International Journal of Mathematics Research.
ISSN 0976-5840 Volume 8, Number 1 (2016), pp. 39-44

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Commuting Well bounded Operators on Hilbert Space

Ms. Rajeshree Nanaware
Pratibha College of Commerce & Computer Studies,
Chinchwad, Pune-411019, India.
Email:rajeshreenanaware.aug1@gmail.com

Dr. S. M. Padhye
Head Of the Department, RLT College, Akola-444001, India.
Email: rltmaths@rediffmail.com

Abstract

The sum & the product of two commuting well-bounded operators on a Hilbert space is not well-bounded. On the other hand, it is shown in a positive direction that, if one of the two well bounded operators is in fact a scalar-type spectral operator, then their sum and product are well-bounded.

Keywords: Self-adjoint Operators,Hilbert Space, Banach Space, Scalar-type Operators, Well-bounded operators.

Subject Code Classification: MSC(2010): 47B40.

Introduction

The theory of scalar-type spectral operators was initiated by N. Dunford[1] in order to generalize the theory of self-adjoint operators to the operators on general Banach spaces..These operators are those which admit an integral representation with respect to a countably additive spectral measure and therefore a functional calculus for bounded measurable functions on their spectrum. In particular the spectral expansion of such an operator converges unconditionally.

Well-bounded operators, introduced by D. R. Smart[2] A bounded linear operator T on a complex reflexive Banach space is said to be well-bounded if it is possible to choose a compact interval J = [a, b] and a positive constant M such that

$$\| p(T) \| \leq M\{ \|p\|_J + \text{var}_J\, p\} \tag{1}$$

for every complex polynomial p, where $\|p\|_J$ denotes sup $\{| p(t) |: t \in J\}$.

They distinction between these classes is that the spectral expansions for a scalar type operator are of an unconditional nature while those for well-bounded operators corresponds to conditional decomposition of the Banach space. They are of interest principally because they admit (and in fact are characterised by) an integral representation similar to, but in general weaker than, the integral representation of a self-adjoint operator on a Hilbert space. T is well-bounded if it is a scalar type spectral operator with real spectrum.

The sum & the product of two commuting well-bounded operators on a Hilbert space is not well-bounded but , if one of the two well bounded operators is in fact a scalar-type spectral operator, then their sum and product are well-bounded.

Notation:

Given a complex Hilbert space H, B(H) denotes the set of all bounded linear operators on H, I is the identity operator on H, and $\sigma(T)$ is the spectrum of an element T of B(H). The strong operator topology on B(H) is denoted by s and, given a sequence $\{T_n\}$ in *B(H)*, we write

$$T = s\text{-} \sum_{n=1}^{\infty} T_n$$

to mean that the series $\sum_{n=1}^{\infty} T_n$ is s- convergent in B(H), with sum T.R is the set of real numbers and *P* is the set of all polynomials in a single variable with complex coefficients.

An example

The example of two commuting well-bounded operators S and T on a Hilbert space such that neither S+T nor ST is well-bounded is based on Example in (4)which sequences $\{x_n\}$ and $\{y_n\}$ in the space l_2 are constructed having the following properties:

(a) $\{x_n\}$ is a (conditional) basis of l_2 ;

(b) $(x_n , y_m) = \delta_{nm}$ for n, m = 1, 2, ...,

(c) $(e_{2n-1}, y_{2m-1}) = \delta_{nm}$ for n, m = 1, 2, ...,;

(d) $\| \sum_{j=1}^{n} n^{-1/2} x_{2j-1}\| \rightarrow \infty$ an $n \rightarrow \infty$

where $e_1 , e_{2,...}$ is the usual orthonormal basis of l_2 and δ_{nm} is the Kronecker delta.

It follows from (b) that the coefficient functional associated with the element x_m of the basis $\{x_n\}$ is given by $x \rightarrow (x, y_m)$ $(x \in l_2)$.

Defining $P_n \in B(l_2)$ by $P_n x = (x, y_n)x_n$ $(x \in l_2)$.for n =1, 2, ..., it is thus seen that each P_n is a projection, as $P_n^2x = P_n(Pn(x)) =$

$= P_n (x, y_n)x_n$

$= (x, y_n) P_n x_n$

$= (x, y_n)x_n$

$= P_n$

and $P_nP_m = P_n (x, y_m)x_m$

$= (x, y_m) P_n x_m$

$= 0$ f or (n # m),

and

$$I = s\text{-} \sum_{n=1}^{\infty} P_n \qquad (2)$$

Also, if $z_n = n^{-1/2} (e_1 + e_2 +\ldots\ldots\ldots\ldots + e_{2n-1\ldots})$then

$\|z_n\|^2 = n^{-1} (e_1 + e_2 +\ldots\ldots\ldots\ldots + e_{2n-1\ldots})^2$

$=\frac{1}{n}.n$

$=1$,

Therefore , z_n is a unit vector and so, using (c)

