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Analytical Solutions of Computer Virus Propagation Model with Anti-virus Software and Time Dependent Connecting Network in Caputo Fractional Derivative Sense

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Abstract

Computer viruses and related malware are major problems for personal and corporate computers and for computer networks. A common approach to prevent the destructive effects of viruses on computers is to install anti-virus programs on individual computers. An alternative approach to prevent the propagation of viruses in computer networks is to use mathematical models similar to those developed for understanding and preventing the spread of epidemics in human, animal or plant populations. In this paper, we focus on the fully susceptible, partially susceptible, infectious, protected (FSSIP) model including Caputo fractional derivative to study the effect of anti-virus software on infection in computer networks. We apply the shifted Chebyshev collocation method to obtain analytical solutions (series solutions) of the FSSIP model with the virus input into each group which is time-dependent function due to the appearance of new viruses and updating of the installed anti-virus software. This method is simple, efficiency, and accurate when comparing the numerical results to Runge-Kutta-Fehlberg method (RKF45) and Cash-Karp method (CK45).

Keywords: Computer viruses, anti-virus programs, Caputo fractional differential equations, FSSIP model, shifted Chebyshev collocation method.

I. INTRODUCTION

With the widespread use of the internet, systems running on networked computers become more vulnerable to digital threats such as computer viruses, ILOVEYOU, Redcode, Melissa and Sasser. These threats pose a serious challenge to information security. Many programmers and anti-virus software companies currently develop and update anti-virus programs against the new viruses, but more computer viruses are continuously created. However, most anti-virus software currently installed in computers can only offer a temporary immunity against viruses because of the rapid development of new viruses and the time lag between the dissemination of the new virus and the dissemination of updated anti-virus software. Due to the fact that the development of anti-virus software often lags behind the appearance of new viruses, many computers are only partially protected and can be susceptible to attacks that can cause the loss of millions of dollars worth of data and loss of productivity. It is therefore important to understand the way that computer viruses spread through computer networks and to work out effective defense measures [1, 2].

In the past several decades, many researchers have used mathematical models for determining the transmission of disease as models to study the spread of computer viruses through computer networks. For example, the SIR classical epidemic model[3, 4] has been used to develop models for the spreading and attacking behaviour of computer viruses in many different phenomena, including virus propagation [1-6], time delay [2], fuzziness [3], effect of anti-virus software [7, 8], virus immunization [9], vaccination [10], quarantine [11, 12], etc. May et al. [13] studied the dynamical behaviour of viruses on scale-free networks, and T. Chen et al. [12] studied the fast quarantining of proactive worms in peer-to-peer (P2P) computer networks. Z. Wang et al. [14, 15] studied the robustness of filtering on non-linearities in packet losses, sensors.

The spread of computer virus phenomenon is unusual and some computer viruses can spread to ten million computers in a short time period. These spreads (revolutions) led to use of mathematical tools based on the derivative and integrals with integer order, and the differential and difference equations. Actually, the present moment new spread changes are actually taking place in modern computer viruses. These changes can be called a revolution of memory and non-locality. It is increasingly obvious in computer viruses when their behaviour may depend on the history of previous changes in computer viruses. Many researchers try to propose fractional generalization models which include Riemann–Liouville, Caputo, Hadamard, Riesz and Grünwald–Letnikov fractional derivative that improved the computer virus models for researching processes with memory or history. The suggested factional models are realized by considering non-integer fractional order $0 < \alpha \leq 1$ instead of positive integer values of α. Fractional modelling is a more advantageous approach which has been used to study the behaviour of diseases or viruses than the integer derivative model which is local in nature, while the fractional derivative is global (non-local). In addition, the fractional derivative is used to increase the stability region of the system.

In 2014, C.M. A. Pinto and J. A. T. Machado [16] proposed a fractional model for computer virus propagation which includes the interaction between computers and removable devices by numerically simulating the model for distinct values of the order of the fractional derivative and for two sets of initial conditions. In 2015, A. H. Handam and A. A. Freihat [17] modified the epidemiological model for computer viruses (SAIR) proposed by J. R. C. Piqueira and V. O. Araujo by fractional derivatives described in the Caputo sense. In 2016, A. M. A. El-Sayed, A. A. M. Arafa, M. Khalil and A. Hassan [18] proposed numerical simulations that are used to show the behaviour of a fractional-order mathematical model with memory for the propagation of computer virus under human intervention. In 2017, E. Bonyah, A. Atangana and M. Khan [19] investigated the analytical solutions of modeling the spread of computer virus via Caputo fractional derivative and the Beta- derivative solved by the Laplace perturbation method and the homotopy decomposition technique. In 2018, J. Singh, D. Kumar, Z. Hammouch and A. Atangana [20] analyzed a moderate fractional epidemiological model to describe computer viruses with an arbitrary order derivative having a non-singular kernel.

Chebyshev method which is a relatively power tool and an emerging area in mathematical researches is greatly useful for solving differential equations, fractional differential equations, integro-differential equations and fractional Volterra integro-differential equations. The properties of Chebyshev are used to make the operational matrix which eventually leads to the coefficient matrix of obtained system. In 2007, G.C. Malachowski, R.M. Clegg and G.I. Redford [21] proposed analytic solutions to model exponential and harmonic functions by using Chebyshev polynomials. In 2015 A. Rivaz, S. J. Ara and F. Yousef [22] studied two-dimensional Chebyshev polynomials for solving two-dimensional integro-differential equations. In 2016, O. B. Arushanyan and S. F. Zaletkin [23] applied Chebyshev series to approximate analytic solution of ordinary differential equations. In 2017, N. Razmjooy and M. Ramezani [24] proposed analytical solution for optimal control by the second kind Chebyshev polynomials expansion. Moreover, many mathematical methods have been proposed for finding analytical solutions based on transform methods or perturbation methods or expansions as series of orthogonal functions. The transformation methods that have been used include Laplace, Fourier and Mellin transforms [25], the Tau method [26]. The perturbation methods include the Adomian decomposition method [27], the variational iteration method [28, 29] and the Sumudu decomposition method [30]. The methods based on expansions in orthogonal functions include expansions in blockpulse functions [31], shifted Chebyshev polynomials [32], shifted Legendre polynomials [33], Chebyshev wavelets [34], Legendre wavelets [35] etc.

II. MODEL FORMULATION

In this paper, we propose a fully and partially susceptible, infectious, protected computer virus propagation model (FSSIP) in Caputo fractional order-α derivative sense, which was originally studied by B. K. Mishra and S. K. Pandey [36]

$$\begin{aligned}
\frac{dS}{dt} &= (1-p)b - \beta SI - \rho S - dS + \gamma S_\gamma + \chi P \\
\frac{dS_\gamma}{dt} &= pb + \rho S - \gamma S_\gamma - dS_\gamma - \xi S_\gamma - (1-\sigma)\beta S_\gamma I \\
\frac{dI}{dt} &= \theta b + \beta SI + (1-\sigma)\beta S_\gamma I - dI - \delta I - \eta I \\
\frac{dP}{dt} &= \xi S_\gamma + \delta I - dP - \chi P.
\end{aligned} \tag{1}$$

In this model, computers can be divided into four groups: fully susceptible (no anti-virus software), partially susceptible (partially effective anti-virus software), infected (infected by virus), and protected (100% effective anti-virus software). Moreover, the connecting network as a function depends on the time between to the appearance of new viruses and updating of the installed anti-virus software. The flow chart of the generalized FSSIP model is shown in Fig. 1.

