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Intuitionistic Ultra Filter and Convergency of Filters

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Abstract

The aim of this paper is to introduce an intuitionistic ultra filter via IS sets and study some of its properties.

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1. Introduction and Preliminaries

The dual notion of filter is an ideal. A filter in normal course can be viewed as locating scheme.A filter space X is the collection of subsets of X, that might contain what one is locating for either a point of X or a subset of X. This basic definition of a filter on a set ensures the finite intersection property of a filter. Filter was introduced in general topology by Henri Cartan in 1937 along with Bourbaki. Cartan idea of filter was obtained as a way of removing the countability restriction by use of sequences. On general topology, the notion of a filter on a topological space X, became one of the basic tools used to describe the convergence in general topological spaces together with the notion of a set. Hence the notion of filters and ultra filters is a powerful tool in topology and analysis. The concept of intuitionisic set was introduced by Coker in[4] it is nothing but the discrete form of intuitionistic fuzzy set introduced by Krassimir T. Atanassov [1]. In this paper the study of intuitionistic filters are extended by defining the ultra filters. Filter points and convergence of filters are defined and studied via intuitionistic sets.

Definition 1.1. [2] Let X be a nonempty fixed set. An intuitionistic set (IS for short) A is an object having the form A = $\prec$ X, A^1, A^2 $\succ$ where A^1 and A^2 are subsets of X satisfying $A^1 \cap A^2 = \phi$. The set A^1 is called the set of members of A, while A^2 is called the set of non members of A.

Definition 1.2. [2] Let X be a nonempty set. $A = \prec$ X, A^1,A^2 $\succ$ and $B = \prec$ X, B^1, B^2 $\succ$ be intuitionistic sets on X and let $\{A_i : i \in J\}$ be an arbitrary family of IS's in X, where A^i = $\prec$ X, ${A_i}^1$, ${A_i}^2$ $\succ$. Then

(a) $A \subseteq B$ if and only if $A^1 \subseteq B^1$ and $B^2 \subseteq A^2$.

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(c) $\cup A_i = \prec$ X, $\cup A_i^1$, $\cap {A_i}^2$ $\succ$.

(d) $\cap A_i = \prec$ X ,$\cap {A_i}^1$, $\cup {A_i}^2$ $\succ$

(e) $\tilde{X} = \prec$ X, X,ϕ $\succ$

(f) $\tilde{\phi} = \prec$ X, ϕ, X $\succ$.

Definition 1.3. [9] An intuitionistic filter ($\mathcal{I_F}$ for short) on a nonempty set X is a family of IS's in X satisfying the following axioms:

(1) $\tilde{\phi} \notin \mathcal{I_F}$

(2) If F $\in \mathcal{I_F}$ and H $\supset$ F, then H $\in \mathcal{I_F}$.

(3) If F $\in \mathcal{I_F}$ and H $\in \mathcal{I_F}$, then F $\cap$ H $\in \mathcal{I_F}$.

In this case, the pair (X, $\mathcal{I_F}$) is called an intuitionistic filter.

Example 1.4. [9] Let $X = \{a, b\}$ and consider the family $\mathcal{I_F} = \{\tilde{X}, A_1, A_2\}$ where $A_1 = \prec X, \{a\}, \{b\} \succ$ and $A_2 = \prec X, \{a\}, \phi \succ$. Then $(X, \mathcal{I_F})$ is an intuitionistic filter on X.

Definition 1.5. [4] Let X be a nonempty set and $p \in X$ be a fixed element in X. Then $\tilde{p} = \prec X, \{p\}, \{p\}^c \succ$ is called an intuitionistic point(IP for short) in X.

Definition 1.6. [2] An intuitionistic topology (IT for short) on a nonempty set X is a family τ of IS's in X satisfying the following axioms:

(T_1) $\tilde{\phi}, \tilde{X} \in \tau$.

(T_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(T_3) $\cup G_i \in \tau$ for any arbitrary family $\{G_i : i \in J\} \subseteq \tau$. In this case the pair (X, τ) is called an intuitionistic topological space(ITS for short) and any IS in τ is known as an intuitionistic open set (IOS for short) in X.

Definition 1.7. [2] Let (X, τ_1), (X, τ_2) be two ITS's on X. Then τ_1 is said to be contained in τ_2 (in symbols $\tau_1 \subseteq \tau_2$ if $G \in \tau_2$ for each $G \in \tau_1$. In this case we also say that τ_1 is coarser than τ_2.