$\| \sum_{j=1}^{n} P_{2j-1}\| \geq \| \sum_{j=1}^{n} P_{2j-1} \ z_n\| =$

$= \|(z_n, y_{2j-1}) x_{2j-1}\|$

$=\| n^{-1/2} (e_1 + e_2 +\ldots\ldots\ldots\ldots + e_{2n-1\ldots}), y_{2j-1}) x_{2j-1}\|$

$= \| \sum_{j=1}^{n} n^{-1/2} \ x_{2j-1}\|$

Hence, by (d),

$\| \sum_{j=1}^{n} P_{2j-1}\| \ \rightarrow \infty$ as $n \rightarrow \infty$

By (2) and the principle of uniform boundedness, the partial sums of the series $\sum_{j=1}^{n} P_n$ are bounded in norm, and so (3) implies that

$$\| \sum_{j=1}^{n} P_{2j} \| \to \infty \text{ as } n \to \infty$$

We will use the following lemmas from[3] in the proof of the theorem.

Lemma 1: Let $\{\lambda_n\}$ be a monotonic bounded sequence in R. Then the series $\sum_{n=1}^{\infty} \lambda_n P_n$ converges strongly in B(l_2) and its sum is a well-bounded operator.

Lemma 2: Let T be a well-bounded operator on H implemented by (M, J),let $\lambda \in$ R, and let K be a compact interval such that λ + t, λt $\in$ K for all t$\in$J. Then λI+ T and λT are well-bounded operators, and are implemented by (M, K).

Lemma 3: Let T be a well-bounded operator on H implemented by (M, J), let S be a self-adjoint operator on H with finite spectrum such that ST = TS, and let K be a compact interval such that λ + t, λt $\in$ K whenever t $\in J$ and $\lambda \in \sigma(S)$.Then S+T and ST are well-bounded operators, and are implemented by (M,K)

Theorem A : If T is well-bounded then for any real number μ there is a unique bounded projection P_μ such that P_μ (β) is the space of eigenvectors of μ. where ,β is reflexive banach space.

Define the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ by putting λ_n = (n+1)/n , $\mu_{2n-1} = \mu_{2n}$ =(2n-1) /2n

For n=1,2,......... and let S = s - $\sum_{n=1}^{\infty} \lambda_n P_n$, T = s - $\sum_{n=1}^{\infty} \mu_n P_n$

Then S and T are well-bounded operators on l_2 by lemma 1, and clearly ST=TS .We show that neither S+T nor ST is well-bounded.
Suppose that S+T is well-bounded. Then , by THEOREM A there is a unique bounded projection Q mapping l_2 onto the eigenspace

$$\{ x \in l_2 (S+T)x = 2x \}$$

such that Q belongs to the bicommutant of S+T. In particular, $QP_n=P_nQ$ for all n &
$\lambda_n + \mu_n$ =2 (n even), $\lambda_n + \mu_n$ > 2 (n odd),

It is easily verified that $QP_n = P_n$ (n even), QP_n = 0 (n odd).

Since I = s- $\sum_{n=1}^{\infty} QP_n$ = s- $\sum_{n=1}^{\infty} P_{2n}$

Therefore the partial sums of the series $\sum_{n=1}^{\infty} P_{2n}$ are bounded in norm by the principle of uniform boundedness, contradicting (4) .Hence S+T is not well-bounded.

Since $\lambda_n \mu_n = 1$ (n odd) and $\lambda_n \mu_n < 1$ (n even), a similar argument shows that ST is not well-bounded, for otherwise (3) would be contradicted.

Throughout this section, let H be a complex Hilbert space. Given a well bounded operator T on H, we say that T is implemented by (M, J) if M is a positive constant and a compact interval such that (1) holds for all p.

Theorem. Let T be a well-bounded operator on H, and let S be a scalar-type spectral operator on H with real spectrum such that ST = TS. Then S+T and ST is well-bounded.

Proof. Since S has real spectrum, then S is similar to a self-adjoint operator. Furthermore, it is easily seen that the property of well-boundedness is invariant under a similarity transformation. Consequently, we may assume without loss of generality that S is self-adjoint.

Let T be implemented by (M, J) and let K be a compact interval such that $\lambda + t$, $\lambda t \in K$ whenever $t \in J$ and $\lambda \in \sigma(S)$ .By the spectral theorem,

$$S = \int \lambda \; F(d\lambda)$$

for some spectral measure F(.) on H with support $\sigma(S)$, and T commutes F(.) since it commutes with S. By approximating the integrand in this integral by step functions, a standard argument gives the existence of a sequence {Sn} of self-adjoint operators, each with finite spectrum contained in $\sigma(S)$ and each commuting with T, such that

$$\| S-Sn \| \to 0 .$$

Let $p \in P$, By Lemma 3 ,

$$\|p(S_n + T\| \leq \{\|p\|_k + \text{var}_K\, p\} \text{ for all n.}$$

Letting $n \to \infty$, we obtain $\|p(S+T)\| \leq M \{\|p\|_k + \text{var}_K\, p \}$,

This shows that S+T is well-bounded. The well-boundedness of ST is obtained similarly.

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