd}-I)(\hat{\alpha}-b)<1 \tag{36}$$

The proof is completed with Lemma 3.2 by the difference (19) and (33) as obtained in equation (27).

3.4 Comparison between the MRTP and D estimator using MSEM criterion.

From (14), the bias, covariance and MSEM of D is given as follows:

$$Bias\big(\hat{\alpha}_D(k,d)\big) = (D_k-I)\hat{\alpha} \tag{37}$$

$$Cov(\hat{\alpha}_D(k,d)) = \sigma^2D_k\Lambda^{-1}D_k' \tag{38}$$

Thus,

$$MSEM\big(\hat{\alpha}_D(k)\big) = \sigma^2D_k\Lambda^{-1}D_k' + (D_k-I)\hat{\alpha}\hat{\alpha}'(D_k-I)' \tag{39}$$

where $D_k = \Lambda(\Lambda+kdI)^{-1}$. The difference between the MSEM of the D estimator and MRTP is given as follows:

$$MSEM(\hat{\alpha}_D(k,d)) - MSEM(\hat{\alpha}_{MRTP}(k,d,b)) = \sigma^2(D_k\Lambda^{-1}D_k' - R_{kd}\Lambda^{-1}R_{kd}') + (D_k - I)\hat{\alpha}\hat{\alpha}'(D_k - I)' - (R_{kd} - I)(\hat{\alpha} - b)(\hat{\alpha} - b)'(R_{kd} - I)' \quad (40)$$

Let k > 0 and 0 < d < 1, the following theorem holds:

Theorem 3.4 Consider two biased competing homogeneous linear estimator $\hat{\alpha}_D(k,d)$ and $\hat{\alpha}_{MRTP}(k,d,b)$. If k > 0 and 0 < d < 1, the estimator $\hat{\alpha}_{MRTP}(k,d,b)$ is superior to estimator $\hat{\alpha}$ using the MSEM criterion, that is, $MSEM(\hat{\alpha}_D(k,d)) - MSEM(\hat{\alpha}_{MRTP}(k,d,b)) > 0$ if and only if

$$(R_{kd} - I)'(\hat{\alpha} - b)'[\sigma^2(D_k\Lambda^{-1}D_k' - R_{kd}\Lambda^{-1}R_{kd}') + (D_k - I)\hat{\alpha}\hat{\alpha}'(D_k - I)']^{-1}(\hat{\alpha} - b)(R_{kd} - I) < 1 \quad (41)$$

Proof: The difference between (19) and (38) was obtained as;

$$\sigma^2(D_k\Lambda^{-1}D_k' - R_{kd}\Lambda^{-1}R_{kd}') = \sigma^2 diag\left\{\frac{\lambda_i}{(\lambda_i + kd)^2} - \frac{\lambda_i}{(\lambda_i + k(1+d))^2}\right\}_{i=1}^{p} \quad (42)$$

From (42), $\lambda_i k(2\lambda_i + k(1+2d)) > 0$, which implies $D_k\Lambda^{-1}D_k' - R_{kd}\Lambda^{-1}R_{kd}' > 0$ with k > 0 and 0< d < 1. By Lemma 3.2, the proof is completed.

3.5 Comparison between the MRTP and MRT using MSEM criterion.

From (15), the bias, covariance and MSEM of MRT is given as follows:

$$Bias(\hat{\alpha}_{MRT}(k,d)) = (R_k - I)\hat{\alpha} \quad (43)$$

$$Cov(\hat{\alpha}_{MRT}(k,d)) = \sigma^2 R_k\Lambda^{-1}R_k' \quad (44)$$

Thus,

$$MSEM(\hat{\alpha}_{MRT}(k,d)) = \sigma^2 R_k\Lambda^{-1}R_k' + (R_k - I)\hat{\alpha}\hat{\alpha}'(R_k - I)' \quad (45)$$

where $R_k = R_{kd} = \Lambda(\Lambda + k(1+d)I)^{-1}$. The difference between the MSEM of the MRT estimator and MRTP is given as follows:

