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MATRIX-GEOMETRIC METHOD FOR QUEUEING MODEL WITH SYSTEM BREAKDOWN, STANDBY SERVER, PH SERVICE AND PH REPAIR

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Abstract

In this paper, we study a system repair problem and present N system with one working and the other in standby. When the system fails it goes to repair and instantaneously a standby server becomes the working one. The repair time of the server and service time of the server are assumed to be of discrete phase-type distribution. We show the process that governs the system is a quasi-birth and death process, we perform steady-state analysis of this model. The time spent by a failed system in service and the total time in the repair facility are shown to be of phase-type. Server performance measures are evaluated.

Keywords: System repair problem, Discrete PH distribution, quasi-birth and death process, Conditional probability of failure, Stationary distribution.

1. Introduction

In this paper, we consider N system and a standby system involving discrete distribution. Most of the research work addressing the machine repair problems has focused on reliable servers. In this research domain, the work by Kness[6], Hsieh[4], Van Der Duyn[15] and Wartenhout[14] provided steady state solution to the machine repair problems with no spares and no standby machines. Models for machine repair problems with standby machines have been developed by Wank and Lee[13], Wank[12], Hsieh and Wank[5] and Sivazlian and Wank[11]. While Sivazlian and Wank modeled failure and repair times of the machine using general distribution, the other three papers used exponential distributions for the same.

Neuts[9] pointed out that all discrete distribution with finite support can be represented by discrete-phase distribution. Alfa and Neuts[2] showed the elapsed time

representation for such distribution. Alfa and Castro[1] considered these results for modelling a system that involved general discrete distribution. Thus, any discrete distribution is phase-type distribution. Given this analysis a general discrete renewal process can be expressed by considering the time among arrivals phase distributed. Also Ruiz catro[10] model a discrete warm standby system and they show that the process covering this one is a discrete level-dependent $M/G/1$.
In this present paper, we study a standby system involving discrete distribution. We consider N system, one working and the other in standby, in repair or waiting for repair. There is a repairman. The standby system do not fail. The working system is subject to internal(non-repairable) and accidental external repairs. When the non-repairable failure occurs the system is removed from the system. When the working server under goes a repairable failure, it goes to repair. In both cases, a standby server becomes the working one instantaneously. When a system is repaired, it re-enters the system and is new. The time of the internal failure has a general distribution and its PH representation is considered. We present a model where the repairability of the working failure can be independent of or dependent on the time system failure. The conditional probability is different types of failure are calculated in a matrix.
This paper is organized as follows. In section 2, we describe the model under study and give a brief review of PH-distribution. In section 3 & 4, dedicated to the Markov chain description methods and its steady state analysis using Matrix-analytical methods are presented. Section 5, stationary probability vector are calculated. In Section 6, we obtain system performance. In Section 7, we calculate the conditional probability of failure. In section 8, we give the numerical examples. Finally, we give the conclusion.

2. Phase-Type distribution

Poisson process and exponential distribution have very nice mathematical properties that make queueing model with these tractable. However, in applications these assumptions are highly restrictive. To get away from Poisson / exponential models. Neuts[9] developed the theory of PH- distribution and related point process. In stochastic modelling, PH-distribution lend themselves naturally to algorithmic. In this section we review the discrete time PH-distribution.

Definition (Discrete phase-type distribution):
Let $\{X_n : n \in N\}$ be a discrete time Markov chine as defined with state space $S = \{1,2,3,\ldots,m,m+1\}$, where the first m states are transient and the last state is absorbing, and transition probability matrix

$$Q = \begin{bmatrix} T & t' \\ 0 & 1 \end{bmatrix}$$

and T is a square matrix of dimension m, t' is a column vector and 0 is a row vector of dimension m. Since Q is a transition matrix, we have $T_{ij} \geq 0$ and $t_i \geq 0 \forall i, j \in S$ and $T1 + t' =$. Where 1 is the column vector of ones of the appropriate dimension m.

The probability distribution of the initial state is denoted with the row vector (α, α_{m+1}). Let $Z = \inf(n \in N : X_n = m+1)$ be the random variable of the time to the absorbing state $m+1$. The distribution of Z is called a discrete phase-type distribution with representation (α, T). Note that the knowledge of (α, T) is sufficient since

$$t' = (I - T)1 \qquad \text{and} \qquad \alpha_{m+1} = 1 - \alpha 1$$

Where I is the identity matrix of dimension m. The dimension m of T is called the order of the PH distribution and the transient states $\{1, 2, 3, \ldots, m\}$ are called the phase. The vector t contains the so-called exit probability.