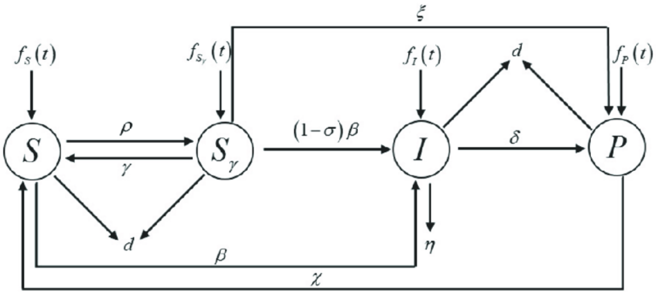


Fig. 1. Flow chart for the generalized FSSIP computer network model.

The model (1) can be extended by the Caputo fractional derivatives of orders $\alpha \in (0,1]$, so the resulting equations still have the problem in which the units of the left-hand side and the right-hand side of the resulting system mismatch, the units of the left-hand side are $\text{time}^{-\alpha}$ while the units of the right-hand side of the system have the dimension time^{-1}. Therefore, we must preserve units as described in [37, 38] on both sides of each equation in the resulting system by changing some of the original parameters.

The propagation of computer virus can be determined and modelled by the system of non-linear Caputo fractional differential equations

$$\begin{aligned}
D_t^\alpha S(t) &= f_S^\alpha(t) - \left[\beta^\alpha I(t) + \rho^\alpha + d^\alpha\right] S(t) + \gamma^\alpha S_\gamma(t) + \chi^\alpha P(t) \\
D_t^\alpha S_\gamma(t) &= f_{S_\gamma}^\alpha(t) + \rho^\alpha S(t) - \left[\gamma^\alpha + d^\alpha + \xi^\alpha + (1-\sigma)\beta^\alpha I(t)\right] S_\gamma(t) \\
D_t^\alpha I(t) &= f_I^\alpha(t) + \left[\beta^\alpha S(t) + (1-\sigma)\beta^\alpha S_\gamma - d - \delta^\alpha - \eta^\alpha\right] I(t) \\
D_t^\alpha P(t) &= f_P^\alpha(t) + \xi^\alpha S_\gamma(t) + \delta^\alpha I(t) - \left[d^\alpha + \chi^\alpha\right] P(t),
\end{aligned} \tag{2}$$

with initial conditions $S(0), S_\gamma(0), I(0)$ and $P(0)$, where $D_t^\alpha(\cdot)$ is Caputo fractional derivative operator, the value of $\alpha \in (0,1]$ represents fractional order of Caputo fractional derivative and the definitions of the variables, parameters, and functions in the model are given in Table 1.

Table 1. Definitions of parameters and functions in the generalized FSSIP model

Parameter	Explanation	Units
$f_S(t)$	Rate of attachment of new fully susceptible S nodes to the network	minite^{-1}
$f_{S_\gamma}(t)$	Rate of attachment of new partially susceptible S_γ nodes to the network	minite^{-1}
$f_I(t)$	Rate of attachment of new infected I nodes to the network	minite^{-1}
$f_P(t)$	Rate of attachment of new fully protected P nodes to the network	minite^{-1}
$d>0$	Rate at which internal nodes are detached from the network	minite^{-1}
$\rho>0$	Rate at which S nodes move to the S_γ group due to installation of anti-virus software	minite^{-1}
$\gamma>0$	Rate at which S_γ nodes move to the S group due to out-of-date anti-virus software	minite^{-1}
$\beta>0$	Rate at which S nodes become infected and move to I group	minite^{-1}
$0<\sigma<1$	Effectiveness factor of anti-virus software in reducing rate at which S_γ nodes become infected	
$\xi>0$	Rate at which S_γ nodes move to P group due to installation of anti-virus software	minite^{-1}
$\delta>0$	Rate at which virus is removed from I nodes by anti-virus software and they move to P group	minite^{-1}
$\eta>0$	Rate at which I nodes crash due to infection by virus	minite^{-1}
$\chi>0$	Rate at which P nodes lose protection and move to S group due to out-of-date anti-virus software	minite^{-1}

III. PRELIMINARIES

III.I Caputo fractional Derivative

In this section we introduce some necessary definitions of Caputo fractional derivative [39].

Definition 1. Let $\alpha \in \mathbb{R}^+, n = \lceil \alpha \rceil$ and $u \in C^n[a,b]$. Then the Caputo fractional derivative of $u(t)$ is defined by

$$D_a^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{u^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx,$$

with some properties of Caputo fractional derivatives;

$$D_a^\alpha C = 0 \quad C \text{ is a constant} \tag{3}$$

$$D_a^\alpha t^\beta = \begin{cases} 0, & \beta < \lceil \alpha \rceil \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & \beta \geq \lceil \alpha \rceil \end{cases} \tag{4}$$

where $\beta \in \mathbb{N} \cup 0$. $\lfloor \alpha \rfloor$ denotes the largest integer less than or equal to α and $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

III.II Existence of solutions for Caputo fractional differential equation

Theorem 2. Assuming a function f in the initial value problem of fractional order with $\alpha \in (0,1]$

$$D_a^\alpha u(t) = f(t, u(t)), \quad u(a) = u_0, \quad u \in \mathbb{R}^n \tag{5}$$

is continuous and bounded. Then there exists a solution [40].

Theorem 3. The function $u(t) \in C[a,b]$ is a solution of the initial value problem in (5) if and only if it is a solution of the nonlinear Volterra integral equation

$$u(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} u_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau, \tag{6}$$

where $m = \lceil \alpha \rceil$ [40].

III.III Shifted Chebyshev polynomials

Definition 4. Chebyshev polynomials $T_n(s)$ of the first kind are polynomials in s of degree $n = 0,1,2,3,\ldots,$ which are defined by the relation [41, 42]:

$$T_n(s) = \cos(n\theta), \quad \text{where } s = \cos(\theta).$$

This leads to shifted Chebyshev polynomials (of the first kind) $T_n^*(t)$ of degree n for $t\in[0,1]$ given by

$$T_n^*(t)=T_n(s)=T_n(2t-1),$$

where $T_0^*(t)=1, T_1^*(t)=2t-1$ and the shifted Chebyshev polynomials $T_{n+1}^*(t)$

$$T_{n+1}^*(t)=2(2t-1)T_n^*(t)-T_{n-1}^*(t),\quad n=1,2,3,\ldots \tag{7}$$

The zeros of the shifted Chebyshev polynomials $T_{N+1}^*(t)$ are given by

$$t_n=\frac{1}{2}\left(1+\cos\left(\frac{[4(N-n)+3]\pi}{4(N+1)}\right)\right),\quad n=0,1,2,\ldots,N. \tag{8}$$