$$\Delta_D = MSEM(\hat{\alpha}_D(k,d)) - MSEM(\hat{\alpha}_{MRTP}(k,d,b))$$

$$= (R_{kd} - I)\hat{\alpha}\hat{\alpha}'(R_{kd} - I)' - (R_{kd} - I)(\hat{\alpha} - b)(\hat{\alpha} - b)'(R_{kd} - I)'$$

$$= \hat{\alpha}\hat{\alpha}' - (\hat{\alpha} - b)(\hat{\alpha} - b)' \quad (46)$$

Let k > 0 and 0 < d < 1, $\Delta_D > 0$ if and only if $\hat{\alpha}\hat{\alpha}' > (\hat{\alpha} - b)(\hat{\alpha} - b)'$. Therefore, the following theorem is postulated:

Theorem 3.5 The modified ridge type estimator with a prior information, $\hat{\alpha}_{MRTP}(k,d,b)$ is superiror to the modified ridge type estimator, $\hat{\alpha}_{MRT}(k,d,b)$ in the MSEM sense if and only if $\hat{\alpha}\hat{\alpha}' - (\hat{\alpha} - b)(\hat{\alpha} - b)' \geq 0$.

4. SELECTION CHOICE OF BIASING PARAMETERS *k* AND *d* FOR MRTP

For the purpose of practical application of this new estimator, the optimum value of k and d are obtained by differenciating the scaler MSE function of the MRTP estimator presented in equation (47);

$$f(k,d) = MSEM(\hat{\alpha}_{MRTP}(k,d,b))$$
$$= \sigma^2 R_{kd}\Lambda^{-1}R_{kd}' + (R_{kd} - I)(\hat{\alpha} - b)(\hat{\alpha} - b)'(R_{kd} - I)'$$
$$= \sigma^2\sum_{i=1}^{p}\frac{\lambda_i}{(\lambda_i + k(1+d))^2} + k^2(1+d)^2\sum_{i=1}^{p}\frac{(\alpha_i - b)^2}{(\lambda_i + k(1+d))^2} \quad (47)$$

Differentiating equation (47) with respect to k and equating to zero yields:

$$-\sigma^2\lambda_i + k(1+d)(\alpha_i - b)^2(\lambda_i + k(1+d)) - k^2(1+d)^2(\alpha_i - b)^2 = 0$$

Consequently,

$$k_{MRTP} = \frac{\sigma^2}{(1+d)(\alpha_i - b)^2} \quad (48)$$

It should be noted that when b = 0 in (48), k_{MRTP} becomes the estimated k_{MRT} obtained for the MRT estimator by Lukman et al (2019) as presented in (49) and when d=0, b=0 in (48), k_{MRTP} becomes the estimated k obtained for RRE by Hoerl and Kennard (1970) as presented in (50).

$$\hat{k}_{MRT} = \frac{\sigma^2}{(1+d)\alpha_i^2} \quad (49)$$

$$\hat{k} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2} \quad (50)$$

Hoerl and Kennard (1975) defined the harmonic version of the ridge parameter in equation (50) as;

$$\hat{k}_{HKB} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2} \quad (51)$$

Differentiating equation (47) with respect to d and equating to zero yields:

$$-\sigma^2\lambda_i + k(1+d)(\alpha_i - b)^2(\lambda_i + k(1+d)) - k^2(1+d)^2(\alpha_i - b)^2 = 0$$

Thus;

$$d_{MRTP} = \frac{\sigma^2}{k(\alpha_i - b)^2} - 1 \quad (52)$$

Following Hoerl and Kennard (1975), we obtain the harmonic mean of parameter k and d as follows;

$$k_{HMRTP} = \frac{p\hat{\sigma}^2}{(1+d)\sum_{i=1}^{p}(\alpha_i - b)^2} \quad (53)$$

and

$$d_{HMRTP} = \frac{p}{\sum_{i=1}^{p} 1/d_{MRTP}} \quad (54)$$

The selection of the parameters *k* and *d* in can be obtained iteratively as follows:

