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On Einstein Nearly Kenmotsu Manifolds

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Abstract

The present paper deals with the study of Einstein nearly Kenmotsu manifold with projective curvature tensor P and conharmonic curvature tensor N satisfying $R(X,Y).P=0$ and $R(X,Y).N=0$ and have shown manifold satisfying these condition is locally isometric to hyperbolic space $H^n(-1)$.

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1. INTRODUCTION

Tanno, S. ([10]) Classified connected almost metric manifold whose automorphism group possess the maximum dimension. for such a manifold M, the sectional curvature of the plane section ξ is constant say c. If c>0, M is a homogeneous Sasakian manifold of constant sectional curvature. If c =0, M is a product of line or a circle with a Kaehler manifold of constant holomorphic section curvature. If c <0, M is wrapped product space $R\times_f C^n$. In 1971, Kenmotsu studied a class of contact Riemannian manifold satisfying some special condition and characterized the differential Geometric properties of the manifolds of class(3); the structure so obtained is now known as Kenmotsu structure.

An almost contact manifold (M,φ,ξ,η,g) is called nearly kenmotsu manifold by Shukla, A. ([9]) if the following relations holds:

(1.1) $$(\nabla_X\varphi)Y+(\nabla_Y\varphi)X=-\eta(Y)\varphi X-\eta(X)\varphi Y$$

Where ∇ is Levi-Civita connection of g. Moreover, if M satisfies

(1.2) $(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$

then it is called a Kenmotsu manifold([4]). It is easy to see that every Kenmotsu manifold is nearly Kenmotsu manifold but converse is not true. Many other author ([6, 7, 8]) have studied nearly Kenmotsu manifold briefly.

In this paper we have study Einstein nearly Kenmotsu manifold with projective curvature tensor P and conharmonic curvature tensor N and have proved four interesting theorems.

2. ON NEARLY KENMOTSU MANIFOLDS

An $n\ (= 2m + 1)$-dimensional differentiable manifold M is called an almost contact Riemannian manifold if, there is an almost contact structure (φ, ξ, η) consisting of a (1, 1) tensor field φ, a vector field ξ and 1-form ξ satisfying ([6, 7, 8])

(2. 1) $\varphi^2(X) = -X + \eta(X)\xi$

(2. 2) $\eta(\xi) = 1, \varphi\xi = 0, \eta(\varphi X) = 0$

Let g be Riemannian metric with (φ, ξ, η), that is,

(2. 3) $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\,\eta(Y)$

Or equivalently

(2. 4) $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$

For all vector fields X, Y, M on M. If an almost contact metric manifold satisfies (1. 1) then M is called nearly Kenmotsu manifold.

In every n –dimensional nearly Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ the following important identities holds. (for more detail see ([7, 8])

(2. 5) $(\nabla_X \eta)Y = g(X, Y) - \eta(X)\,\eta(Y)$

(2. 6) $R(X, Y)\xi = \eta(X)Y - \eta(Y)X$

(2. 7) $R(\xi, X)Y = -g(X, Y)\xi + \eta(Y)X$

(2. 8) $S(X, \xi) = -(n-1)\eta(X)$

(2. 9) $S(\varphi X, \varphi Y) = S(X, Y) + (n-1)\eta(X)\,\eta(Y)$

(2. 10) $\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X)$

Where R is the Riemannian curvature and S is the Ricci tensor of g.

Let $(M, \varphi, \xi, \eta, g)$ be a nearly Kenmotsu manifold. In ([6]) it is proven that

(2. 11) $\nabla_X \xi = X - \eta(X)\xi, \nabla_\xi \xi = 0.$

The Projective curvature tensor P ([1]) and Conharmonic curvature tensor N ([5]) on Riemannian manifold are defined respectively as

(2. 12) $P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]$

(2. 13) $N(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y$
$+g(Y, Z)r(X) - g(X, Z)r(Y)]$

Where R is the Riemannian curvature and S is the Ricci tensor and r is the scalar curvature.

A Riemannian manifold M is said to be Einstein manifold if its Ricci tensor S of type $(0, 2)$ is of the form

(2. 14) $S(X, Y) = kg(X, Y)$

For all $X, Y \in \chi(M)$ and k is certain scalar function on M.