By shifted Chebyshev method, the solution $u(t)$ can be expressed in terms of shifted Chebyshev polynomials as

$$u(t)=\sum_{n=0}^{\infty}a_nT_n^*(t) \tag{9}$$

where the coefficients a_n are given by

$$\begin{aligned}a_0&=\frac{1}{\pi}\int_0^1\frac{u(t)T_0^*(t)}{\sqrt{t-t^2}}dt,\\ a_n&=\frac{2}{\pi}\int_0^1\frac{u(t)T_n^*(t)}{\sqrt{t-t^2}}dt,\qquad n=1,2,3,\ldots\end{aligned} \tag{10}$$

In practice, the first (N+1)-terms of shifted Chebyshev polynomials are only considered, so we have an approximate analytical solution as:

$$u^N(t)=\sum_{n=0}^{N}a_nT_n^*(t). \tag{11}$$

Theorem 5. Let $u^N(t)$ be approximated by Chebyshev polynomials as (11) and $0<\alpha<1$; then the Caputo fractional derivative of $u^N(t)$ is obtained by [43]

$$D^{\alpha}(u^N(t))=\sum_{n=\lceil\alpha\rceil}^{N}\sum_{k=\lceil\alpha\rceil}^{n}a_nw_{n,k}^{(\alpha)}t^{k-\alpha}, \tag{12}$$

where $w_{n,k}^{(\alpha)}$ is given by

$$w_{n,k}^{(\alpha)}=(-1)^{n-k}\frac{2^{2k}n(n+k-1)!\Gamma(k+1)}{(2k)!(n-k)!\Gamma(k+1-\alpha)}. \tag{13}$$

III.IV Shifted Chebyshev analytical solution

In the shifted Chebyshev method, the approximate analytical solution of the FSSIP model can be written as a sum of shifted Chebyshev polynomials as the form:

$$\begin{aligned}S^N(t)&=\sum_{n=0}^{N}a_n^{(1)}T_n^*(t),\qquad S_{\gamma}^N(t)=\sum_{n=0}^{N}a_n^{(2)}T_n^*(t),\\ I^N(t)&=\sum_{n=0}^{N}a_n^{(3)}T_n^*(t),\qquad P^N(t)=\sum_{n=0}^{N}a_n^{(4)}T_n^*(t),\end{aligned} \tag{14}$$

where the real-numbers $a_n^{(k)},\ k=1,2,3,4,\ 0\le n\le N$ are unknown coefficients which can be determined later. By constructing some matrices in (14), we have

$$\begin{aligned}S^{(N)}(t)&=\mathbf{A_1}\mathbf{T}^*(t),\qquad S_{\gamma}^{(N)}(t)=\mathbf{A_2}\mathbf{T}^*(t),\\ I^{(N)}(t)&=\mathbf{A_3}\mathbf{T}^*(t),\qquad P^{(N)}(t)=\mathbf{A_4}\mathbf{T}^*(t),\end{aligned} \tag{15}$$

where the shifted Chebyshev column vector $\mathbf{T}^*(t)=[T_0^*(t)\ \ T_1^*(t)\ \ \ldots\ \ T_N^*(t)]^T$ and the coefficient row vectors $\mathbf{A_1},\mathbf{A_2},\mathbf{A_3},\mathbf{A_4}$ are defined by

$$\begin{aligned}\mathbf{A_1}&=[a_0^{(1)}\ \ a_1^{(1)}\ \ \ldots\ \ a_N^{(1)}],\ \mathbf{A_2}=[a_0^{(2)}\ \ a_1^{(2)}\ \ \ldots\ \ a_N^{(2)}],\\ \mathbf{A_3}&=[a_0^{(3)}\ \ a_1^{(3)}\ \ \ldots\ \ a_N^{(3)}],\mathbf{A_4}=[a_0^{(4)}\ \ a_1^{(4)}\ \ \ldots\ \ a_N^{(4)}].\end{aligned} \tag{16}$$

From (7), each term $t^i, i=1,2,3,\ldots,N$ can be rewritten as the combination of the shifted Chebyshev functions.

$$\begin{aligned}t^0&=T_0^*\\ t^1&=2^{-1}(T_0^*+T_1^*)\\ t^2&=2^{-3}(6T_0^*+4T_1^*+T_2^*)\\ &\ \vdots\\ t^N&=2^{-2N+1}\sum_{k=0}^{N}\binom{2N}{k}T_{N-k}^*(t),\quad 0\le t\le 1.\end{aligned} \tag{17}$$

Define $\mathbf{Y}(t)=[1\ \ t\ \ \ldots\ \ t^N]^T$, then it obtains

$$\mathbf{Y}(t)=\mathbf{C}\mathbf{T}^*(t),\quad\text{or}\quad \mathbf{T}^*(t)=\mathbf{C}^{-1}\mathbf{Y}(t), \tag{18}$$

where the coefficient shifted Chebyshev matrix $\mathbf{C}$ is given as [44]

$$\mathbf{C}=\begin{bmatrix}2^0\binom{0}{0} & 0 & \cdots & 0\\ 2^{-2}\binom{2}{1} & 2^{-1}\binom{2}{0} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ k\binom{2N}{N} & 2k\binom{2N}{N-1} & \cdots & 2k\binom{2N}{0}\end{bmatrix}, \tag{19}$$

and $k=2^{-2N}$. Therefore, the solutions in (15) can be rewritten in terms of the product of matrices as

$$\begin{aligned}S^{(N)}(t)&=\mathbf{A_1}\mathbf{C}^{-1}\mathbf{Y}(t),\qquad S_{\gamma}^{(N)}(t)=\mathbf{A_2}\mathbf{C}^{-1}\mathbf{Y}(t),\\ I^{(N)}(t)&=\mathbf{A_3}\mathbf{C}^{-1}\mathbf{Y}(t),\qquad P^{(N)}(t)=\mathbf{A_4}\mathbf{C}^{-1}\mathbf{Y}(t).\end{aligned} \tag{20}$$

Applying the Caputo fractional derivative in (20), we also construct the operational matrix for the Caputo fractional derivative of $\mathbf{Y}(t)$ as

$$D_t^{\alpha}\mathbf{Y}(t)=\mathbf{B}_{\alpha}(t)\mathbf{Y}(t),$$

where the Caputo Chebyshev fractional derivative matrix $\mathbf{B}_{\boldsymbol{\alpha}}(t)$ is given by

$$\mathbf{B}_{\alpha}(t)=t^{-\alpha}\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} & 0 & \cdots & 0\\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N+1)}{\Gamma(N+1-\alpha)}\end{bmatrix}. \quad (21)$$

Now, the Caputo fractional derivatives of each solution are given by the $(N+1)\times(N+1)$ matrices

$$\begin{aligned}D_t^{\alpha}S^{(N)}(t)=\mathbf{A_1}\mathbf{C}^{-1}\mathbf{B}_{\alpha}(t)\mathbf{Y}(t), D_t^{\alpha}S_{\gamma}^{(N)}(t)=\mathbf{A_2}\mathbf{C}^{-1}\mathbf{B}_{\alpha}(t)\mathbf{Y}(t),\\ D_t^{\alpha}I^{(N)}(t)=\mathbf{A_3}\mathbf{C}^{-1}\mathbf{B}_{\alpha}(t)\mathbf{Y}(t), D_t^{\alpha}P^{(N)}(t)=\mathbf{A_4}\mathbf{C}^{-1}\mathbf{B}_{\alpha}(t)\mathbf{Y}(t).\end{aligned} \quad (22)$$