Step 1: calculate $\hat{k}_{HKB}$

Step 2: estimate d_{HMRTP} using $\hat{k}_{HKB}$

Step 3: If $d_{HMRTP} > 1$ or $d_{HMRTP} < 0$, use $\hat{d} = min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right)$

Step 4: estimate k_{HMRTP} using the choice of d selected in step 3

5. MONTE-CARLO SIMULATION

A simulation study was conducted following the study of Mc Donald and Galerneau (1975) and Kibria (2003). The following equation was used to generate the data:

$$x_{ij} = (1-\rho^2)^{1/2} z_{ij} + \rho z_{i,p+1} \quad (47)$$

where x_{ij} denotes the explanatory variables with $i = 1,2,\dots,n$ and $j = 1,2,\dots,p$. z_{ij} represent the independent standard normal distribution mean zero and unit variance. ρ represent the correlation between explanatory variables and z_{ij} are pseudo-random numbers from the standard normal distribution. The coefficients, $\beta_1, \beta_2, \dots, \beta_p$ are selected as the normalized eigenvectors corresponding to the largest eigenvalue of $X'X$ so that we have $\beta'\beta = 1$, which is a common restriction in simulation studies of this type (Newhouse and Oman, 1971; Lukman et. al., 2017). The dependent variable are then determined by

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_p x_{ip} + \varepsilon_i, \quad i = 1,2,\dots,n \quad (48)$$

where independent $\varepsilon_i's$ are generated from $N(0,\sigma^2)$. The number of parameter was fixed at p = 3 and other parameters such as ρ, σ *and* n were varied; their values considered in this study are given by:

- $\rho = 0.9,\ 0.99$
- $\sigma = 1,\ 5,\ 10$
- $n = 30,\ 50,\ 100$

The biasing parameters *k* and *d* were also varied as (0.3, 0.7 and 0.9) and (0.2, 0.5 and 0.8) respectively. The experiment is replicated 2000 times by generating new pseudo-random numbers and the estimated mse calculated as:

$$mse(\hat{\alpha}) = \frac{1}{2000}\sum_{j=1}^{2000}\left(\hat{\alpha}_{ij} - \alpha_i\right)'\left(\hat{\alpha}_{ij} - \alpha_i\right) \quad (49)$$

R programme was used for the simulation study, the results are shown in Table (1-3). It was observed that the MSEs of the estimators increases as the level of correlation (ρ) and error variance (σ) increases. As the sample size (n), the biasing parameters (k and d) increases, a decrease in the MSEs was noticed. Overall, the MRTP estimator was observed to outperform other estimators considered in this study possessing the smallest MSE when compared with other estimators. As expected, the OLS estimator performs least due to multicollinearity introduced into the simulation. Finally, the simulation results are consistent with the theoretical results.

6. APPLICATION TO A REAL LIFE DATA

The Portland cement which was originally adopted by Wood et al. (1932) and recently used in the Lukman et al. (2019) is used to illustrate the performance of this new estimator. The data is presented below;

$$X = \begin{pmatrix} 7 & 26 & 6 & 60 \\ 1 & 29 & 15 & 52 \\ 11 & 56 & 8 & 20 \\ 11 & 31 & 8 & 47 \\ 7 & 52 & 6 & 33 \\ 11 & 55 & 9 & 22 \\ 3 & 71 & 17 & 6 \\ 1 & 31 & 22 & 44 \\ 2 & 54 & 18 & 22 \\ 21 & 47 & 4 & 26 \\ 1 & 40 & 23 & 34 \\ 11 & 66 & 9 & 12 \\ 10 & 68 & 8 & 12 \end{pmatrix}, \quad Y = \begin{pmatrix} 78.5 \\ 74.3 \\ 104.3 \\ 87.6 \\ 95.9 \\ 109.2 \\ 102.7 \\ 72.5 \\ 93.1 \\ 115.9 \\ 83.8 \\ 113.3 \\ 109.4 \end{pmatrix}$$