The cumulative distribution function of the discrete PH distribution is

$$F_Z(z) = 1 - \alpha T^z 1 \qquad \text{for} \qquad z = 0, 1, 2, \ldots$$

and its probability mass function is

$$f_Z(0) = \alpha_{m+1} f_Z(z) = \alpha T^{z-1} t' \qquad \text{for} \quad z = 1, 2, 3, \ldots$$

Assumption 1. The failure times of the system are assumed to be exponential with parameter λ. The distribution of the service denoted by F_r, follows a phase-type distribution $\mathrm{PH}(\alpha, T)$ of order m.

$$F_r = \alpha T^{r-1} t', \quad r \in N, \qquad F_0 = \alpha_{m+1} = 0$$

Assumption 2. The repair time distribution is phase-type distribution $\mathrm{PH}(\beta, S)$ of order N given by

$$G_r = \beta S^{r-1} S^0, \quad r \in N, \qquad G_0 = \beta_{n+1} = 0$$

Assumption 3. The random variable, the repair time and the working server are independent.

Under this assumption, the system is governed by a Markov chine with $n+1$ states.

3. The Model Description

In this section, we first describe the system model. Then we derive a quasi-birth and death process of the system

The notation $\otimes$ will stand for the kronecker product of two matrices. Thus, if A is a matrix of order $m_1 \times m_2$ and if B is a matrix of order $n_1 \times n_2$, then $A \otimes B$ will denote a matrix of order $m_1 n_1 \times m_2 n_2$ whose $(i, j)^{\text{th}}$ block matrix is given by $a_{ij} B$. For more details on kronecker products, we refer to Bellman [3]. Before we describe the Markov chain the repairman model, we represent a review of PH-distribution.

The assumption of the system model are as follows

- When the working server fails and goes to repairs(repairman is idle), and a standby server becomes a working system. The block that represents this transition is $(t'\alpha \otimes \beta)$ of order $(m \times mk)$.
- When the main server fails and if has not completed any repair. The corresponding block is given by $(t' \otimes S)$.
- The working server fails and repair time is busy, fails system joins the queue. Once repaired, the server is returned back to normal working condition. The

corresponding block is $(t'\alpha \otimes S)$ of order $(mk \times mk)$

- Assumed that the repairman is subject to failure, which can occur even when the repairman is idle and the repaired goes to standby. The block governs this transition is $(T\alpha \otimes S')$ of order $(mk \times m)$.
- All server ar e in r epaired a nd o ne o f t hem is r epaired. T he co rresponding block is given $(\alpha \otimes S'\beta)$
- The repairman remains busy, the block is $(T \otimes S'\beta)$ of order $(mk \times mk)$.
- There is neither failure n or co mpleted r epairs. In t he first case t here is a transition among the service phase or the repair phase separately. The block governed by T. The transition $N \to n$ is governed by the matrix S.
- There is neither fail arrives nor a r epair is co mpleted or at the same time, a failure an d a r epair o ccur. T he bl ock is $(T \otimes S + t'\alpha \otimes S'\beta)$ of order is $(mk \times mk)$.

4. Quasi-Birth and Death Process

This matrix represents a d iscrete Quasi-birth and death process was developed Neuts [9] to solve the stationary state probability for the vector state Markov process with repetitive st ructure. W e de velop t he st eady-state pr obability. T he corresponding transition rate matrix Q at this Markov chain has the block-tridiogonal form. Consider the generator matrix Q as shown below

$$Q = \begin{bmatrix} B_0 & C_0 & & & \\ B_1 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & \ddots & \ddots & \ddots & \\ & & A_2 & A_1 & C_{N-1,N} \\ & & & B_{N,N-1} & B_{N,N} \end{bmatrix} \quad (1)$$

where

$B_0 = T, C_0 = (t'\alpha \otimes \beta), B_1 = (T \otimes S'), A_1 = (T \otimes S) + (t'\alpha \otimes S'\beta), A_2 = (T \otimes S'\beta),$

