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Some Results on Fuzzy Semimodular Lattice

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Abstract:

In this Paper, Fuzzy Semimodular Lattice – Definition of Fuzzy semimodular Lattice- Fuzzy Complement of μ (a), Finite Fuzzy Semimodular Lattice, Characterization theorem are given.

Keywords:Fuzzy Lattice, Fuzzy Modular Lattice, Fuzzy Distributive Lattice, Fuzzy Semimodular Lattice, Finite Fuzzy Semimodular Lattice.

Introduction

The Concept of Fuzzy Lattice was already introduced by Ajmal,N[1], S.Nanda[4] and WilCox,L.R [5] explained modularity in the theory of Lattices, G.Gratzer[2], BarDalo, G.H and Rodrigues,E[3] Stern,m[6] explained semimodular Lattices, M.Mullai and B.Chellappa[7] explained Fuzzy L-ideal. A few definitions and results are listed that the fuzzy semimodular lattice using in this paper we explain fuzzy semimodular lattice, Definition of fuzzy semimodular lattice, Characterization theorem of Fuzzy semimodular lattice and some examples are given, Fuzzy Modular Lattice satisfy P_2 and P_3.

Definition: 1.1

A Fuzzy lattice L and $\mu(a) \in L$. We say that $\mu(a') \in L$ is a Fuzzy complement of $\mu(a)$ if $\mu(a \wedge a') = \mu(0)$ and $\mu(a \vee a') = \mu(1)$ we denote the Fuzzy set of Fuzzy complements of $\mu(a)$ by $C_{\mu(a)}$.

Theorem 1.1

Let L be a Finite Fuzzy Semimodular lattice $\mu(a), \mu(b) \in L$, $\mu(a) < \mu(b)$,
$\mu(b') \in C_{\mu(b)} \setminus C_{\mu(a)}$ then $\mu[(a \vee b') \wedge b] = \mu(a)$ and $\mu(a) \vee \mu(b')$ is a co-atom.

Proof

Let L be a finite Fuzzy Semimodular lattice and $\mu(a), \mu(b) \in L$, $\mu(a) < \mu(b)$ and let $\mu(b') \in C_{\mu(b)} \setminus C_{\mu(a)}$

$\Rightarrow \mu(a) \wedge \mu(b') < \mu(b) \wedge \mu(b'), \mu(b') \in C_{\mu(b)} \setminus C_{\mu(a)}$

$\Rightarrow \mu(a) \wedge \mu(b') = \mu(0)$ and $\mu(b') \notin C_{\mu(a)}$

Also $\mu(a) < \mu(b) \Rightarrow \mu(a) \vee \mu(b') < \mu(b) \vee \mu(b')$

$\Rightarrow \mu(a) \vee \mu(b') < \mu(l)$

We have $\mu(a \vee b') \vee \mu(b) = \mu(a) \vee \mu(b' \vee b)$

$= \mu(a) \vee \mu(l)$

$= \mu(a \vee l)$

$= \mu(l)$

$\mu(a \vee b') \wedge \mu(b) < \mu(b)$

Also $\mu(a) < \mu(a) \vee \mu(b'), \mu(a) < \mu(b)$

$\Rightarrow \mu(a) \wedge \mu(a) < \mu(a \vee b') \wedge \mu(b)$

$\Rightarrow \mu(a) \leq \mu(a \vee b') \wedge \mu(b) < \mu(b)$ and $\mu(a) < \mu(b)$

$\Rightarrow \mu(a) \leq \mu(a \vee b') \wedge \mu(b)$

As L is Fuzzy Semimodular

$\mu(a) = \mu(a \vee b') \wedge \mu(b) < \mu(b)$

$\Rightarrow \mu(a) \vee \mu(b') > \mu(b) \vee \mu(b') = \mu(l)$

$\Rightarrow \mu(a) \vee \mu(b')$ is a co-atom.

Theorem 1.2

Let L be a Finite Fuzzy Semimodular lattice $\mu(a), \mu(b) \in L$, $\mu(a) < \mu(b)$, $\mu(b') \in C_{\mu(b)} \setminus C_{\mu(a)}$. If there exists $\mu(c) \in L$. Such that $\mu(b') < \mu(c)$ and $\mu(a) \vee \mu(c) = \mu(1)$ then $\mu(c) \in C_{\mu(a)}$.