The regression model is defined as follows:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon_i \quad (55)$$

where Y is the heat evolved after one hundred and eighty (180) days of curing measurement in calories per gram of cement, X_1 represents tricalcium aluminate, X_2 represents tricalcium silicate, X_3 represents tetracalcium aluminoferrite and X_4 represents β-dicalcium silicate. The eigenvalues of $X'X$ are 44676.20, 5965.42, 809.95, 105.42 for $\lambda_1, \lambda_2, \lambda_3$ and λ_4 respectively. Its condition number was obtained to be 3.662×10^7 which indicates the presence of severe multicollinearity. The shrinkage parameter were estimated as $\hat{k}_{HKB} = 0.007676, \hat{k}_{MRT} = 0.007664, k_{HMRTP} = 0.007136$ and $\hat{d} = 0.00153$.

Tables 1. Estimted MSEs when n=30

Rho			**0.9**						**0.99**					
k	**d**	**Sigma**	**OLSE**	**RE**	**MRE**	**D**	**MRT**	**MRTP**	**OLSE**	**RE**	**MRE**	**D**	**MRT**	**MRTP**
0.3	0.2	1	0.8171	0.6309	0.5511	0.7711	0.6041	0.0003	9.5095	1.6424	1.0368	5.1597	1.3956	0.0007
		5	20.4290	15.5412	13.3026	19.2536	14.8038	0.0074	237.7363	37.9600	21.9083	128.1597	31.4656	0.0157
		10	81.7159	62.1140	53.1201	77.0059	59.1527	0.0296	950.9452	151.3116	87.0093	512.4163	125.2988	0.0626
	0.5	1	0.8172	0.6309	0.4696	0.7112	0.5687	0.0003	9.5095	1.6424	0.6875	2.9125	1.1405	0.0006
		5	20.4290	15.5412	10.8558	17.6963	13.8068	0.0069	237.7363	37.9600	12.3832	70.8811	24.6882	0.0123
		10	81.7159	62.1140	43.2732	70.7618	55.1467	0.0276	950.9452	151.3116	48.8312	283.1345	98.1477	0.0491
	0.8	1	0.8172	0.6309	0.4163	0.6605	0.5380	0.0003	9.5095	1.6424	0.5376	1.9939	0.9673	0.0005
		5	20.4290	15.5412	9.1096	16.3454	12.9222	0.0065	237.7363	37.9600	8.1496	47.1417	20.0378	0.0100
		10	81.7159	62.1140	36.2321	65.3419	51.5904	0.0258	950.9452	151.3116	31.8515	188.0824	79.5137	0.0398