3. AN EINSTEIN NEARLY KENMOTSU MANIFOLD SATISFYING $\boldsymbol{R(X,Y).P=0}$

In this section we assume that

(3. 1) $R(X,Y).P(U,V)W = 0$

Let the Riemannian Manifold M be an Einstein manifold, then (2. 12) gives

(3. 2) $P(X,Y)Z = R(X,Y)Z - \frac{k}{n-1}[g(Y,Z)X - g(X,Z)Y]$

Now, (3. 2) can be written as

(3.3) $'P(X,Y,Z,W) =' R(X,Y,Z,W) - \frac{k}{n-1}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$

Where $'P(X,Y,Z,W) = g(P(X,Y)Z,W)$ and $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$

Using (2. 10) in (3. 3), we get

(3. 4) $\eta(P(X,Y)Z) = \left(1 + \frac{k}{n-1}\right)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]$

Taking $X = \xi$ in (3. 4) and using (2. 4) we get

(3. 5) $\eta(P(\xi,Y)Z) = \left(1 + \frac{k}{n-1}\right)[\eta(Y)\eta(Z) - g(Y,Z)]$

Again taking $Z = \xi$ in (3. 4) and using (2. 4), we get

(3. 6) $\eta(P(X,Y)\xi) = 0$

Now

$(R(X,Y)P(U,V)W = R(X,Y)P(U,V)W - P(R(X,Y)U,V)W$
$-P(U,R(X,Y)V)W - P(U,V)R(X,Y)W$

In view of (3. 1), we have

$R(X,Y)P(U,V)W - P(R(X,Y)U,V)W$
$-P(U,R(X,Y)V)W - P(U,V)R(X,Y)W = 0$

Therefore,

(3. 7) $g[R(X,Y)P(U,V)W,\xi] - g[P(R(X,Y)U,V)W,\xi]$
$-g[P(U,R(X,Y)V)W,\xi] - g[P(U,V)R(X,Y)W,\xi] = 0$

In view of (2. 7) and (3. 4) it follows that

(3. 8) $-{}'P(U,V,W,Y) + \eta(Y)\eta(P(U,V)W) - \eta(U)\eta(P(Y,V)W) -$
$\eta(V)\eta(P(U,Y)W)$
$-\eta(W)\eta(P(U,V)Y) + g(Y,U)\eta(P(\xi,V)W) + g(Y,V)\eta(P(U,\xi)W) = 0$

Taking $Y = U$ in (3. 8), we get

(3. 9) $-{}'P(U,V,W,U)$- $\eta(V)\eta(P(U,U)W) - \eta(W)\eta(P(U,V)U)$
$+g(U,U)\eta(P(\xi,V)W) + g(U,V)\eta(P(U,\xi)W) = 0$

Let $\{e_i\}, i = 1,2,3\ldots\ldots\ldots n$ be an orthonormanl basis of tangent space at any point. Then the sum for $1 \le i \le n$ of the relation (3. 9) for $U = e_i$, gives

(3. 10) $\eta(P(\xi,Y)Z) = \frac{1}{n}S(V,W) - \left[\frac{k}{n} + \frac{1}{n} + \frac{k}{n(n-1)}\right]g(V,W)$

$$+\left[\frac{k}{n} + \frac{1}{n} + \frac{(n-1)}{n}\right]\eta(V)\eta(W)$$

Using (3. 5) in (3. 10) and after simplification, we get

(3. 11) $S(V,W) = -(n-1)g(V,W)$

This gives

(3. 12) $k = -(n-1)$

Using (3. 12) in (3. 8), we get

$-{}'P(U,V,W,Y) = 0$

From above it follows that

(3. 13) $P(U,V)W = 0$

Hence, we can state the following theorem:

Theorem 1. 1. If in an Einstein nearly Kenmotsu manifold M, the relation $R(X,Y).P = 0$ holds, then the manifold is projectively flat.

Next, let us suppose that the Einstein nearly Kenmosu manifold is projectively flat, that is $P(X,Y)Z = 0$. Then from (3. 3) we have

(3. 14) $'R(X,Y,Z,W) = \frac{k}{n-1}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$

Where $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$

Putting $X = W = \xi$ in (3. 14) and using (2. 4) and (2. 7), we get

(3. 15) $\left(1 + \frac{k}{n-1}\right)[(\eta(Y)\eta(Z) - g(Y,Z)] = 0$

This shows that either $k = -(n-1)$ or $\eta(Y)\eta(Z) = g(Y,Z)$.

But if $g(Y,Z) = \eta(Y)\eta(Z)$, then from (2. 3) we get $g(\varphi Y, \varphi Z) = 0$, which is not possible.

Therefore, $k = -(n-1)$. Now putting $k = -(n-1)$ in (3. 2) and using $P(X,Y)Z = 0$, we get

(3. 16) $R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y]$

Therefore the manifold is of constant scalar curvature -1.

Hence, we can state the following theorem:

Theorem1. 2. A projectively flat Einstein nearly Kenmotsu manifold is locally isometric to hyperbolic space $H^n(-1)$.

