

The Role of q-sets in Topology

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Abstract

The interior and closure operators in topological spaces play a vital role in the generalization of closed sets and open sets in topological spaces. These operators have application to Rough set theory, Data mining and Digital image processing. Thangavelu and Chandrasekhara Rao introduced the concept of q-sets in topology and studied the basic properties of q-sets. The purpose of this paper is to characterize the further properties of q-sets in topology.

1. Introduction

The interior and closure operators in topological spaces play a dominant role in the generalization of closed sets and open sets in topological spaces. These operators have application to Rough set theory, Data mining and Digital image processing. Thangavelu and Chandrasekhara Rao [6] introduced the concept of q-sets in topology and studied the basic properties of q-sets. The purpose of this paper is to characterize the further properties of q-sets in topology. The basic properties of the interior and the closure operators are discussed in section 2 and the section 3 is dealt with the topologies generated by the q-sets. In section 4, the q-interior and q-closure operators are introduced. Finally the q-sets in the real line are investigated in section 5.

2. Preliminaries

Throughout this section X is a topological space and A, B are the subsets of X . The notations clA and $intA$ denote the closure of A and interior of A in X respectively. The following definitions and lemmas will be useful in sequel.

Definition 2.1

- i) Regular open [6] if $A = int\ cl\ A$ and regular closed if $A = cl\ int\ A$.
- ii) α -open[5] if $A \subseteq int\ cl\ int\ A$ and α -closed if $cl\ int\ cl\ A \subseteq A$.

- iii) β - open [1] if $A \subseteq cl\ int\ cl\ A$ and β -closed if $int\ cl\ int\ A \subseteq A$.
 - iv) Pre-open [4] if $A \subseteq int\ cl\ A$ and pre-closed if $cl\ int\ A \subseteq A$.
 - v) A q-set [7] if $cl\ int\ A \supseteq int\ cl\ A$.
 - vi) semi-open[3] if $A \subseteq cl\ int\ A$ and semi-closed if $int\ cl\ A \subseteq A$.
- A subset of a topological space X is clopen if it is both open and closed .. Similarly regular clopen, α -clopen, β -clopen, pre-clopen and semi-clopen sets are defined.

Lemma 2.2 [2]

If A is a subset of X then

(i) $\alpha\ int\ \alpha\ cl\ A = int\ cl\ A$

(ii) $\alpha\ cl\ \alpha\ int\ A = cl\ int\ A$

where α interior of A is denoted by $\alpha\ int\ A$ and α closure of A is denoted by $\alpha\ cl\ A$.

Lemma 2.3 [6]

If A is clopen and B is a q-set then $A \cap B$ is a q-set and $A \cup B$ is a q-set.

Lemma 2.4 [6]

A is a q-set if and only if $X - A$ is a q-set.

3. Topology generated by q-sets

Lemma 3.1

If A is a subset of a topological X , then the following are equivalent

- (i) A is clopen
- (ii) A is regular clopen
- (iii) A is α -clopen.

Thus we have the following diagram

Diagram 3.2

Regular open \Rightarrow open \Rightarrow α -open \Rightarrow semi- open \Rightarrow q-set

Regular closed \Rightarrow closed \Rightarrow α - closed \Rightarrow semi- closed \Rightarrow q-set

Proposition 3.3

Let (X, τ) be a topological space. Suppose A is clopen and B is a q-set. Then $N = \{\phi, A \cap B, A, A - (A \cap B), X\}$ is a topology and every member of N is a q-set.

Proof

$N = \{\phi, A \cap B, A, A - \{A \cap B\}, X\}$ is clearly a topology on X . Clearly ϕ, A, X are all q-sets. Since A is clopen and B is a q-set, by using Lemma 2.3, $A \cap B$ is a q-set. Since complement of a q-set is again a q-set, $X - (A \cap B)$ is a q-set that implies $A \cap (X - (A \cap B))$ is also a q-set. This proves that $A - (A \cap B) = A \cap (X - (A \cap B))$ is a q-set. Hence every member of N is a q-set.