0.7	0.2	1	0.8172	0.4929	0.4494	0.7174	0.4612	0.0002	9.5095	0.7695	0.6251	3.0660	0.6606	0.0003
		5	20.4290	11.5763	10.2114	17.8584	10.5896	0.0053	237.7363	14.6511	10.6363	74.8227	11.6320	0.0058
		10	81.7159	46.1750	40.6766	71.4119	42.2006	0.0211	950.9452	57.9236	41.8262	298.9148	45.8190	0.0229
	0.5	1	0.8171	0.4929	0.4027	0.6084	0.4238	0.0002	9.5094	0.7695	0.5064	1.4314	0.5559	0.0002
		5	20.4290	11.5763	8.6357	14.9224	9.3644	0.0046	237.7363	14.6510	7.2432	32.4113	8.6750	0.0043
		10	81.7159	46.1750	34.3185	59.6291	37.2606	0.0186	950.9452	57.9236	28.2145	129.0872	33.9594	0.0170
	0.8	1	0.8172	0.4929	0.3707	0.5318	0.3954	0.0002	9.5095	0.7695	0.4417	0.9363	0.4904	0.0002
		5	20.4290	11.5763	7.4492	12.7391	8.3721	0.0042	237.7363	14.6511	5.3269	19.1991	6.7762	0.0034
		10	81.71588	46.17498	29.5221	29.5221	33.2538	0.0166	950.9452	57.9236	20.5223	76.1529	26.3401	0.01317
0.9	0.2	1	0.8172	0.4494	0.4163	0.6934	0.4193	0.0002	9.5095	0.6251	0.5376	2.5286	0.5447	0.0003
		5	20.4290	10.2114	9.1096	17.2250	9.2100	0.0046	237.7363	10.6363	8.1496	60.9949	8.3534	0.0042
		10	81.7159	40.6766	36.2321	68.8715	36.6375	0.0183	950.9452	41.8262	31.8515	243.5532	32.6689	0.0163
	0.5	1	0.8172	0.4494	0.3801	0.5687	0.3853	0.0002	9.5095	0.6251	0.4595	1.1405	0.4696	0.0002
		5	20.4290	10.2114	7.8103	13.8068	8.0029	0.0040	237.7363	10.63632	5.861347	24.68821	6.163167	0.003081584
		10	81.7159	40.6766	30.9829	55.1467	31.7618	0.0159	950.9452	41.8262	22.6680	98.1477	23.8796	0.011939
	0.8	1	0.8172	0.4494	0.3549	0.4880	0.3608	0.0002	9.5095	0.6251	0.4138	0.7512	0.4239	0.0002
		5	20.4290	10.2114	6.8111	11.4255	7.0540	0.0035	237.7363	10.6363	4.4754	14.1459	4.7862	0.0024
		10	81.71588	40.6766	26.9386	45.5678	27.9225	0.0140	950.9452	41.8262	17.1023	55.8982	18.3509	0.0092