Proof

To Prove $\mu(a) \wedge \mu(c) = \mu(0)$

If $\mu(a) \wedge \mu(c) > \mu(0)$

Then there is $\mu(a_1)$ such that

$\mu(0) \leq \mu(a_1) \leq \mu(a) \wedge \mu(c)$

Also $\mu(a_1) \leq \mu(a) < \mu(b)$

$\Rightarrow \mu(a_1) \wedge \mu(b') = \mu(0)$

$\mu(o) \leq \mu(a_1) \Rightarrow \mu(b') < \mu(a_1) \vee \mu(b')$

on the other hand $\mu(a_1) < \mu(c)$ and $\mu(b') < \mu(c)$

$\Rightarrow \mu(a_1) \vee \mu(b') \leq \mu(c)$

$\Rightarrow \mu(c) = \mu(a_1) \vee \mu(b')$

We conclude $\mu(a) \vee \mu(b') = \mu(a \vee a_1) \vee \mu(b')$

$= \mu(a) \vee \mu(a_1 \vee b')$

$= \mu(a) \vee \mu(c)$

$= \mu(1)$

But from $\mu\,(a) < \mu\,(b)$ and $\mu\,(b) \wedge \mu\,(b') = \mu\,(0)$
We get $\mu\,(a) \wedge \mu\,(b') = \mu\,(0)$ so $\mu\,(b') \in C_{\mu\,(a)}$

Which is a contradiction
Hence $\mu\,(a) \wedge \mu\,(c) = \mu\,(0)$

Theorem: 1.3
Let L be a finite Fuzzy Semimodular lattice $\mu\,(a), \mu\,(b) \in L$.
$\mu\,(a) < \mu\,(b)$, $\mu\,(b') \in C_{\mu\,(b)} \setminus C_{\mu\,(a)}$. If there exists an atom $\mu\,(a_1) \in L$. such that $\mu\,(a_1) \vee \mu\,(a \vee b') = \mu\,(1)$ then $\mu\,(a_1 \vee b')$ is a Fuzzy complement of $\mu\,(a)$ and $\mu\,(b') < \mu\,(a_1 \vee b')$.

Proof
We have $\mu\,(a \wedge b') = \mu\,(0)$ and
$\mu\,(b') \notin C_{\mu\,(a)} \Rightarrow \mu\,(a \vee b') < \mu\,(1)$
From $\mu\,(a_1) \vee \mu\,(a \vee b') = \mu\,(1)$
We have $\mu\,(a_1) \leq \mu\,(a \vee b')$ and so
$\mu\,(a_1) \wedge \mu\,(a \vee b') = \mu\,(0)$
This implies $\mu\,(a_1 \wedge b') = \mu\,(0)$
So, by the Fuzzy Semimodular Property
$\mu\,(b') < \mu\,(a_1 \vee b')$
Now by using theorem 1.2
We Conclude $\mu\,(a_1 \vee b') \in C_{\mu\,(a)}$

Definition: 1.2
Let L be a Fuzzy Modular Complemented Fuzzy lattice then
P: $\forall\ \mu(a) \neq \mu(b) \in L$, if $C_{\mu\,(a)} \neq \phi$ and $C_{\mu\,(b)} \neq \phi$ then $C_{\mu\,(a)} \neq C_{\mu\,(b)}$.

Definition: 1.3
Let L be a Fuzzy lattice consider a pair $(\mu(a), \mu(b)) \in L^2$ such that
$\mu(a) < \mu(b)$ $C_{\mu\,(a)} \neq \phi$ and $C_{\mu\,(b)} \neq \phi$. If L is Fuzzy distributive then
Q_1 : there exists $(\mu\,(a'), \mu(b')) \in C_{\mu\,(a)} \times C_{\mu\,(b)}$ such that $\mu(b') \leq \mu(a')$.

Definition: 1.4
Let L be a Fuzzy lattice consider a pair $(\mu(a), \mu(b)) \in L^2$
such that $\mu(a) < \mu(b)$, $C_{\mu\,(a)} \neq \phi$ and $C_{\mu\,(b)} \neq \phi$. If L is Fuzzy distributive then,
Q_2 : $\forall\ \mu(b') \in C_{\mu\,(b)}$ there exists $\mu(a') \in C_{\mu\,(b)}$ such that $\mu(b') \leq \mu(a')$.