Proposition 3.4

Let (X, τ) be a topological space. Suppose A is clopen and B is a q -set. Then $\mathcal{N} = \{\phi, A, B, A \cap B, A \cup B, X\}$ is a topology and every member of \mathcal{N} is a q -set.

Proof

$\mathcal{N} = \{\phi, A, B, A \cap B, A \cup B, X\}$ is clearly a topology on X . Clearly ϕ, A, X are all q -sets. Since A is clopen and B is a q -set, by using Lemma 2.3, $A \cap B$ and $A \cup B$ are q -sets. Hence every member of \mathcal{N} is a q -set.

In the rest of the section, $q(\tau)$ denotes the collection of q -sets in (X, τ) .

Proposition 3.5

Let (X, τ) be a space with point inclusion topology. Then $q(\tau)$ is a discrete topology on X .

Proof

Fix $a \in X$. Let $\tau = \{\phi\} \cup \{A \subseteq X : a \in A\}$. Let $B \subseteq X$. The

$$int B = \begin{cases} \phi & \text{if } B = \phi \\ B & \text{if } a \in B \\ \phi & \text{if } a \notin B \end{cases} \quad \text{and} \quad cl B = \begin{cases} \phi & \text{if } B = \phi \\ X & \text{if } a \in B \\ B & \text{if } a \notin B \end{cases}$$

$$cl int B = \begin{cases} \phi & \text{if } B = \phi \\ X & \text{if } a \in B \\ \phi & \text{if } a \notin B \end{cases} \quad \text{and} \quad int cl B = \begin{cases} \phi & \text{if } B = \phi \\ X & \text{if } a \in B \\ \phi & \text{if } a \notin B \end{cases}$$

Then it follows $q(\tau) = \wp(X)$, the power set of X , that is the discrete topology on X .

Proposition 3.6

Let (X, τ) be a space with set inclusion topology. Then $q(\tau)$ is not a topology on X .

Proof

Let $A \subseteq X, A \neq \phi, \tau_A = \{\phi\} \cup \{B \subseteq X ; B \supseteq A\}$. Then $B \subseteq X$ is closed if and only if $X - B$ is open iff $X - B = \phi$ or $X - B \supseteq A$ iff $B = X$ or $B \subseteq X - A$ iff $B = X$ or $B \cap A = \phi$.

$$int B = \begin{cases} \phi & \text{if } B = \phi \\ \phi & \text{if } B \subset A, B \neq A \\ A & \text{if } B = A \\ B & \text{if } B \supset A \\ \phi & \text{if } B \cap A = \phi, B \neq \phi \\ \phi & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ \phi & \text{if } B = X \end{cases}$$

$$cl B = \begin{cases} \phi & \text{if } B = \phi \\ X & \text{if } B \subset A, B \neq A \\ X & \text{if } B = A \\ X & \text{if } B \supset A \\ B & \text{if } B \cap A = \phi, B \neq \phi \\ X & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

$$cl \text{ int } B = \begin{cases} \phi & \text{if } B = \phi \\ \phi & \text{if } B \subset A, B \neq A \\ X & \text{if } B = A \\ X & \text{if } B \supset A \\ \phi & \text{if } B \cap A = \phi, B \neq \phi \\ \phi & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

$$\text{int } cl B = \begin{cases} \phi & \text{if } B = \phi \\ \phi & \text{if } B \subset A, B \neq A \\ X & \text{if } B = A \\ X & \text{if } B \supset A \\ \phi & \text{if } B \cap A = \phi, B \neq \phi \\ \phi & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

Thus $q(\tau) = \{ \phi, X \} \cup \{ B : B \supseteq A \} \cup \{ B : B \subseteq X - A \}$ is not a topology on X .

Proposition 3.7

Let (X, τ) be a space with point exclusion topology. Then $q(\tau)$ is the discrete topology on X .