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Tables 2. Estimated MSEs when n=50

Rho			**0.9**						**0.99**					
k	**d**	**Sigma**	**OLSE**	**RE**	**MRE**	**D**	**MRT**	**MRTP**	**OLSE**	**RE**	**MRE**	**D**	**MRT**	**MRTP**
0.3	0.2	1	0.3731	0.3334	0.3116	0.3644	0.3265	0.0002	3.9499	1.5381	1.0476	3.0885	1.3522	0.0007
		5	9.3282	8.3245	7.7525	9.1116	8.1456	0.0041	98.7495	37.65815	24.67898	77.2007	32.78672	0.0164
		10	37.3126	33.3002	31.0104	36.4472	32.5842	0.0163	394.998	150.5768	98.56483	308.8264	131.0618	0.0655
	0.5	1	0.3731	0.3334	0.2844	0.3521	0.3167	0.0002	3.9500	1.5381	0.7046	2.2781	1.1402	0.0006
		5	9.3282	8.3245	7.0106	8.8022	7.8891	0.0039	98.7495	37.6582	15.2253	56.7057	27.1648	0.0136
		10	37.3126	33.3002	28.0373	35.2107	31.5575	0.0158	394.9978	150.5768	60.6308	226.8324	108.5309	0.0543
	0.8	1	0.3731	0.3334	0.2625	0.3406	0.3076	0.0002	3.9500	1.5381	0.5410	1.7754	0.9834	0.0005
		5	9.3282	8.3245	6.3819	8.5102	7.6460	0.0038	98.7495	37.6582	10.4660	43.8153	22.9418	0.0115
		10	37.3126	33.3002	25.5137	34.0430	30.5840	0.0153	394.9978	150.5768	41.5027	175.2341	91.5982	0.0458
0.7	0.2	1	0.3731	0.2928	0.2766	0.3534	0.2812	0.0001	3.9500	0.7898	0.6377	2.3492	0.6759	0.0003
		5	9.3282	7.2440	6.7898	8.8357	6.9208	0.0035	98.7495	17.6210	13.3096	58.5161	14.4093	0.0072
		10	37.3126	28.9732	27.1511	35.3446	27.6771	0.0138	394.9978	70.2499	52.9351	234.0769	57.3533	0.0287
	0.5	1	0.3731	0.2928	0.2562	0.3276	0.2658	0.0001	3.9500	0.7898	0.5055	1.3803	0.5615	0.0003
		5	9.3282	7.2440	6.1934	8.1749	6.4799	0.0032	98.74945	17.62103	9.391818	33.5257	11.0790	0.0055
		10	37.3126	28.9732	24.7561	32.7017	25.9072	0.0130	394.9978	70.2499	37.1809	134.0226	43.9683	0.0220
	0.8	1	0.3731	0.2928	0.2396	0.3057	0.2526	0.0001	3.9500	0.7898	0.4309	0.9541	0.4872	0.0002
		5	9.3282	7.2440	5.6812	7.5937	6.0848	0.0030	98.7495	17.6210	7.0458	22.1452	8.8298	0.0044
		10	37.3126	28.9731	22.6960	30.3746	24.3195	0.0122	394.9978	70.2499	27.7327	88.4033	34.9187	0.0175
0.9	0.2	1	0.3731	0.2766	0.2625	0.3482	0.2638	0.0001	3.9500	0.6377	0.5410	2.0855	0.5490	0.0003
		5	9.3282	6.7898	6.3819	8.7030	6.4208	0.0032	98.7495	13.3096	10.4660	51.7894	10.7045	0.0054
		10	37.3126	27.1511	25.5136	34.8141	25.6699	0.0128	394.9978	52.9351	41.5027	207.1560	42.4621	0.0212
	0.5	1	0.3731	0.2766	0.2447	0.3167	0.2475	0.0001	3.9500	0.6377	0.4516	1.1402	0.4633	0.0002
		5	9.3282	6.7898	5.8436	7.8891	5.9279	0.0030	98.7495	13.3096	7.7111	27.1648	8.0829	0.0040
		10	37.3126	27.1511	23.3497	31.5575	23.6885	0.0118	394.9978	52.9351	30.4133	108.5309	31.9110	0.0160
	0.8	1	0.3731	0.2766	0.2303	0.2911	0.2339	0.0001	3.9500	0.6377	0.3983	0.7710	0.4101	0.0002
		5	9.3282	6.7898	5.3787	7.1963	5.4963	0.0027	98.7494	13.3096	5.9666	17.0959	6.3634	0.0032
		10	37.3126	27.1511	21.4773	28.7818	21.9513	0.0110	394.9978	52.9351	30.4133	108.5309	31.9110	0.0160