Substituting these results into Eq. (2), we have the following system

$$\begin{aligned}&\mathbf{A_1}\mathbf{C}^{-1}\mathbf{B}_{\alpha}(t)\mathbf{Y}(t)+\beta\mathbf{A_1}\mathbf{C}^{-1}\mathbf{Y}(t)\mathbf{A_3}\mathbf{C}^{-1}\mathbf{Y}(t)\\ &\gamma\mathbf{A_2}\mathbf{C}^{-1}\mathbf{Y}(t)-\chi\mathbf{A_4}\mathbf{C}^{-1}\mathbf{Y}(t)+(\rho+d)\mathbf{A_1}\mathbf{C}^{-1}\mathbf{Y}(t)=f_S(t)\\ &\mathbf{A_2}\mathbf{C}^{-1}\mathbf{B}_{\alpha}(t)\mathbf{Y}(t)+(1-\sigma)\beta\mathbf{A_2}\mathbf{C}^{-1}\mathbf{Y}(t)\mathbf{A_3}\mathbf{C}^{-1}\mathbf{Y}(t)\\ &\rho\mathbf{A_1}\mathbf{C}^{-1}\mathbf{Y}(t)+(\gamma+d+\xi)\mathbf{A_2}\mathbf{C}^{-1}\mathbf{Y}(t)=f_{S_\gamma}(t)\\ &(d+\delta+\eta)\mathbf{A_3}\mathbf{C}^{-1}\mathbf{Y}(t)-\beta\mathbf{A_1}\mathbf{C}^{-1}\mathbf{Y}(t)\mathbf{A_3}\mathbf{C}^{-1}\mathbf{Y}(t)\\ &-(1-\sigma)\beta\mathbf{A_2}\mathbf{C}^{-1}\mathbf{Y}(t)\mathbf{A_3}\mathbf{C}^{-1}\mathbf{Y}(t)+\mathbf{A_3}\mathbf{C}^{-1}\mathbf{B}_{\alpha}(t)\mathbf{Y}(t)=f_I(t)\\ &\mathbf{A_4}\mathbf{C}^{-1}\mathbf{B}_{\alpha}(t)\mathbf{Y}(t)+(d+\chi)\mathbf{A_4}\mathbf{C}^{-1}\mathbf{Y}(t)\\ &-\xi\mathbf{A_2}\mathbf{C}^{-1}\mathbf{Y}(t)-\delta\mathbf{A_3}\mathbf{C}^{-1}\mathbf{Y}(t)=f_P(t)\end{aligned} \quad (23)$$

with the following initial conditions :

$$\begin{aligned}S(0)=\mathbf{A_1}\mathbf{C}^{-1}\mathbf{Y}(0),\ S_{\gamma}(0)=\mathbf{A_2}\mathbf{C}^{-1}\mathbf{Y}(0),\\ I(0)=\mathbf{A_3}\mathbf{C}^{-1}\mathbf{Y}(0),\ P(0)=\mathbf{A_4}\mathbf{C}^{-1}\mathbf{Y}(0),\end{aligned} \quad (24)$$

where the matrices $\mathbf{A}_i, i=1,2,3,4$ in (16), $\mathbf{Y}(t)=[1\ \ t\ \ \ldots\ \ t^N]^T$ in (18) and the matrix $\mathbf{C}$ in (19). The system in Eq. (23) and initial conditions (24) can be evaluated at the specific t_i of the $N+1$ roots $(t_0,t_1,\ldots,t_N)$ in Eq. (8). We next define the $(N+1)\times(N+1)$ matrix $\mathcal{Y}$, the $1\times(N+1)$ matrices $\mathbf{F}_S,\mathbf{F}_{S_\gamma},\mathbf{F}_I,\mathbf{F}_P$, and an $1\times 4(N+1)$ matrix $\mathbf{F}$ defined as follows:

$$\begin{aligned}\mathcal{Y}(t^{(N)})&=[\mathbf{Y}(0)\ \ \mathbf{Y}(t_1)\ \ \ldots\ \ \mathbf{Y}(t_N)],\\ \mathbf{F}_S(t^{(N)})&=[S(0), f_S(t_1),\ldots, f_S(t_N)],\\ \mathbf{F}_{S_\gamma}(t^{(N)})&=[S_\gamma(0), f_{S_\gamma}(t_1),\ldots, f_{S_\gamma}(t_N)],\\ \mathbf{F}_I(t^{(N)})&=[I(0), f_I(t_1),\ldots, f_I(t_N)],\\ \mathbf{F}_P(t^{(N)})&=[P(0), f_P(t_1),\ldots, f_P(t_N)],\\ \mathbf{F}(t^{(N)})&=[\mathbf{F}_S(t^{(N)})\ \ \mathbf{F}_{S_\gamma}(t^{(N)})\ \ \mathbf{F}_I(t^{(N)})\ \ \mathbf{F}_P(t^{(N)})],\end{aligned}$$

and we also construct the $4(N+1)\times 4(N+1)$ matrices $\overline{\mathcal{Y}},\overline{\mathbf{B}},\overline{\mathbf{C}}$ and $\overline{\mathbf{T}^*}$

$$\overline{\mathcal{Y}}=\begin{bmatrix}\mathcal{Y} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathcal{Y} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathcal{Y} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{Y}\end{bmatrix},\quad \overline{\mathbf{B}}=\begin{bmatrix}\mathbf{B}_\alpha & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_\alpha & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{B}_\alpha & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_\alpha\end{bmatrix},$$

$$\overline{\mathbf{C}}=\begin{bmatrix}\mathbf{C}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}^{-1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{C}^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{C}^{-1}\end{bmatrix},\quad \overline{\mathbf{T}^*}=\begin{bmatrix}\mathbf{T}^*\mathbf{A_3} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{T}^*\mathbf{A_3} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{T}^*\mathbf{A_3} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}$$

and $1\times 4(N+1)$ matrix $\mathbf{A}=[\mathbf{A_1}\ \ \mathbf{A_2}\ \ \mathbf{A_3}\ \ \mathbf{A_4}]$, finally the system in Eq. (23) can be rewritten in the matrix form as

$$\mathbf{A}\left(\overline{\mathbf{C}}\ \overline{\mathbf{B}}+\mathcal{K}_0\overline{\mathbf{T}^*}\ \overline{\mathbf{C}}+\mathcal{K}_1\overline{\mathbf{C}}\right)\overline{\mathcal{Y}}=\mathbf{F}, \quad (25)$$

where $\mathbf{I}$ is identity matrix and the matrices $\mathcal{K}_0$ and $\mathcal{K}_1$ are given by

$$\mathcal{K}_0=\begin{bmatrix}\beta\mathbf{I} & \mathbf{0} & -\beta\mathbf{I} & \mathbf{0}\\ \mathbf{0} & (1-\sigma)\beta\mathbf{I} & -(1-\sigma)\beta\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},$$

$$\mathcal{K}_1=\begin{bmatrix}(\alpha+d)\mathbf{I} & -\alpha\mathbf{I} & \mathbf{0} & \mathbf{0}\\ -\gamma\mathbf{I} & (\gamma+d+\xi)\mathbf{I} & \mathbf{0} & -\xi\mathbf{I}\\ \mathbf{0} & \mathbf{0} & (d+\delta+\eta)\mathbf{I} & -\delta\mathbf{I}\\ -\chi\mathbf{I} & \mathbf{0} & \mathbf{0} & (d+\chi)\mathbf{I}\end{bmatrix}.$$