Proof

Fix $a \in X$. Let $\tau = \{ X \} \cup \{ A \subseteq X, a \notin A \}$. Let $B \subseteq X$.

$$\text{int } B = \begin{cases} \phi & \text{if } B = \phi \\ B - a & \text{if } a \in B \\ B & \text{if } a \notin B \\ \phi & \text{if } B = \{ a \} \\ X - a & \text{if } B = X \end{cases}$$

$$cl B = \begin{cases} \phi & \text{if } B = \phi \\ B & \text{if } a \in B \\ B \cup \{a\} & \text{if } a \notin B \\ \{a\} & \text{if } B = \{a\} \\ X & \text{if } B = X \end{cases}$$

$$cl \text{ int } B = \begin{cases} \phi & \text{if } B = \phi \\ B & \text{if } a \in B \\ B \cup \{a\} & \text{if } a \notin B \\ \phi & \text{if } B = \{a\} \\ X & \text{if } B = X \\ X & \text{if } B = X - a \end{cases}$$

$$\text{int } cl B = \begin{cases} \phi & \text{if } B = \phi \\ B - \{a\} & \text{if } a \in B \\ B & \text{if } a \notin B \\ \phi & \text{if } B = \{a\} \\ X & \text{if } B = X \\ X & \text{if } B = X - a \end{cases}$$

$q(\tau) = \wp(X)$, the power set of X , is the discrete topology on X .

Proposition 3.8

Let (X, τ) be a space with set exclusion topology. Then $q(\tau)$ is the discrete topology on X .

Proof

Let $A \subseteq X, A \neq \phi$.

Let $\tau = \{X\} \cup \{B \subseteq X: B \cap A = \phi\} = \{X\} \cup \{B \subseteq X: B \subseteq X - A\}$. Let $B \subseteq X$ is closed iff $X - B$ is open iff $X - B = X$ or $X - B \subseteq X - A$ iff $B = \phi$ or $B \supseteq A$.

$$\text{int } B = \begin{cases} \phi & \text{if } B = \phi \\ \phi & \text{if } B \subset A, B \neq A \\ \phi & \text{if } B = A \\ B - A & \text{if } B \supset A \\ B & \text{if } B \cap A = \phi, B \neq \phi \\ B - A & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

$$clB = \begin{cases} \phi & \text{if } B = \phi \\ A & \text{if } B \subset A, B \neq A \\ A & \text{if } B = A \\ B \cup A & \text{if } B \supset A \\ B \cup A & \text{if } B \cap A = \phi, B \neq \phi \\ B \cup A & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

$$cl\text{int } B = \begin{cases} \phi & \text{if } B = \phi \\ \phi & \text{if } B \subset A, B \neq A \\ \phi & \text{if } B = A \\ B \cup A & \text{if } B \supset A \\ B \cup A & \text{if } B \cap A = \phi, B \neq \phi \\ B \cup A & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

$$\text{int } clB = \begin{cases} \phi & \text{if } B = \phi \\ \phi & \text{if } B \subset A, B \neq A \\ \phi & \text{if } B = A \\ B - A & \text{if } B \supset A \\ B - A & \text{if } B \cap A = \phi, B \neq \phi \\ B - A & \text{if } B \cap A \neq \phi, B \cap (X - A) \neq \phi \\ X & \text{if } B = X \end{cases}$$

$q(\tau) = \wp(X)$, the power set of X , is the discrete topology on X .

Proposition 3.8

Let X be an infinite set. $\tau = \{\phi\} \cup \{A \subseteq X: X - A \text{ is finite}\}$. $q(\tau)$ is not a topology on X .