Theorem: 1.4
Let L be a Finite Fuzzy Semimodular lattice. $\mu(a), \mu(b) \in L$ Such that
$\mu(a) < \mu(b)$ and $C_{\mu\,(b)} \neq \phi$. If $C_{\mu\,(a)} \neq C_{\mu\,(b)}$ then $\forall\ \mu\,(b') \in C_{\mu\,(b)} / C_{\mu\,(a)}$. There exists
$\mu(c) \in C_{\mu\,(a)}$ such that $\mu\,(b') < \mu(c)$.

Proof

Let L be as stated and let $\mu(a), \mu(b) \in L$.
$\mu(a) < \mu(b)$ and $\mu(a') \in C_{\mu(a)} / C_{\mu(b)}$.
Let $\mu(b') \in C_{\mu(b)} / C_{\mu(a)}$
We have $\mu(a' \vee b) = \mu(1)$
And $\mu(a') \notin C_{\mu(b)}$
$\Rightarrow \mu(o) < \mu(a' \wedge b)$
Let $\mu(a_1)$ be an atom such that
$\mu(a_1) \leq \mu(a' \wedge b)$
From $\mu(a \wedge a') = \mu(o)$ and $\mu(a_1) \leq \mu(a')$
We have $\mu(a \wedge a_1) = \mu(o)$ and as $\mu(o) < \mu(a_1)$
And L is Fuzzy Semimodular we conclude
$\mu(a) < \mu(a \vee a_1)$
Since $\mu(a) < \mu(b)$ and $\mu(a_1) \leq \mu(b)$
We get $\mu(a \vee a_1) \leq \mu(b)$ and as $\mu(a) < \mu(b)$
We have $\mu(c) \in C_{\mu(a)}$
Let $\mu(c) = \mu(a_1 \vee b')$
Then $\mu(a \vee c) = \mu(a \vee (a_1 \vee b'))$
$= \mu((a \vee a_1) \vee b')$
$= \mu(b \vee b')$
$= \mu(1)$
By the theorem 1.3
We Conclude $\mu(c) \in C_{\mu(a)}$.

Corollary 1.1

Let L be a finite Fuzzy Semimodular lattice. If $\mu(a), \mu(b) \in L$. Such that $\mu(a) < \mu(b)$ and $C_{\mu(b)} \neq \phi$ and $C_{\mu(a)} \neq C_{\mu(b)}$ then $(\mu(a), \mu(b))$ satisfies Q_2 .

Proof

Let L be a finite Fuzzy Semimodular lattice.
Let $\mu(a), \mu(b) \in L$ be such that $\mu(a) < \mu(b)$, $C_{\mu(b)} \neq \phi$ and $C_{\mu(a)} \neq C_{\mu(b)}$
Let $\mu(b') \in C_{\mu(b)}$
If $\mu(b') \in C_{\mu(b)} / C_{\mu(a)}$ then by the theorem 4.8 there exists $\mu(a') \in C_{\mu(a)}$ such that $\mu(b') < \mu(a')$ in particular $\mu(b') \leq \mu(a') \in C_{\mu(a)}$ then take $\mu(a') = \mu(b')$.

Corollary 1.2

(i) Finite Complemented Fuzzy Semimodular lattice Satisfy Q_2 .
(ii) Let L be Finite Fuzzy Semimodular lattice $\mu(a), \mu(b) \in L$ such that $C_{\mu(a)} \neq \phi$, $C_{\mu(b)} \neq \phi$ and $\mu(a) < \mu(b)$. Then $(\mu(a), \mu(b))$ Satisfies Q_1.

Proof

Proof of (i):

Let L be a Finite Complemented Fuzzy Semimodular lattice and $(\mu(a), \mu(b)) \in L^2$.