4. AN EINSTEIN NEARLY KENMOTSU MANIFOLD SATISFYING $\boldsymbol{R(X,Y).N = 0}$

In this section we assume that

(4. 1) $R(X,Y).N(U,V)W = 0$

Let the Riemannian Manifold M be an Einstein manifold, then (2. 12) gives

(4. 2) $N(X,Y)Z = R(X,Y)Z - \frac{2k}{n-2}[g(Y,Z)X - g(X,Z)Y]$

Now, (4. 2) can be written as

(4. 3) $'N(X,Y,Z,W) =' R(X,Y,Z,W) - \frac{2k}{n-2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$

Where $'N(X,Y,Z,W) = g(N(X,Y)Z,W)$ and $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$

Using (2. 10) in (4. 3), we get

(4. 4) $\eta(N(X,Y)Z) = \left(1 + \frac{2k}{n-2}\right)[(g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]$

Taking $X = \xi$ in (3. 4) and using (2. 4) we get

(4. 5) $\eta(N(\xi,Y)Z) = \left(1 + \frac{2k}{n-2}\right)[(\eta(Y)\eta(Z) - g(Y,Z)]$

Again taking $Z = \xi$ in (3. 4) and using (2. 4), we get

(4. 6) $\eta(N(X,Y)\xi) = 0$

Now

$(R(X,Y)N(U,V)W = R(X,Y)N(U,V)W - N(R(X,Y)U,V)W$

$-N(U,R(X,Y)V)W - N(U,V)R(X,Y)W$
In view of (4. 1), we have
$R(X,Y)N(U,V)W - N(R(X,Y)U,V)W$
$-N(U,R(X,Y)V)W - N(U,V)R(X,Y)W = 0$
Therefore,
(4. 7) $g[R(X,Y)N(U,V)W,\xi] - g[N(R(X,Y)U,V)W,\xi]$
$-g[N(U,R(X,Y)V)W,\xi] - g[N(U,V)R(X,Y)W,\xi] = 0$
In view of (2. 7) and (4. 4) it follows that
(4. 8) $-'N(U,V,W,Y) + \eta(Y)\eta(N(U,V)W) - \eta(U)\eta(N(Y,V)W) -$
$\eta(V)\eta(N(U,Y)W)$
$\eta(W)\eta(N(U,V)Y) + g(Y,U)\eta(N(\xi,V)W) + g(Y,V)\eta(N(U,\xi)W) = 0$
Taking $Y = U$ in (4. 8), we get
(4. 9) $-'N(U,V,W,U)$- $\eta(V)\eta(N(U,U)W) - \eta(W)\eta(N(U,V)U)$
$+g(U,U)\eta(N(\xi,V)W) + g(U,V)\eta(N(U,\xi)W) = 0$
Let $\{e_i\}, i = 1,2,3 \ldots\ldots\ldots n$ be an orthonormanl basis of tangent space at any point. Then the sum for $1 \le i \le n$ of the relation (3. 9) for $U = e_i$, gives
(4. 10) $\eta(N(\xi,Y)Z) = \frac{1}{n}S(V,W) - \frac{1}{n}\left[1 + \frac{2nk}{(n-2)}\right]g(V,W)$
$+\frac{1}{n}\left[n + \frac{2k}{(n-2)} + \frac{2(n-1)k}{n-2}\right]\eta(V)\eta(W)$
Using (4. 5) in (4. 10) and after simplification, we get
(4. 11) $S(V,W) = -(n-1)g(V,W) - \mathrm{n(n-1)}\left[1 + \frac{2k}{(n-2)}\right]\eta(V)\eta(W)$
Putting $W = \xi$ in above and using (2. 12, yields
(4. 12) $k = -(n-2)/2$
Using (4. 12) in (4. 8), we get
$-'N(U,V,W,Y) = 0$
From above it follows that
(4. 13) $N(U,V)W = 0$
Hence, we can state the following theorem:

Theorem 1. 3. If in an Einstein nearly Kenmotsu manifold M, the relation $R(X,Y).N = 0$ holds, then the manifold is projectively flat.
Next, let us suppose that Einstein nearly Kenmosu manifold is conhormonicaly flat, that is $N(X,Y)Z = 0$. Then from (4. 3) we have
(4. 14) $'R(X,Y,Z,W) = \frac{2k}{n-2}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$
Where $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$
Putting $X = W = \xi$ in (4. 14) and using (2. 4) and (2. 7), we get
(4. 15) $\left(1 + \frac{2k}{n-2}\right)[(\eta(Y)\eta(Z) - g(Y,Z)] = 0$
This shows that either $k = -(n-2)/2$ or $\eta(Y)\eta(Z) = g(Y,Z)$.
But if $g(Y,Z) = \eta(Y)\eta(Z)$, then from (2. 3) we get $g(\varphi Y,\varphi Z) = 0$, which is not possible.
Therefore, $k = -(n-2)/2$. Now putting $k = -(n-2)/2$ in (4. 2) and using $N(X,Y)Z = 0$, we get

(4. 16) $R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y]$

Therefore the manifold is of constant scalar curvature -1.

Hence, we can state the following theorem:

Theorem1. 4. A conharmonically flat Einstein nearly Kenmotsu manifold is locally isometric to hyperbolic space $H^n(-1)$.

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