$A_0 = (t'\alpha \otimes S), C_{N-1,N} = (t' \otimes S), B_{N,N-1} = (\alpha \otimes S'\beta),\ B_{N,N} = S.$

5. Stationary Probability Vector

We start to discuss the method to obtain matrix R and stationary distribution vector X. F rom re ference N euts [9] a nd L atouch a nd R amaswami [7]. L et $X = \{X_0, X_1, X_2, \ldots, X_N\}$ b e th e s tationary p robability vector c orresponding to th e transition probability matrix(1). T he vector X satisfies the matrix equation

$$XQ = 0, \quad X1 = 1 \tag{2}$$

By de veloping t his equa lity in t erms o f t he bl ocks (2) to (6), we h ave t he equat ion system,

$$X_0 = X_0 B_0 + X_1 B_1 \tag{3}$$

$$X_1 = X_0 C_0 + X_1 A_1 + X_2 A_2 \tag{4}$$

$$X_2 = X_1 A_0 + X_2 A_1 + X_3 A_2$$

$$X_{N-1} = X_{N-2} A_0 + X_{N-1} A_1 + X_N B_{N,N} \tag{5}$$

$$X_N = X_{N-1} C_{N-1,N} + X_N B_{N,N} \tag{6}$$

Normalizing condition is

$$\sum_{i=1}^{N} X_i 1 = 1 \tag{7}$$

Solving this equat ion s ystem, app lied t o the d iscrete cas e, L et $R^1, R^2, R^3, \ldots, R^{n-2}$ be matrices of order $mk \times mk$

$$R^{j-1} = A_0 + R^{j-1} A_1 + R^{j-1} R^j A_2, \qquad J = 2,3,\ldots,N-2$$

$$(A_i + R^j A_2), J = 2,3,\ldots,N-2$$

The probability vector $X_1, X_2, \ldots, X_{N-1}$ verify

$$X_j = X_{j-1} A_0 + X_j A_1 + X_{j+1} A_2, \qquad J = 2,3,\ldots,N-2$$

If and only if

$$X_j = X_{j-1} R^{j-1}, \qquad 1 \le j \le N-1$$

By successive substitutions, we have

$$X_j = X_1 \prod_{k=1}^{j-1} R^k, \qquad 1 \le j \le N-1 \tag{8}$$

Also we can express X_{N-1} and X_N in a matrix geometry form as

$$\begin{aligned} X_{N-1} &= X_{N-2} R^{N-2} \\ X_N &= X_{N-1} R^{N-1} \\ R^{N-1} &= B_{N-1} (I - B_{N,N})^{-1} \\ R^{N-2} &= A_0 (I - A_1 - R^{N-1} B_{N,N-1})^{-1} \end{aligned} \tag{9}$$

The rest of the matrices $R, R^2, R^3, \ldots, R^{n-3}$ can be calculated recursively by the following expression. Under the assumption that the matrix $(I - A_1 - R_{N-1}B_{N,N-1})$ is non-singular.

$$R^{j-1} = A_0(I - A_1 - R^j A_2)^{-1}, \qquad 1 \le j \le N-1$$

The normalizing condition can be expressed as follows:

$$X_0 = X_0B_0 + X_1B_1 \text{ and } X_1 = X_0C_0 + X_1[A_1 + R^1A_2]$$
$$1 = X_0 1 + X_1\left[\sum_{i=2}^{n}\prod_{j=1}^{i-1} R^j 1 + 1\right] \qquad (10)$$

The stationary probability vector is determined on calculation X_0 and X_1

$$(X_0, X_1)\begin{bmatrix} B_0 & C_0 & 1 \\ B_1 & A_1 + R_1A_2 & U \end{bmatrix} = (X_0, X_1, 1)$$

$$(X_0, X_1) = (0,1)\begin{bmatrix} I - B_0 & C_0 & 1 \\ -B_1 & A_1 + R_1A_2 - I & U \end{bmatrix}^{-1}, \quad U = \sum_{i=2}^{N}\prod_{j=1}^{i-1} R^j 1 + 1 \qquad (11)$$

6. System Performance

We calculate the probability that the system is service at epoch k. The probability matrix at epoch is P^k, it is clear that,

$$A(k) = \sum_{i=0}^{N-1} P_i^k 1 = 1 - P_N^k 1 \qquad (12)$$

The probability P_i^k are determined from the Markov chain and

$$P_0^k = \left[(\alpha,0)P^k\right]_{1\times m}, \quad P_i^k = \left[(\alpha,0)P^k\right]_{m[(i-1)k+1]+1:m(ik+1)}$$
$$P_N^k = \left[(\alpha,0)P^k\right]_{m[(n-1)k+1]+1:m((N-1)k+1)+k}$$