Taking transpose, the equation (25) becomes

$$\overline{\mathcal{Y}}^{\mathbf{T}}\left(\overline{\mathbf{B}}^{\mathbf{T}}\ \overline{\mathbf{C}}^{\mathbf{T}}+\mathcal{K}_0^{\mathbf{T}}\overline{\mathbf{C}}^{\mathbf{T}}\ \overline{\mathbf{T}^*}^{\mathbf{T}}+\mathcal{K}_1^{\mathbf{T}}\overline{\mathbf{C}}^{\mathbf{T}}\right)\mathbf{A}^{\mathbf{T}}=\mathbf{F}^{\mathbf{T}}, \quad (26)$$

which can be solved the matrix $\mathbf{A}^{\mathbf{T}}$ and then determine the coefficient matrices $\mathbf{A}_i,\ i=1,2,3,4$ in Eq. (16).

IV. SHIFTED CHEBYSHEV APPROXIMATE ANALYTICAL SOLUTIONS

In this section we applied Shifted Chebyshev method to solve analytical solutions of the FSSIP model in Eq. (20) with some fractional orders $\alpha=0.6,0.8,1$ and several types of functions for connecting network: $f_S^{\alpha}(t)$, $f_{S_\gamma}^{\alpha}(t)$, $f_I^{\alpha}(t)$, $f_P^{\alpha}(t)$ and using some parameters in Table 2.

Table 2. Values of parameters used in the numerical simulations

Parameter	Value	Parameter	Value	Parameter	Value
b	0.05	p	0.03	d^{α}	0.02
ρ^{α}	0.09	γ^{α}	0.07	η^{α}	0
ξ^{α}	0.03	δ^{α}	0	θ	0
χ^{α}	0	β^{α}	0.0005	σ^{α}	0.5

Case 1. Functions for connecting network are given by $f_S^\alpha(t)=(1-p)b$, $f_{S_\gamma}^\alpha(t)=pb$, $f_I^\alpha(t)=\theta b$, $f_P^\alpha(t)=0$ [36] which mean that the new fully (S) and partially susceptible (S_γ), and infected nodes (I) connected to the network with constant rates. Assuming the shifted Chebyshev analytical solutions $(N=6)$ of the FSSIP model (2) as

$$S(t)=\sum_{r=0}^{6}a_r^{(1)}T_r^*(t),\quad S_\gamma(t)=\sum_{r=0}^{6}a_r^{(2)}T_r^*(t),$$
$$I(t)=\sum_{r=0}^{6}a_r^{(3)}T_r^*(t),\quad P(t)=\sum_{r=0}^{6}a_r^{(4)}T_r^*(t). \tag{27}$$

By shifted Chebyshev method, the zeros of $T_6^*(t)$ in Eq. (8) can be calculated as

$$t_0=0.0125,\quad t_1=0.1091,\quad t_2=0.2831,\quad t_3=0.5000,\quad t_4=0.7169,\quad t_5=0.8909,\quad t_6=0.9875. \tag{28}$$

Calculating $\mathbf{C}^{-1}$ in Eqs. (19)–(21) and the shifted Chebyshev matrices $\mathbf{Y}(t_i)$, $\mathbf{B}_\alpha(t_i)$, $i=0,1,\cdots,6$ as

$$\mathbf{Y}=\begin{bmatrix}1.0 & 1.0 & 1.0 & \cdots & 1.0\\ 0.0125 & 0.1091 & 0.2831 & \cdots & 0.9875\\ 0.0002 & 0.0119 & 0.0801 & \cdots & 0.9751\\ 1.9e-6 & 0.0013 & 0.0227 & \cdots & 0.9629\\ 2.5e-8 & 0.0001 & 0.0064 & \cdots & 0.9508\\ 3.0e-10 & 2.0e-5 & 0.0018 & \cdots & 0.9389\\ 3.8e-12 & 1.7e-6 & 0.0005 & \cdots & 0.9271\end{bmatrix},$$

$$\mathbf{B}_\alpha(t_i)=t_i^{-\alpha}\begin{bmatrix}0 & 0 & 0 & \cdots & 0\\ 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha)} & 0 & \cdots & 0\\ 0 & 0 & \frac{\Gamma(3)}{\Gamma(3-\alpha)} & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \frac{\Gamma(7)}{\Gamma(7-\alpha)}\end{bmatrix},$$

$$\mathbf{C}^{-1}=\begin{bmatrix}1 & 0 & 0 & \cdots & 0\\ -1 & 2 & 0 & \cdots & 0\\ 1 & -8 & 8 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & -72 & 840 & \cdots & 2048\end{bmatrix}.$$

The symbolic computation of the shifted Chebyshev method has already developed to compute the coefficients matrices $\mathbf{A_i}$ with $\alpha=1$ as

$$\mathbf{A_1}=\left[10.1410\ \ -15.9771\ \ 10.7495\ \ -6.4387\ \ \cdots\ \ 1.0234\right]^T,$$
$$\mathbf{A_2}=\left[5.4365\ \ -6.9952\ \ 2.5502\ \ 0.6439\ \ \cdots\ \ -1.0339\right]^T,$$
$$\mathbf{A_3}=\left[10.0527\ \ -10.0267\ \ 1.5776\ \ 0.7105\ \ \cdots\ \ -0.0692\right]^T,$$
$$\mathbf{A_4}=\left[5.3473\ \ -0.9537\ \ -3.1931\ \ 2.2911\ \ \cdots\ \ 0.0880\right]^T.$$

then the shifted Chebyshev approximate analytical solutions $(N=6)$ which provide a finite series are given by

$$\begin{aligned}S^{(N)}(t)&=50-3.5049t+0.1178t^2-0.0021t^3+\left(2.0\times10^{-5}\right)t^4\\&\quad-\left(9.6575\times10^{-8}\right)t^5+\left(1.8401\times10^{-10}\right)t^6\\ S_\gamma^{(N)}(t)&=10+1.2419t-0.0797t^2+0.0018t^3-\left(1.9\times10^{-5}\right)t^4\\&\quad+\left(9.4486\times10^{-8}\right)t^5-\left(1.8590\times10^{-10}\right)t^6\\ I^{(N)}(t)&=20+0.1195t-0.0126t^2+0.0002t^3-\left(1.8\times10^{-6}\right)t^4\\&\quad+\left(7.3821\times10^{-9}\right)t^5-\left(1.2447\times10^{-11}\right)t^6\\ P^{(N)}(t)&=6.6521\times10^{-10}+0.5949t-0.0102t^2+\left(2.2\times10^{-4}\right)t^3\\&\quad+\left(7.5\times10^{-7}\right)t^4-\left(6.5\times10^{-9}\right)t^5+\left(1.6\times10^{-11}\right)t^6.\end{aligned} \tag{29}$$

All graphs of their solutions are shown in Fig 2.