2. Intuitionistic ultra filter

In this chapter, we introduced the intutionistic ultra filter and study some of its basic properties.

Definition 2.1. An intuitionistic filter $\mathcal{I_F}$ on X is called an intuitionistic ultra filter ($\mathcal{I_{UF}}$ for short) on X if and only if $\mathcal{I_F}$ is not properly contained in any other intuitionistic filter on X. In other words there does not exist any other intuitionistic filter which is strictly finer than $\mathcal{I_F}$.

That is $\mathcal{I_F}$ is an intutionistic ultra filter $\Leftrightarrow \mathcal{I_{F*}} \supset \mathcal{I_F} \Rightarrow \mathcal{I_{F*}} = \mathcal{I_F}$ for each intuitionistic filter $\mathcal{I_{F*}}$ on X.

Remark 2.2. From above, an intutionistic ultra filter on X is a maximal element of the collection of all intutionistic filters on X partially ordered by the inclusion relation $\mathcal{C}$.

Example 2.3. Let $X = \{a, b, c\}$ and consider the family $\mathcal{I_{UF}} = \{\tilde{X}, A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}, A_{16}, A_{17}, A_{18}\}$, where $A_1 = \prec X, \phi, \{a, b\} \succ$, $A_2 = \prec X, \{a\}, \phi \succ$, $A_3 = \prec X, \{b\}, \phi \succ$, $A_4 = \prec X, \{c\}, \phi \succ$, $A_5 = \prec X, \{a, b\}, \phi \succ$, $A_6 = \prec X, \{a, c\}, \phi \succ$, $A_7 = \prec X, \{b, c\}, \phi \succ$, $A_8 = \prec X, \{a\}, \{b\} \succ$, $A_9 = \prec X, \{b\}, \{a\} \succ$, $A_{10} = \prec X, \{c\}, \{b\} \succ$, $A_{11} = \prec X, \{c\}, \{a\} \succ$, $A_{12} = \prec X, \{b, c\}, \{a\} \succ$, $A_{13} = \prec X, \{a, c\}, \{b\} \succ$, $A_{14} = \prec X, \{c\}, \{a, b\} \succ$, $A_{15} = \prec X, \phi, \{a\} \succ$, $A_{16} = \prec X, \phi, \{b\} \succ$, $A_{17} = \prec X, \phi, \phi \succ$, $A_{18} = \prec X, X, \phi \succ$. Then $\mathcal{I_{UF}}$ is an intuitionistic ultra filter on X.

Example 2.4. Let $X = \{a, b, c\}$ and consider the family $\mathcal{I_{F1}} = \{\tilde{X}, A_1\}$ where $A_1 = \prec X, \{a, b\}, \phi \succ$. It is not an intuitionistic ultra filter because $\mathcal{I_{F1}}$ is properly contained in any other intuitionistic filter on X.

Theorem 2.5. Every intuitionistic filter on a nonempty set X is contained in an intuitionistic ultra filter on X.

Proof. Let $\mathcal{I_F}$ be an intuitionistic filter on X and $\mathcal{C}$ be the collection of all intuitionistic filters on X which contains $\mathcal{I_F}$. So that $\mathcal{C}$ is nonempty as atleast $\mathcal{I_F} \in \mathcal{C}$. By Remark 2.2, $\mathcal{C}$ is partially ordered by the inclusion relation. Let $\mathcal{D}$ be a linearly ordered intuitionistic subset of $\mathcal{C}$, so that for any two intuitionistic members $\mathcal{I_{F1}}$ and $\mathcal{I_{F2}}$ of $\mathcal{C}$, then either $\mathcal{I_{F1}} \subset \mathcal{I_{F2}}$ or $\mathcal{I_{F2}} \subset \mathcal{I_{F1}}$. Let $\mathcal{E} = \cup\, \{\mathcal{I_{F\alpha}} : \mathcal{I_{F\alpha}} \in \mathcal{D}\}$. Clearly, $\mathcal{E}$ is nonempty. Now, as $\mathcal{I_{F\alpha}}$ is an intuitionistic filter on X then $\tilde{\phi} \notin \mathcal{I_{F\alpha}}$ for any $\mathcal{I_{F\alpha}} \in \mathcal{D}$.