Tables 3. Estimated MSEs when n=100

Rho			**0.9**						**0.99**					
k	**d**	**Sigma**	**OLSE**	**RE**	**MRE**	**D**	**MRT**	**MRTP**	**OLSE**	**RE**	**MRE**	**D**	**MRT**	**MRTP**
0.3	0.2	1	0.2181	0.2042	0.1967	0.2150	0.2018	0.0001	2.5506	1.2584	0.9441	2.1272	1.1423	0.0006
		5	5.4523	5.0676	4.8370	5.3713	4.9964	0.0025	63.7640	30.1485	21.4437	52.9902	26.9734	0.0135
		10	21.8091	20.2594	19.3277	21.4833	19.9719	0.0100	255.0558	120.3330	85.4018	211.8975	107.5949	0.0538
	0.5	1	0.2181	0.2042	0.1877	0.2107	0.1985	0.0001	2.5506	1.2584	0.7042	1.6928	1.0055	0.0005
		5	5.4523	5.0676	4.5242	5.2537	4.8928	0.0024	63.7640	30.1485	14.4785	41.7275	23.1753	0.0116
		10	21.8091	20.2594	18.0612	21.0098	19.5537	0.0098	255.0558	120.3330	57.4259	166.7623	92.3528	0.0462
	0.8	1	0.2181	0.2042	0.1809	0.2067	0.1954	0.0001	2.5506	1.2584	0.5787	1.4022	0.9008	0.0005
		5	5.4523	5.0676	4.2459	5.1406	4.7931	0.0024	63.7640	30.1485	10.6433	34.0273	20.2121	0.0101
		10	21.8091	20.2594	16.9307	20.5540	19.1502	0.0096	255.0558	120.3330	42.0077	135.8898	80.4569	0.0402
0.7	0.2	1	0.2181	0.1905	0.1852	0.2112	0.1867	0.0001	2.5506	0.7660	0.6542	1.7325	0.6830	0.0003
		5	5.4523	4.6243	4.4279	5.2666	4.4852	0.0022	63.7640	16.3112	12.9701	42.7679	13.8411	0.0069
		10	21.8091	18.4671	17.6703	21.0616	17.9030	0.0090	255.0558	64.7899	51.3633	170.9326	54.8642	0.0274
	0.5	1	0.2181	0.1905	0.1790	0.2022	0.1819	0.0001	2.5506	0.7660	0.5498	1.1600	0.5951	0.0003
		5	5.4523	4.6243	4.1599	5.0081	4.2901	0.0021	63.7640	16.3112	9.7308	27.4616	11.1551	0.0056
		10	21.8091	18.4671	16.5806	20.0194	17.1106	0.0086	255.0558	64.7899	38.3368	109.5536	44.0660	0.0220
	0.8	1	0.2181	0.1905	0.1744	0.1947	0.1780	0.0001	2.5506	0.7660	0.4857	0.8808	0.5346	0.0003
		5	5.4523	4.6243	3.9199	4.7714	4.1098	0.0021	63.7640	16.3112	7.6527	19.6410	9.2444	0.0046
		10	21.8091	18.4671	15.6017	19.0624	16.3765	0.0082	255.0558	64.7899	29.9741	78.1637	36.3801	0.0182
0.9	0.2	1	0.2181	0.1852	0.1809	0.2094	0.1813	0.0001	2.5506	0.6542	0.5787	1.5836	0.5851	0.0003
		5	5.4523	4.4279	4.2459	5.2155	4.2635	0.0021	63.7640	12.9701	10.6433	38.8529	10.8432	0.0054
		10	21.8091	17.6703	16.9307	20.8560	17.0023	0.0085	255.0558	51.3633	42.0077	155.2388	42.8116	0.0214
	0.5	1	0.2181	0.1852	0.1758	0.1985	0.1765	0.0001	2.5506	0.6542	0.5040	1.0055	0.5142	0.0003
		5	5.4523	4.4279	3.9970	4.8928	4.0367	0.0020	63.7640	12.9701	8.2555	23.1753	8.5875	0.0043
		10	21.8091	17.6703	15.9167	19.5537	16.0784	0.0080	255.0558	51.3633	32.4007	92.3528	33.7365	0.0169
	0.8	1	0.2181	0.1852	0.1721	0.1899	0.1730	0.0001	2.5506	0.6542	0.4556	0.7525	0.4667	0.0002
		5	5.4523	4.4279	3.7734	4.6040	3.8308	0.0019	63.7640	12.9701	6.6468	15.9141	7.0210	0.0035
		10	21.8091	17.6703	15.0030	18.3847	15.2377	0.0076	255.0558	51.3633	25.9236	63.1943	27.4307	0.0137

Table 4. Regression coefficients and MSEs of the portland data.