Proof

Let $B \subseteq X$.

$$\text{int } B = \begin{cases} B & \text{if } B = \phi \text{ or } X \\ B & \text{if } B \text{ is infinite, } X - B \text{ is finite} \\ \phi & \text{if } B \text{ is finite} \\ \phi & \text{if } B \text{ is infinite, } X - B \text{ is infinite} \end{cases}$$

$$\text{cl } B = \begin{cases} B & \text{if } B = \phi \text{ or } X \\ X & \text{if } B \text{ is infinite, } X - B \text{ is finite} \\ B & \text{if } B \text{ is finite} \\ X & \text{if } B \text{ is infinite, } X - B \text{ is infinite} \end{cases}$$

$$\text{cl int } B = \begin{cases} B & \text{if } B = \phi \text{ or } X \\ X & \text{if } B \text{ is infinite, } X - B \text{ is finite} \\ \phi & \text{if } B \text{ is finite} \\ \phi & \text{if } B \text{ is infinite, } X - B \text{ is infinite} \end{cases}$$

$$\text{int cl } B = \begin{cases} B & \text{if } B = \phi \text{ or } X \\ X & \text{if } B \text{ is infinite, } X - B \text{ is finite} \\ \phi & \text{if } B \text{ is finite} \\ X & \text{if } B \text{ is infinite, } X - B \text{ is infinite} \end{cases}$$

Then $q(\tau) = \{B : B \text{ is finite or } X - B \text{ is finite}\}$ is not a topology.

4. q -interior and q -closure

Definition 4.1

Let A be a subset of a topological space X . The q -interior of A denoted by $q\text{-int}(A)$ is the union of all q -sets contained in A and the q -closure of A denoted by $q\text{-cl}(A)$ is the intersection of all q -sets containing A .

Since the collection of all q -sets is not closed under union and intersection it follows that $q\text{-int}(A)$ need not be a q -set and $q\text{-cl}(A)$ need not be a q -set. But $q\text{-int}(A) \subseteq A \subseteq q\text{-cl}(A)$ is always true for any subset A of a topological space.

Proposition 4.2

- i) $q\text{-int}(\phi) = \phi$, $q\text{-cl}(\phi) = \phi$
- ii) $q\text{-int}(X) = X$, $q\text{-cl}(X) = X$
- iii) $q\text{-int } A \subseteq A \subseteq q\text{-cl } A$
- iv) $A \subseteq B \Rightarrow q\text{-int } A \subseteq q\text{-int } B$ and $q\text{-cl } A \subseteq q\text{-cl } B$.
- v) $q\text{-int} (A \cap B) \subseteq q\text{-int } A \cap q\text{-int } B$.
- vi) $q\text{-cl} (A \cap B) \subseteq q\text{-cl } A \cap q\text{-cl } B$
- vii) $q\text{-cl} (A \cup B) \supseteq q\text{-cl } A \cup q\text{-cl } B$
- viii) $q\text{-int} (A \cup B) \supseteq q\text{-int} A \cup q\text{-int} B$
- ix) $q\text{-int}(q\text{-int } A) \subseteq q\text{-int } A$
- x) $q\text{-cl}(q\text{-cl } A) \supseteq q\text{-cl } A$
- xi) $q\text{-int}(q\text{-cl } A) \supseteq q\text{-int } A$
- xii) $q\text{-cl}(q\text{-int } A) \subseteq q\text{-cl } A$

Proof

(i), (ii), (iii), (iv) are obvious.

Since $A \cap B \subseteq A$, $A \cap B \subseteq B$, by using (iv), we have $q\text{-int} (A \cap B) \subseteq q\text{-int } A \cap q\text{-int}$

B and $q-cl(A \cap B) \subseteq q-cl A \cap q-cl B$. Also since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ it follows from (iv) that $q-cl A \cup q-cl B \subseteq q-cl(A \cup B)$ and $q-int A \cup q-int B \subseteq q-int(A \cup B)$. This proves (v), (vi), (vii) and (viii). Since $q-int A \subseteq A$, $A \subseteq q-cl A$ and $q-int A \subseteq q-cl A$ it follows from (iv) that $q-int(q-int A) \subseteq q-int A$, $q-cl(q-cl A) \supseteq q-cl A$. $q-int(q-cl A) \supseteq q-int A$ and $q-cl(q-int A) \subseteq q-cl A$. This proves (ix), (x), (xi) and (xii). This completes the proof of the proposition.