Hence, $\tilde{\phi} \notin \mathcal{E}$. Let $F = \prec X, {F_1}^1, {F_1}^2 \succ \in \mathcal{E}$. Then F belongs to $\mathcal{I_{F\alpha}}$ for atleast one $\mathcal{I_{F\alpha}} \in \mathcal{D}$. Since each $\mathcal{I_{F\alpha}}$ is an intuitionistic filter and if H $\supseteq$ F, then H $\in \mathcal{I_{F\alpha}}$. Hence H $\in \cup\, \{\mathcal{I_{F\alpha}} : \mathcal{I_{F\alpha}} \in D\} = \mathcal{E}$.

Let $\prec$ X, ${F_1}^1$, ${F_1}^2$ $\succ$ and $\prec$ X, ${F_2}^1$, ${F_2}^2$ $\succ$ $\in \mathcal{E}$. Then $\prec$ X, ${F_1}^1$, ${F_1}^2$ $\succ$ $\in \mathcal{I_{F\alpha}}$ and $\prec$ X, ${F_1}^1$, ${F_1}^2$ $\succ$ $\in \mathcal{I_{F\beta}}$ for some $\mathcal{I_{F\alpha}}$ and $\mathcal{I_{F\beta}}$ respectively in $\mathcal{D}$. As $\mathcal{D}$ is linearly ordered in IS set, either $\mathcal{I_{F\alpha}} \subset \mathcal{I_{F\beta}}$ or $\mathcal{I_{F\beta}} \subset \mathcal{I_{F\alpha}}$. Hence, both $\prec$ X, ${F_1}^1$, ${F_1}^2$ $\succ$ and $\prec$ X, ${F_2}^1$, ${F_2}^2$ $\succ$ are contained in either in $\mathcal{I_{F\alpha}}$ or in $\mathcal{I_{F\beta}}$. Since $\mathcal{I_{F\alpha}}$ and $\mathcal{I_{F\beta}}$ is an intuitionistic filter. Therefore, $\prec$ X, ${F_1}^1$, ${F_1}^2$ $\succ \cap \prec$ X, ${F_2}^1$, ${F_2}^2$ $\succ$ belongs to either in $\mathcal{I_{F\alpha}}$ or in

$\mathcal{I}_{\mathcal{F}\beta}$. Hence $\prec$ X, ${F_1}^1$, ${F_1}^2$ $\succ \cap \prec$ X, ${F_2}^1$, ${F_2}^2$ $\succ \in \mathcal{E}$. Therefore $\mathcal{E}$ is an intuitionistic filter on X. As $\mathcal{E}$ is finer than every member of $\mathcal{D}$ and as such $\mathcal{E}$ is an upperbound of $\mathcal{D}$. Thus it is proved that in a partially ordered nonempty set $\mathcal{C}$ every linearly ordered intuitionistic subset has an upperbound. Hence $\mathcal{C}$ must contain a maximal element say $\prec$ X, ${F^*}^1$, ${F^*}^2$ $\succ$ which implies, By definition 2.1, $\prec$ X, ${F^*}^1$, ${F^*}^2$ $\succ$ is an intuitionistic ultra filter containing $\mathcal{I}_{\mathcal{F}}$. ■

3. Characterization of intuitionistic ultra filters

Theorem 3.1. An intuitionistic filter $\mathcal{I}_{\mathcal{F}}$ on a nonempty set X is an intuitionistic ultra filter on X if and only if $\mathcal{I}_{\mathcal{F}}$ contains all those intuitionistic subsets of X which intersect every member of $\mathcal{I}_{\mathcal{F}}$.

Proof. Let $\mathcal{I}_{\mathcal{F}}$ be an intuitionistic filter on X,such that $\mathcal{I}_{\mathcal{F}}$ contains all those intuitionistic subsets of X which intersect every member of $\mathcal{I}_{\mathcal{F}}$. To prove that $\mathcal{I}_{\mathcal{F}}$ is an intuitionistic ultra filter on X, it is enough to prove that there does not exist any other intuitionistic filter on X which is strictly finer than $\mathcal{I}_{\mathcal{F}}$.

If possible, let ${\mathcal{I}_{\mathcal{F}}}^*$ be an intuitionistic filter on X, which is strictly finer than $\mathcal{I}_{\mathcal{F}}$. Now let $F^* = \prec X$, ${F^*}^1$, ${F^*}^2$ $\succ \in {\mathcal{I}_{\mathcal{F}}}^*$ and being an intuitionistic filter on X, each member of ${\mathcal{I}_{\mathcal{F}}}^*$ intersects every member of ${\mathcal{I}_{\mathcal{F}}}^*$. Hence F^* intersects every member of ${\mathcal{I}_{\mathcal{F}}}^*$ and as $\mathcal{I}_{\mathcal{F}} \subset {\mathcal{I}_{\mathcal{F}}}^*$, it is concluded that F^* intersects every member of $\mathcal{I}_{\mathcal{F}}$ and so F^* $\in \mathcal{I}_{\mathcal{F}}$,which implies ${\mathcal{I}_{\mathcal{F}}}^* \subset \mathcal{I}_{\mathcal{F}}$. But this is a contradiction to the assumption that ${\mathcal{I}_{\mathcal{F}}}^*$ is an intuitionistic filter on X which is strictly finer than $\mathcal{I}_{\mathcal{F}}$. Hence $\mathcal{I}_{\mathcal{F}}$ is an intuitionistic ultra filter on X.