Estimators	Coefficients					MSEs
	α_0	α_1	α_2	α_3	α_4	
$\hat{\alpha}_{OLS}$	62.4054	1.5511	0.5102	0.1019	-0.1441	4912.09
$\hat{\alpha}_{RRE}(\hat{k}_{HKB})$	8.5415	-1.6371	-0.2099	-0.9160	-1.8400	2989.80
$\hat{\alpha}_{MRRE}(\hat{k}_{HKB})$	8.8989	-1.6371	-0.2099	-0.9172	-1.8574	785.70
$\hat{\alpha}_{D}(\hat{k}_{MRT}, \hat{d})$	61.7741	-1.6371	-0.2099	-0.9160	-1.8401	4818.48
$\hat{\alpha}_{MRT}(\hat{k}_{MRT}, \hat{d})$	8.5415	-1.6371	-0.2099	-0.9160	-1.8400	2980.84
$\hat{\alpha}_{MRTP}(\hat{k}_{MRT}, \hat{d})$	15.9134	-1.6371	-0.2099	-0.9160	-1.8401	148.38
$\hat{\alpha}_{MRTP}(k_{HMRTP}, \hat{d})$	16.3799	-1.6371	-0.2099	-0.9160	-1.8401	159.28

The same shrinkage parameter used by Lukman et al. (2019) was adopted for the MRTP estimator, and we observed that MSE of the MRTP estimator in both cases is smaller than the MSE of the ridge estimator, modified ridge estimator, two-parameter estimator by Dorugade and MRT estimator. The MSE of the modified ridge estimator was next observed to the smallest while OLS estimator performed least as expected.

CONCLUSION

In this article, a new estimator was proposed by adding prior information to the modified ridge-type parameter to overcome the problem of multicollinearity in a linear regression model. We established the superiority of the new estimator with other existing estimators under the mean square error criterion. This new estimator was shown to include the modified ridge-type estimator, ridge estimator and the ordinary least squared estimator as individual cases. The estimators were compared by applying it to real life data and a simulation study, which further proves the superiority of the proposed estimator (MRTP) as compared to others.

REFERENCES

[1] Dorugade AV. Modified two parameter estimator in linear regression. *J Stat Trans New Ser*. 2014;15(1):23-36.

[2] Farebrother RW. “Further results on the mean square error of ridge regression,” *Journal of the Royal Statistical Society: Series B (Methodological)*. 1976;38(3):248–250.

[3] Hoerl AE, Kennard RW. Ridge regression: biased estimation for nonorthogonal problems. *Dent Tech*. 1970;12(1):55-67.

[4] Hoerl AE, Kennard RW, Baldwin KF. Ridge regression: some simulation. *Commun Stat*. 1975;4(2):105-123.

[5] Liu K. A new class of biased estimate in linear regression. *Commun Stat Theor Meth*. 1993;22:393-402.

[6] Lukman AF, Ayinde K. Review and classifications of the ridge parameter estimation techniques. *Hacettepe Math Stat*. 2017;46(5):953-967.

[7] Lukman AF, Ayinde K, Ajiboye SA. Monte-Carlo study of some classification-based ridge parameter estimators. *J Mod Appl Stat Methods*. Q9 2017;16(1):428-451.

[8] Lukman AF, Ayinde K, Binuomote S, Clement OA. Modified ridge-type estimator to combat multicollinearity: Applications to chemical data. *Journal of Chemometrics*. 2019;3125

[9] Massy WF. “Principal components regression in exploratory statistical research,” *Journal of the American Statistical Association*. 1965;60(309): 234–266.

[10] Newhouse JP, Oman SD. An evaluation of ridge estimators. *Rand Rep*. 1971;R-716-PR:1-28.

[11] Stein C. “Inadmissibility of the usual estimator for mean ofmultivariate normal distribution,” in *Proceedings of the Third Berkley SymSymposium on Mathematical and Statistics Probability*, J. Neyman, Ed. Berkeley, CA, USA. 1956;1:197–206.

[12] Swindel FF. “Good ridge estimators based on prior information,”*Communications in Statistics-Theory and Methods*. 1976;11:1065–1075.

[13] Trenkler G, Toutenburg H. “Mean squared error matrix comparisons between biased estimators - an overview of recent results,” *Statistical Papers*. 1990;31(1):165–179,.

[14] Woods H, Steinour HH, Starke HR. Effect of composition of Portland cement on heat evolved during hardening. *Ind Eng Chem*. 1932;24(11):1207-1241.