Taking $\lim_{k\to\infty}$, we have the stationary availability

$$A = \sum_{i=0}^{N-1} X_i 1 = X_0 1 + X_1\left(\sum_{i=1}^{N-1}\prod_{k=1}^{i-1} R^k\right)1 = 1 - X_N 1 \qquad (13)$$

The non-availability is given by $\overline{A} = 1 - A = X_N$. Which is the probability that all the units are in repair or waiting for repair.

7. Conditional Probability of Failure

The failure occurs in the discrete distribution PH (α, T) and the repairman is idle, this is expressed by $P_0^{k-1}t'$. When $i > 1$, the repairs does not effect to this measure and in this case it is expressed as $P_i^{k-1}(t' \otimes 1)$.

The conditional probability of failure is given by the expression

$$b_k = P_0^{k-1}t' + \sum_{i=1}^{N-1} P_i^{k-1}(t' \otimes 1) \quad (14)$$

and stationary case it is equal to

$$b = \mathsf{X}_0 t' + \sum_{i=1}^{N-1} \mathsf{X}_i (t' \otimes 1) = \mathsf{X}_0 t' + \mathsf{X}_1 \left(\sum_{i=1}^{N-1} \prod_{k=1}^{i-1} \mathsf{R}^k \right)(t' \otimes 1) \quad (15)$$

Next we co nsider that a failure of the system occurs when all the systems are non-service and we can write this new measure in transient regime as

$$b_s^k = P_{N-1}^{k-1}(t' \otimes 1)$$

and in the stationary case

$$b_s = \mathsf{X}_{N-1}(t \otimes 1) = \mathsf{X}_1 \left(\prod_{k=1}^{N-2} \mathsf{R}^k \right)(t' \otimes 1) \quad (16)$$

8. Numerical Examples

In t his se ction, we app ly t he c alculation p erformed a bove t he pr actical case. T he methods were i mplemented i n m atlab programme. W e c onsider the f ollowing represents for the PH-distribution of the ser vice time and repair time $\alpha(1,0,0)$ and $\beta(1,0,0)$

$$T = \begin{pmatrix} 0.7 & 0.3 & 0 \\ 0 & 0.4 & 0.6 \\ 0 & 0 & 0.6 \end{pmatrix}; \quad S = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1.5 & 1.5 \\ 0 & 0 & 2 \end{pmatrix}$$

The matrix T indicates t he w orking s ystem has a n se rvice t ime t hat u ndergoes successive decorating phases. The order of T and S are, respectively $m = 2$ and $n = 3$. Consequently, the matrices $\mathsf{A}_0, \mathsf{A}_1$ and A_2 are of order 6×6; C_0 and B_1 are of order 2×6 and 6×2. By using the euation (11) R matrix, the sub-vector of the stationary probability vector are given below

$$\mathsf{X}_0 = (0.2756, 0.1543, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)$$

$$\mathsf{X}_1 = (0.0843, 0.0469, 0.0413, 0.0226, 0.0011, 0.0041, 0.0028, 0.0003, 0.0000)$$

$$\vdots$$

The per formance measures ca lculated in t his model depe nd o n t he sub-vectors (X_0, X_1), they can be explicitly determined, and they are $b = 0.001$

- The number of queued and waiting for repair is $M_q = 0.0776$
- The mean total expected in the repair facility is $M_L = 0.3651$

Conclusion

In this paper, we have a comparative analysis for a system repair problem where both the systems and the repairman can fail. The repair times of the systems and the service times o f t he r epairman ar e modelled us ing d ifferent PH- distribution. Us ing Q BD process, we obtained the stationary probability distribution. We find that the behavior of t he d ifferent per formance measures co nsidered in t he paper is incentive t o the distribution o f t he ser vice t imes o f t he r epairman. Our r esults can be t reated as performance evaluation tool for the concerned system which may be suited to many congestion situations arising in many practical applications encountered in computer and c ommunication systems, di stribution and s ervice s ectors, p roduction and manufacturing system, etc.,

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