Conversely, assume $\mathcal{I}_{\mathcal{F}}$ be an intuitionistic ultra filter on X. Now choose A to be any arbitrary intuitionistic subset of X, which intersects every member of $\mathcal{I}_{\mathcal{F}}$. Consider a collection ${\mathcal{I}_{\mathcal{F}}}^* = \{F^* : F^* \supset A \cap F$ for some $F \in I_F\}$. If F $\in \mathcal{I}_{\mathcal{F}}$, then F $\supset$ A $\cap$ F, so F $\in {\mathcal{I}_{\mathcal{F}}}^*$, which implies $\mathcal{I}_{\mathcal{F}} \subset {\mathcal{I}_{\mathcal{F}}}^*$.

As per the construction, A intersects every member of $\mathcal{I}_{\mathcal{F}}$. Hence A $\cap$ F $\neq \tilde{\phi}$ for all F $\in \mathcal{I}_{\mathcal{F}}$ and $F^* \supset$ A $\cap$ F. Therefore $F^* \neq \tilde{\phi}$ for all $F^* \in {\mathcal{I}_{\mathcal{F}}}^*$. Thus $\tilde{\phi} \notin {\mathcal{I}_{\mathcal{F}}}^*$.

Let $F^* \in {\mathcal{I}_{\mathcal{F}}}^*$. Then $F^* \supset$ A $\cap$ F for some F $\in \mathcal{I}_{\mathcal{F}}$. If G^* is an IS set and $G^* \supset F^*$, then obviously $G^* \supset$ A $\cap$ F for some F $\in \mathcal{I}_{\mathcal{F}}$ implies $G^* \in {\mathcal{I}_{\mathcal{F}}}^*$. That is, all the super set of F^* also is in ${\mathcal{I}_{\mathcal{F}}}^*$. Let ${F_1}^*$ and ${F_2}^*$ be both members of ${\mathcal{I}_{\mathcal{F}}}^*$. Then ${F_1}^* \supset$ A $\cap$ F_1 and ${F_2}^* \supset$ A $\cap$ F_2 for some F_1, $F_2 \in \mathcal{I}_{\mathcal{F}}$. Hence ${F_1}^* \cap {F_2}^* \supset$ (A $\cap$ F_1) $\cap$ (A $\cap$ F_2) = A $\cap$($F_1 \cap F_2$) = A $\cap$ F, where F = $F_1 \cap F_2 \in \mathcal{I}_{\mathcal{F}}$. Thus ${F_1}^* \cap {F_2}^* \supset$ A $\cap$ F for some F $\in {\mathcal{I}_{\mathcal{F}}}^*$ and so ${F_1}^* \cap {F_2}^* \in {\mathcal{I}_{\mathcal{F}}}^*$. Thus ${\mathcal{I}_{\mathcal{F}}}^*$ is an intuitionistic filter on X such that ${\mathcal{I}_{\mathcal{F}}}^*$ contains $\mathcal{I}_{\mathcal{F}}$. But it is given that $\mathcal{I}_{\mathcal{F}}$ is an intuitionistic ultra filter on X. Therefore ${\mathcal{I}_{\mathcal{F}}}^*$ $\supset \mathcal{I}_{\mathcal{F}} \Rightarrow {\mathcal{I}_{\mathcal{F}}}^* = \mathcal{I}_{\mathcal{F}}$. Further $\tilde{X} \in \mathcal{I}_{\mathcal{F}}$ and A $\cap$ $\tilde{X}$ = A and A $\supset$ A $\cap$ $\tilde{X}$ implies A $\in {\mathcal{I}_{\mathcal{F}}}^*$ or A $\in \mathcal{I}_{\mathcal{F}}$. Thus $\mathcal{I}_{\mathcal{F}}$ contains all those intuitionistic subsets of X which intersect every member of $\mathcal{I}_{\mathcal{F}}$. ■

Theorem 3.2. A class $\mathcal{F} = \{\prec X, {A_i}^1, {A_i}^2 \succ : i \in J\}$ of nonempty intuitionistic subsets of X is an intuitionistic ultra filter on X if the following conditions are satisfied (i) $\mathcal{F}$ has

F.I.P. (ii) For some intuitionistic set $\prec$ X, A^1, $A^2 \succ$ of X, either $\prec$ X, A^1, $A^2 \succ \in \mathcal{F}$ (or) $\tilde{X}- \prec$ X, A^1, $A^2 \succ \in \mathcal{F}$.