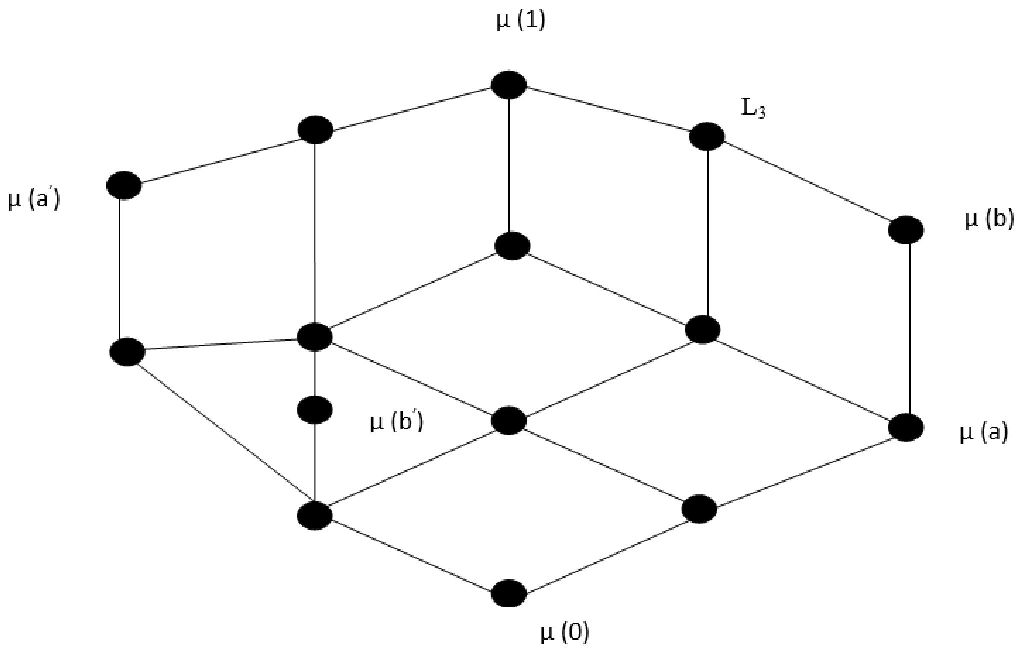
First we note, that on a Finite Complemented Fuzzy lattice if we show that Q_2 holds when μ (a) < μ (b) then this Property is also valid when μ (a) < μ (b).
Let μ (a) < μ (b).
Then Property holds when $\mu (b') \in C_{\mu (a)}$
Suppose $\mu (b') \in C_{\mu (b)} / C_{\mu (a)}$.
By the theorem 1.1, the Fuzzy element $\mu (a) \vee \mu (b')$ is a co-atom.
It is easy to see that there is a complement $\mu (a_1)$ of $\mu (a) \vee \mu (b')$ which is an atom.
So by theorem 1.3, $\mu (b') < \mu (b') \vee \mu (a_1)$ and Q_2 is Satisfied.

Proof of (ii)
Let L be a Finite Fuzzy Semimodular lattice μ (a), μ (b) $\in$ L. such that $C_{\mu (a)} \neq \phi$, $C_{\mu (b)} \neq \phi$ and μ (a) < μ (b).

If $C_{\mu (a)} \neq C_{\mu (b)}$ then, by corollary 1.1(μ (a), μ (b)) Satisfies Q_2 and therefore Satisfies Q_1.
If $C_{\mu (a)} \subseteq C_{\mu (b)}$ then choose $\mu (a') \in C_{\mu (a)}$ and consider the pair
$(\mu (a'), \mu (a')) \in C_{\mu (a)} \times C_{\mu (b)}$

Example: 1.1
Here is an example for Finite Fuzzy Semi modular lattices which do not satisfy Q_2 .

Verification:



In this figure $\mu (b') \not\leq \mu (a')$
$\therefore Q_2$ is not statisfied.

Definition: 1.5

Let L be a Fuzzy lattice consider a pair $(\mu(a), \mu(b)) \in L^2$. Such that $\mu(a) < \mu(b)$, $C_{\mu(a)} \neq \phi$ and $C_{\mu(b)} \neq \phi$. If L is Fuzzy distributive then

P_1 : There exists $(\mu(a'), \mu(b')) \in C_{\mu(a)} \times C_{\mu(b)}$ Such that $\mu(b') < \mu(a')$

Definition: 1.6

Let L be a Fuzzy lattice consider a pair $(\mu(a), \mu(b)) \in L^2$. Such that $\mu(a) < \mu(b)$, $C_{\mu(a)} \neq \phi$ and $C_{\mu(b)} \neq \phi$. If L is Fuzzy distributive then

$P_2 : \forall \mu(b') \in C_{\mu(b)}$ there exists $\mu(a') \in C_{\mu(a)}$ such that $\mu(b') < \mu(a')$.