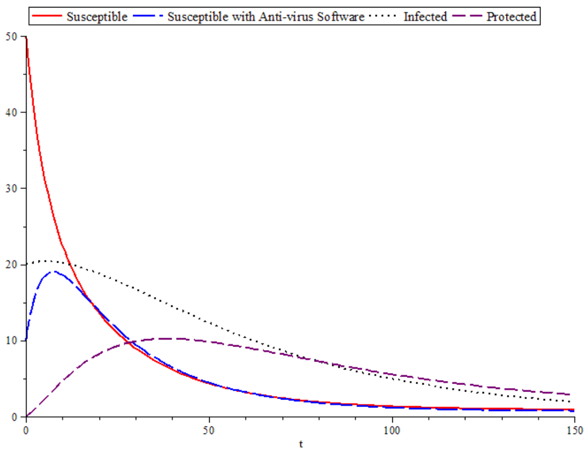
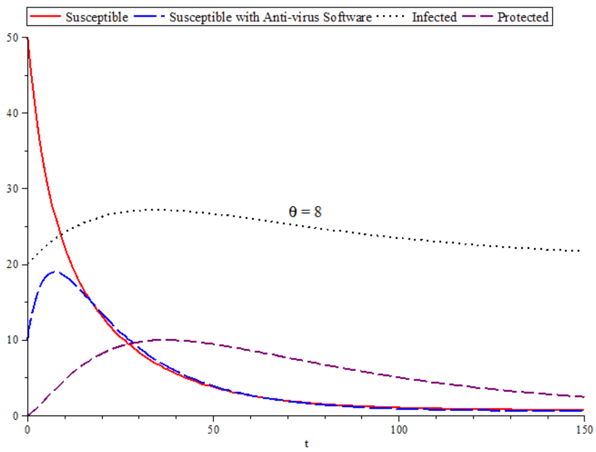


Fig. 2. Shifted Chebyshev solutions with $\alpha=1$, $S(0)=50$, $S_\gamma(0)=10$, $I(0)=20$, $P(0)=0$.

In Fig. 2, the solution of anti-virus susceptible nodes (S_γ) and protected nodes (P) are increasing during short time period after that decreasing and approaching the equilibrium point which is an asymptotically stable.



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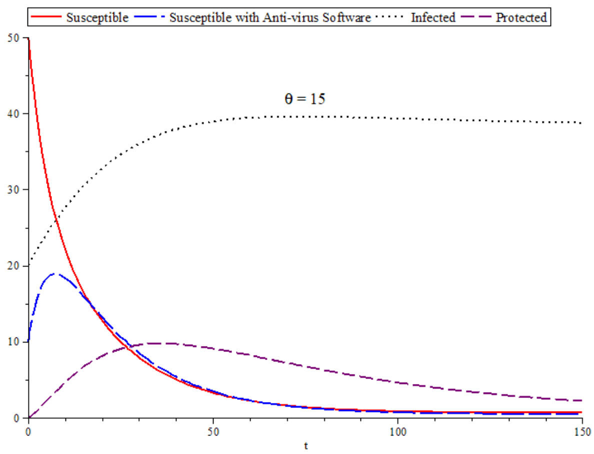
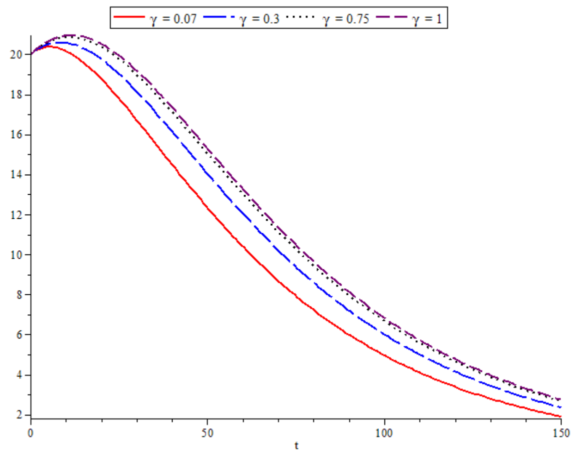


Fig. 3. Shifted Chebyshev solutions for (a) $\theta = 8$, (b) $\theta = 15$, where $S(0) = 50$, $S_\gamma(0) = 10$, $I(0) = 20$, $P(0) = 0$.

In Fig. 3, if the function of connecting new infected I nodes to the network ($f_I^\alpha(t) = \theta b$) increases by increasing the value of $\theta = 8$ and 15. The numerical results show that increasing the value of θ will lead to more numbers of computer virus at infected nodes $I(t)$.

The system of non-linear Caputo fractional differential equations (2) will be used as our model for analysis for order $\alpha = 1$ to find equilibrium points of the system. We set $D_t S(t), D_t S_\gamma(t), D_t I(t), D_t P(t) = 0$ and then solve the obtained equations for the disease free equilibrium point $E_0 = (S^*, S_\gamma^*, 0, P^*) = (0.85869565, 0.65652174, 0, 0.98478261)$. The local stability of E_0 determined by modulus of the eigenvalues are $\lambda_1 = -0.019406521, \lambda_2 = -0.020000001$, $\lambda_3 = -0.035470131$ and $\lambda_4 = -0.194529869$.