Proof. Given $\mathcal{F}$ is a nonempty family of intuitionistic subsets of X and $\mathcal{F}$ has F.I.P, then by a Theorem 2.8 in [9], there exists atleast one intuitionistic filter say $\mathcal{I_F}^*$ ={$\prec X, {B_j}^1, {B_j}^2 \succ$: $j \in k$} containing $\mathcal{F}$. To prove that $\mathcal{F}$ is an intuitionistic ultra filter on X, it is enough to prove that there is no other intuitionistic filter on X, which is strictly finer than $\mathcal{F}$.

If possible, let $\mathcal{I_F}^*$ be an intuitionistic filter on X, which is strictly finer than $\mathcal{F}$ and take an intuitionistic sets {$\prec X, {B_j}^1, {B_j}^2 \succ$: $j \in k$} $\in \mathcal{I_F}^*$, so that $\tilde{X}$ - {$\prec X, {B_j}^1, {B_j}^2 \succ$} $\notin \mathcal{I_F}^*$ for some j. Since $\mathcal{I_F}^*$ is strictly finer than $\mathcal{F}$, $\tilde{X}$ - $\prec$ X, ${B_j}^1$, ${B_j}^2 \succ \notin \mathcal{F}$. Then {$\prec X, {B_j}^1, {B_j}^2 \succ$} $\in \mathcal{F}$, which shows that every member of $\mathcal{I_F}^*$ is a member of $\mathcal{F}$. But this is a contradiction to our assumption that $\mathcal{I_F}^*$ is an intuitionistic filter on X strictly finer than $\mathcal{F}$. Therefore $\mathcal{F}$ is an intuitionistic ultra filter on X. ■

Theorem 3.3. An intuitionistic filter $\mathcal{I_F}$ on a set X is an intuitionistic ultra filter if and only if for any two intuitionistic subsets A = $\prec$ X, A^1, $A^2 \succ$ and B = $\prec$ X, B^1, $B^2 \succ$ of X such that $\prec$ X, A^1, $A^2 \succ \cup \prec$ X, B^1, $B^2 \succ \in \mathcal{I_F}$, either $\prec$ X, A^1, $A^2 \succ \in \mathcal{I_F}$ or $\prec$ X, B^1, $B^2 \succ \in \mathcal{I_F}$.

Proof. Suppose that $\mathcal{I_F}$ is an intuitionistic ultra filter on X and let $\prec$ X, A^1, $A^2 \succ \cup \prec$ X, B^1, $B^2 \succ \in \mathcal{I_F}$. If possible, let $\prec$ X, A^1, $A^2 \succ \notin \mathcal{I_F}$ and $\prec$ X, B^1, $B^2 \succ \notin \mathcal{I_F}$. Now both $\prec$ X, A^1, $A^2 \succ$ and $\prec$ X, B^1, $B^2 \succ$ can not be empty for otherwise $\prec$ X, A^1, $A^2 \succ \cup \prec$ X, B^1, $B^2 \succ$ will also be empty and belong to $\mathcal{I_F}$. Assume that B $\neq \prec$ X, ϕ, X $\succ$.

Now, consider the collection $\mathcal{G_{I_F}}$ = {$G = \prec X, {G_1}^1, {G_1}^2 \succ$ and $G \cup A \in \mathcal{I_F}$}. Clearly $\mathcal{G_{I_F}}$ is nonempty because atleast B $\in \mathcal{G_{I_F}}$, since $\prec$ X, A^1, $A^2 \succ \cup \prec$ X, B^1, $B^2 \succ \in \mathcal{I_F}$. As $\prec$ X, ϕ, X $\succ \cup \prec$ X, A^1, $A^2 \succ = \prec$ X, A^1, $A^2 \succ \notin \mathcal{I_F}$ implies $\prec$ X, ϕ, X $\succ \notin \mathcal{G_{I_F}}$. If $\prec$ X, ${G_1}^1$, ${G_1}^2 \succ \in \mathcal{G_{I_F}}$, then G $\cup$ A $\in \mathcal{I_F}$. So if $\prec$ X, H^1, $H^2 \succ \supset \prec$ X, ${G_1}^1$, ${G_1}^2 \succ$, then $\prec$ X, H^1, $H^2 \succ \cup \prec$ X, A^1, $A^2 \succ \supset \prec$ X, ${G_1}^1$, ${G_1}^2 \succ \cup \prec$ X, A^1, $A^2 \succ \in \mathcal{I_F}$, as $\mathcal{I_F}$ is an intuitionistic filter. Hence $\prec$ X, H^1, $H^2 \succ \in \mathcal{G_{I_F}}$.