Definition: 1.7

Let L be a Fuzzy lattice consider a pair $(\mu(a), \mu(b)) \in L^2$. Such that $\mu(a) < \mu(b)$, $C_{\mu(a)} \neq \phi$ and $C_{\mu(b)} \neq \phi$. If L is Fuzzy distributive then

$P_3 : \forall \mu(a') \in C_{\mu(a)}$ there exists $\mu(b') \in C_{\mu(b)}$ such that $\mu(b') < \mu(a')$.

Theorem: 1.5

Fuzzy modular lattices satisfy P_2 and P_3

Proof:

It is enough to show P_2 because this implies P_3 by Fuzzy duality.

Let L be a Fuzzy modular lattice.

Consider $\mu(a), \mu(b) \in L$ Such that $\mu(a) < \mu(b)$, $C_{\mu(a)} \neq \phi$ and $C_{\mu(b)} \neq \phi$.

For all $\mu(b') \in C_{\mu(b)}$ choose $\mu(a') \in C_{\mu(a)}$. We will prove that

$\mu(a'') = \mu(a' \wedge b) \wedge \mu(b')$ is a Fuzzy

Complement of $\mu(a)$ greater than $\mu(b')$

In fact $\mu(a \vee a'') \geq \min\{\mu(a), \mu(a'')\}$

$\geq \min\{\mu(a), \mu(a' \wedge b) \vee \mu(b')\}$

$\geq \min\{\mu(a) \vee \mu(a' \wedge b), \mu(b')\}$

$\geq \min\{\mu(a \vee a') \wedge \mu(b), \mu(b')\}$

$\geq \min\{\mu(b), \mu(b')\}$

$= \mu(b \vee b')$

$= \mu(1)$

Also $\mu(a) \wedge ((\mu(a' \wedge b) \vee \mu(b')) = \mu(a) \wedge (((\mu(a' \wedge b) \vee \mu(b')) \wedge \mu(b))$

$= \mu(a) \wedge ((\mu(a' \wedge b) \vee \mu(b' \wedge b))$

$= \mu(a) \wedge \mu(a') \wedge \mu(b)$

$= \mu(0)$

We have $\mu(b') \leq \mu(a'')$ and if $\mu(b') = \mu(a'')$

Then we would have $\mu(a) < \mu(b)$,

$\mu(a \wedge b') = \mu(b \wedge b') = \mu(0)$ and

$\mu(a \vee b') = \mu(b \vee b') = \mu(l)$

which contradicts the Fuzzy modularity of L.

Conclusion

The paper is proved that Let L be a Finite Fuzzy Semimodular lattice $\mu(a)$, $\mu(b) \in L$, $\mu(a) < \mu(b)$, $\mu(b') \in C_{\mu(b)} \setminus C_{\mu(a)}$ then $\mu[(a \wedge b') \wedge b] = \mu(a)$ and $\mu(a) \vee \mu(b')$ is a co-atom, Let L be a Finite Fuzzy Semimodular lattice $\mu(a)$, $\mu(b) \in L$, $\mu(a) < \mu(b)$, $\mu(b') \in C_{\mu(b)} \setminus C_{\mu(a)}$. If there exists $\mu(c) \in L$. Such that $\mu(b') < \mu(c)$ and $\mu(a) \vee \mu(c) = \mu(1)$ then $\mu(c) \in C_{\mu(a)}$, Let L be a finite Fuzzy Semimodular lattice $\mu(a)$, $\mu(b) \in L$. $\mu(a) < \mu(b)$, $\mu(b') \in C_{\mu(b)} \setminus C_{\mu(a)}$. If there exists an atom $\mu(a_1) \in L$. such that $\mu(a_1) \vee \mu(a \vee b') = \mu(1)$ then $\mu(a_1 \vee b')$ is a Fuzzy complement of $\mu(a)$ and $\mu(b') < \mu(a_1 \vee b')$, Let L be a Finite Fuzzy Semimodular lattice. $\mu(a)$, $\mu(b) \in L$ Such that $\mu(a) < \mu(b)$ and $C_{\mu(b)} \neq \phi$. If $C_{\mu(a)} \neq C_{\mu(b)}$ then $\forall\ \mu(b') \in C_{\mu(b)} / C_{\mu(a)}$. There exists $\mu(c) \in / C_{\mu(a)}$ such that $\mu(b') < \mu(c)$ and Fuzzy modular lattices satisfy P_2 and P_3.

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