However, the reverse inclusions in (v), (vi), (vii), (viii), (ix), (x), (xi) and (xii) of Proposition 4.2 are not generally true. The following results can be easily established.

Proposition 4.3

If A is a q -set then $q-int A = A = q-cl A$.

Proposition 4.4

If A is a q -set then $q-cl q-int A = A = q-int q-cl A$.

Proposition 4.5

If A is a q -set and B is semi-open (resp. α -open, open) or semi-closed (resp. α -closed, closed) then $q-int(A \cap B) = q-int A \cap q-int B$ and $q-cl(A \cap B) = q-cl A \cap q-cl B$.

Proposition 4.6

If A is a q -set and B is semi-open (resp. α -open, open) or semi-closed (resp. α -closed, closed) then $q-int(B-A) = B-A = q-cl(B-A)$.

Proposition 4.7

If A is a q -set and B is semi-open (resp. α -open, open) or semi-closed (resp. α -closed, closed) then $q-int(A \cup B) = q-int A \cup q-int B$ and $q-cl(A \cup B) = q-cl A \cup q-cl B$.

Proposition 4.8

If A is a q -set and B is semi-open (resp. α -open, open) or semi-closed (resp. α -closed, closed) then

- i) $q-int(q-cl(A \cap B)) = q-int(q-cl(A)) \cap q-int(q-cl(B))$.
- ii) $q-cl(q-int(A \cap B)) = q-cl(q-int(A)) \cap q-cl(q-int(A \cap B))$.
- iii) $q-int(q-cl(A \cup B)) = q-int(q-cl(A)) \cup q-int(q-cl(B))$.
- iv) $q-cl(q-int(A \cup B)) = q-cl(q-int(A)) \cup q-cl(q-int(B))$

Proposition 4.9

For any subset $Y \subseteq X$, $q-int(Y) = X - q-cl(X - Y)$.

Proposition 4.10

- (i) $q-int(q-int A) \subseteq q-int(q-cl(p-int A)) \subseteq q-cl(q-int A) \subseteq q-cl(q-int(q-cl A)) \subseteq q-cl(q-cl A)$
- (ii) $q-int(q-int A) \subseteq q-int(q-cl(p-int A)) \subseteq q-int(q-cl A) \subseteq q-cl(q-int(q-cl A)) \subseteq q-cl(q-cl A)$

Proposition 4.11

Let (X, τ) be topological space. Then X is a union of two non empty disjoint q -sets if and only if it has a non trivial proper q -set.

Proposition 4.12

Suppose $A \subseteq B$ with $cl A = cl B$. If A is a q -set then B is a q -set.

Proposition 4.13

Suppose $A \subseteq B$ with $int A = int B$. If B is a q -set then A is a q -set.

5. q -sets In The Real Line

Any generalization of sets in topology must have applications to the real line \mathbb{R}^1 . In this section the nature of q -sets in the real line is briefly discussed.

Proposition 5.1

- (i) Every singleton set in \mathbb{R}^1 is a q -set.
- (ii) For every real x , $\mathbb{R} - \{x\}$ is a q -set in \mathbb{R}^1 .
- (iii) Every interval is a q -set in \mathbb{R}^1 .
- (iv) The set Q of all rational numbers is not a q -set.
- (v) The set Q of all irrational numbers is not a q -set.

Proposition 5.2

If $q-int(A) = A$ then A need not be a q -set.

Proof.

In \mathbb{R}^1 , take $A = Q$. Then from Proposition 5.1 (i), it follows that $q-int(A) = A$ and from Proposition 5.1 (iv), A is not a q -set.

Proposition 5.3

If $q-cl(A) = A$ then A need not be a q -set.

Proof.

In \mathbb{R}^1 , take $A = \mathbb{R} - Q$. Then from Proposition 5.1 (i), it follows that $q-cl(A) = A$ and from Proposition 5.1 (iv), A is not a q -set.

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