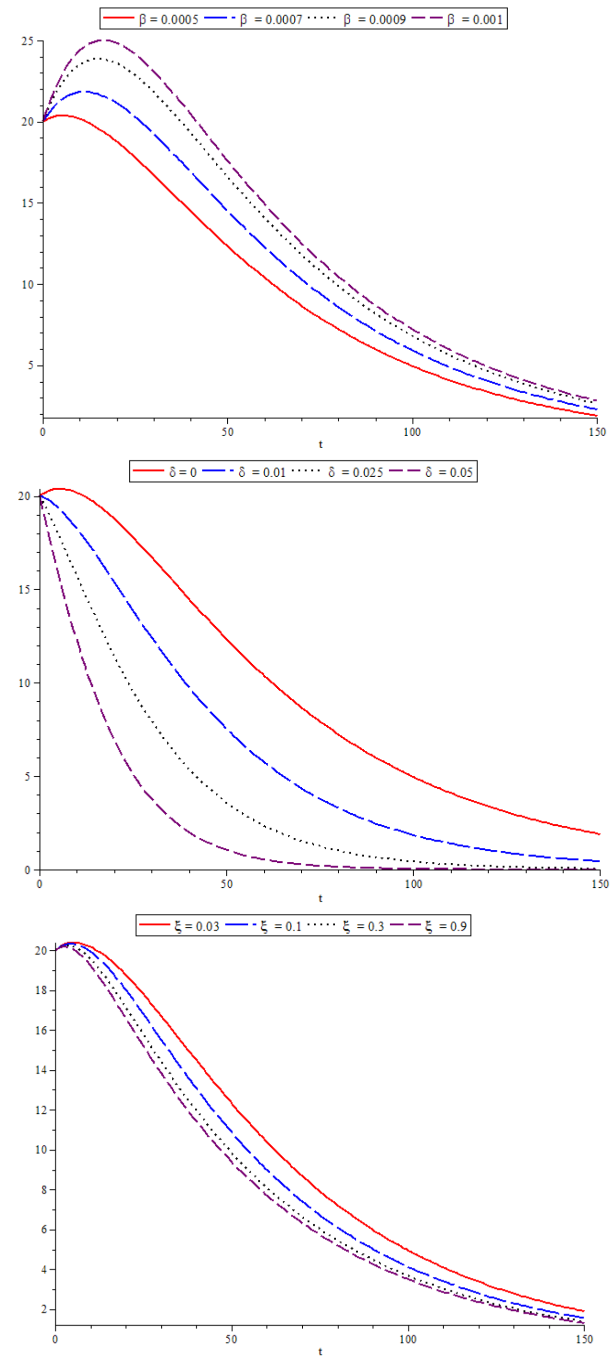


Fig. 4. Shifted Chebyshev solutions for infected I where change $\gamma^\alpha, \beta^\alpha, \delta^\alpha$ and ξ^α with parameters in Table 2.

For Fig. 4 we have also observed that as the value of γ^α and β^α increases, the numbers of computer virus at infected nodes $I(t)$ increases, and the value of δ^α and ξ^α increases, but the numbers of computer virus at infected nodes $I(t)$ decreases.

Cases 2: Assume that $f_S^\alpha(t) = 1.5 + \frac{1}{t+1}\sin(-1.8\pi t)$ and $\alpha = 1$, so the susceptible nodes connect to network periodically. By symbolic shifted Chebyshev program in

Maple with $\alpha = 1$, it obtains the approximate analytical solutions with $N = 7$ as

$$\begin{aligned}
S^{(N)}(t) &= 99 - 328.9520t - 186.8663t^2 + 4606.9740t^3 \\
&\quad - 11411.5371t^4 + 11049.1621t^5 - 3798.2528t^6 \\
S_{\gamma}^{(N)}(t) &= 10 + 694.2143t - 4750.1061t^2 + 14034.1476t^3 \\
&\quad - 20834.8340t^4 + 15176.5517t^5 - 4307.9513t^6 \\
I^{(N)}(t) &= 70 - 139.9807t + 139.7558t^2 - 92.0643t^3 \\
&\quad + 43.2946t^4 - 13.7305t^5 + 2.1985t^6 \\
P^{(N)}(t) &= 92.0408t + 227.1809t^2 - 1348.8316t^3 \\
&\quad + 2408.7456t^4 - 1943.9731t^5 + 598.5447t^6
\end{aligned} \tag{30}$$

with the graphs of solutions in Fig. 5.

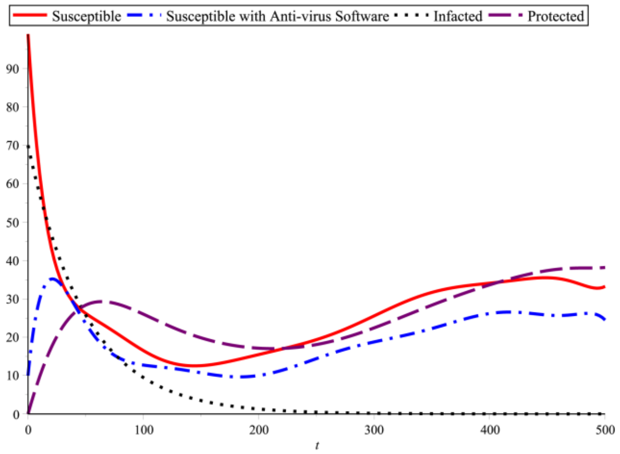


Fig. 5. Shifted Chebyshev solutions when $f_S^{\alpha}(t) = 1.5 + \dfrac{1}{t+1}\sin(-1.8\pi t)$, $f_{S_\gamma}^{\alpha}(t) = pb$, $f_I^{\alpha}(t) = \theta b$, $f_P^{\alpha}(t) = 0$, where $S(0) = 99$, $S_\gamma(0) = 10$, $I(0) = 70$, $P(0) = 0$ with parameters in Table 2.

Cases 3: The susceptible nodes $\left(f_S^{\alpha}(t) = \dfrac{2}{1 + e^{-\frac{t}{2}}} \right)$ connect to the network as an increasing function with several Caputo fractional orders $\alpha = 0.6, 0.8, 1$ and parameters in Table 3.

Table 3. Values of parameters used in the numerical simulations (adapted from [36])

Parameter	Value	Parameter	Value	Parameter	Value
b	0.05	p	0.01	d^{α}	0.05
ρ^{α}	0.09	γ^{α}	0.07	η^{α}	0.02
ξ^{α}	0.06	δ^{α}	0.06	θ	0.15
χ^{α}	0.03	β^{α}	0.01	σ^{α}	0.02

Fig. 6 shows shifted Chebyshev solutions for S, S_γ, I, P where $\alpha = 0.6, 0.8$ and 1.

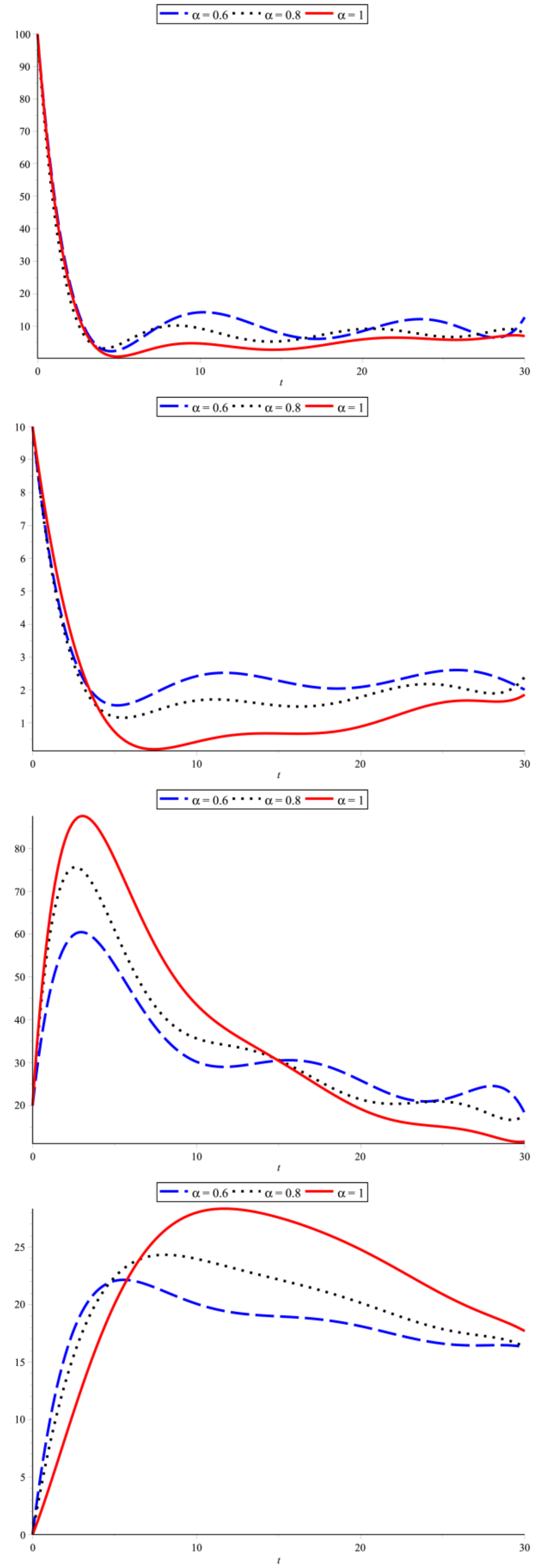


Fig. 6. Shifted Chebyshev solutions for S, S_γ, I, P where $\alpha = 0.6, 0.8$ and 1.

V. COMPARING NUMERICAL RESULTS

In addition, we investigate efficiency and accuracy of this method by comparing shifted Chebyshev solutions to some numerical solutions with different numerical methods; Runge-Kutta-Fehlberg method (RKF45) [45, 46] and the Cash-Karp method (CK45) [47]. The results are shown in these Tables 4 – 7.

Table 4. Comparison between Shifted Chebyshev solutions $S(t)$, $S_\gamma(t)$ and RKF45, $\alpha = 1$.

time	susceptible nodes $(S(t))$			susceptible nodes with anti-virus $(S_\gamma(t))$		
	RKF45	Cheb.	$\lvert \text{RKF45} - \text{Cheb.} \rvert$	RKF45	Cheb.	$\lvert \text{RKF45} - \text{Cheb.} \rvert$
10	22.53172	22.53172	2.75e-06	18.58171	18.58171	5.43e-06
30	8.97418	8.97418	3.25e-06	9.46600	9.46600	8.51e-06
50	4.36336	4.36336	8.54e-06	4.48766	4.48766	7.93e-06
70	2.41071	2.41071	6.45e-06	2.34904	2.34904	7.54e-06
90	1.55149	1.55149	3.27e-06	1.41209	1.41209	8.51e-06
110	1.16583	1.16583	7.65e-05	0.99279	0.99279	9.43e-06
130	0.99138	0.99138	1.27e-06	0.80316	0.80316	1.49e-06
150	0.91284	0.91284	3.57e-05	0.71754	0.71754	2.74e-05

Table 5. Comparison between Shifted Chebyshev solutions $I(t)$, $P(t)$ and RKF45, $\alpha = 1$.

time	infected nodes $(I(t))$			protected nodes $(P(t))$		
	RKF45	Cheb.	$\lvert \text{RKF45} - \text{Cheb.} \rvert$	RKF45	Cheb.	$\lvert \text{RKF45} - \text{Cheb.} \rvert$
10	20.17238	20.17238	3.21e-11	4.66581	4.66581	2.43e-06
30	16.72093	16.72093	2.66e-10	9.87177	9.87177	2.22e-07
50	12.34355	12.34355	7.44e-09	9.81606	9.81606	1.76e-06
70	8.68865	8.68865	3.98e-09	8.16285	8.16285	4.69e-06
90	5.99152	5.99152	3.66e-08	6.35554	6.35554	3.63e-06
110	4.09413	4.09413	2.18e-08	4.83448	4.83448	9.75e-05
130	2.78613	2.78613	3.59e-07	3.67552	3.67552	6.77e-05
150	1.89249	1.89249	3.86e-07	2.83560	2.83560	8.42e-05

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Table 6. Comparison between Shifted Chebyshev solutions $S(t)$, $S_\gamma(t)$ and CK45, $\alpha = 1$.

time	susceptible nodes $(S(t))$			susceptible nodes with anti-virus $(S_\gamma(t))$		
	CK45	Cheb.	$\lvert \text{CK45} - \text{Cheb.} \rvert$	CK45	Cheb.	$\lvert \text{CK45} - \text{Cheb.} \rvert$
10	22.53172	22.53172	4.02e-06	18.58171	18.58171	5.32e-06
30	8.97418	8.97418	2.10e-06	9.46600	9.46600	1.36e-06
50	4.36336	4.36336	4.32e-07	4.48766	4.48766	3.56e-07
70	2.41071	2.41071	7.43e-06	2.34904	2.34904	5.12e-06
90	1.55149	1.55149	6.57e-06	1.41209	1.41209	5.11e-06
110	1.16583	1.16583	9.11e-05	0.99279	0.99279	4.68e-06
130	0.99138	0.99138	7.54e-06	0.80316	0.80316	6.99e-06
150	0.91284	0.91284	5.53e-05	0.71754	0.71754	3.11e-05

Table 7. Comparison between Shifted Chebyshev solutions $I(t)$, $P(t)$ and CK45, $\alpha = 1$.

time	infected nodes $(I(t))$			protected nodes $(P(t))$		
	CK45	Cheb.	$\lvert \text{CK45} - \text{Cheb.} \rvert$	CK45	Cheb.	$\lvert \text{CK45} - \text{Cheb.} \rvert$
10	20.17238	20.17238	3.45e-10	4.66581	4.66581	1.22e-06
30	16.72093	16.72093	3.18e-10	9.87177	9.87177	3.12e-07
50	12.34355	12.34355	2.75e-09	9.81606	9.81606	6.37e-08
70	8.68865	8.68865	4.31e-08	8.16285	8.16285	2.54e-06
90	5.99152	5.99152	4.87e-08	6.35554	6.35554	3.71e-07
110	4.09413	4.09413	2.69e-07	4.83448	4.83448	4.92e-05
130	2.78613	2.78613	2.54e-07	3.67552	3.67552	5.32e-05
150	1.89249	1.89249	1.86e-07	2.83560	2.83560	5.21e-05

VI. CONCLUSION

Chebyshev method is firstly proposed to provide approximated analytical solutions of the FSSIP model in Caputo fractional derivative sense. The method has been applied to three types of connection, which are based on each node connecting to the network as a function. Approximate and analytical solutions of examples are compared. For Case 1, where each node connects to the network as a constant and when comparing with present results, it is clear that the results obtained by the proposed method approaches numerical solutions which are computed by Runge-Kutta-Fehlberg method (RKF45) in Table 4 – 5 and Cash-Karp method (CK45) in Table 6 – 7 with the maximum absolute errors 3.21e -11 and 3.45e-10, respectively. Case 2 presents the approximated analytical solutions where each node connects to network as the function $1.5 + \frac{1}{t+1}\sin(-1.8\pi t)$. Our method obtains the solutions in terms of power expansion. Example 3 presents that applications of this method are very simple and very convenient for solving the Caputo fractional derivative SSIP model with order $\alpha = 0.6, 0.8, 1$ and the connecting network as the functions $\frac{2}{1+e^{-\frac{t}{2}}}$. The proposed method can be applied to solve analytical solutions for various fractional derivatives or other problems.

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