Let $\prec$ X, ${G_1}^1$, ${G_1}^2 \succ$ and $\prec$ X, ${G_2}^1$, ${G_2}^2 \succ$ be in $\mathcal{G_{I_F}}$. Then $\prec$ X, ${G_1}^1$, ${G_1}^2 \succ \cup \prec$ X, A^1, $A^2 \succ \in \mathcal{I_F}$ and $\prec$ X, ${G_2}^1$, ${G_2}^2 \succ \cup \prec$ X, A^1, $A^2 \succ \in \mathcal{I_F}$. Since $\mathcal{I_F}$ is an intuitionistic filter their intersection also belongs to $\mathcal{I_F}$. So ($\prec$ X, ${G_1}^1$, ${G_1}^2 \succ \cup \prec$ X, A^1, $A^2 \succ$) $\cap$ ($\prec$ X, ${G_2}^1$, ${G_2}^2 \succ \cup \prec$ X, A^1, $A^2 \succ$) $\in \mathcal{I_F}$. That is ($\prec$ X, ${G_1}^1$, ${G_1}^2 \succ \cap \prec$ X, ${G_2}^1$, ${G_2}^2 \succ$) $\cup \prec$ X, A^1, $A^2 \succ \in \mathcal{I_F}$. Hence, $\prec$ X, ${G_1}^1$, ${G_1}^2 \succ \cap \prec$ X, ${G_2}^1$, ${G_2}^2 \succ \in \mathcal{G_{I_F}}$. Thus, $\mathcal{G_{I_F}}$ is an intuitionistic filter on X.

If F = $\prec$ X, F^1, $F^2 \succ \in \mathcal{I_F}$ then $\prec$ X, F^1, $F^2 \succ \cup \prec$ X, A^1, $A^2 \succ$ being a super set of $\prec$ X, F^1, $F^2 \succ$ is also in $\mathcal{I_F}$. By the definition of $\mathcal{G_{I_F}}$, F $\in \mathcal{G_{I_F}}$. Therefore $\mathcal{I_F} \subseteq \mathcal{G_{I_F}}$, which implies that $\mathcal{G_{I_F}}$ is an intuitionistic filter finer than $\mathcal{I_F}$. But this is a contradiction.So either $\prec$ X, A^1, $A^2 \succ \in \mathcal{I_F}$ or $\prec$ X, B^1, $B^2 \succ \in \mathcal{I_F}$.

Conversely, assume for any two intuitionistic subsets A = $\prec$ X, A^1, $A^2 \succ$ and B = $\prec$ X, B^1, $B^2 \succ$ of X such that $\prec$ X, A^1, $A^2 \succ \cup \prec$ X, B^1, $B^2 \succ \in \mathcal{I_F}$ then either $\prec$ X, A^1, $A^2 \succ \in \mathcal{I_F}$ or $\prec$ X, B^1, $B^2 \succ \in \mathcal{I_F}$.

Let A be any intuitionistic subset of X. As $\mathcal{I_F}$ is an intuitionistic filter, $\prec$ X, X,$\phi \succ$

$\in \mathcal{I_F}$ or $\prec$ X, X, $\phi \succ = \prec$ X, A^1, $A^2 \succ \cup (\prec$ X, X, $\phi \succ$ - $\prec$ X, A^1, $A^2 \succ) \in \mathcal{I_F}$ by assumption, either $\prec$ X, A^1, $A^2 \succ \in \mathcal{I_F}$ or $\tilde{X} - \prec$ X, A^1, $A^2 \succ \in \mathcal{I_F}$. Hence by Theorem 3.2, $\mathcal{I_F}$ is an intuitionistic ultra filter on X. ■

Theorem 3.4. Every intuitionistic filter $\mathcal{I_F}$ on a set X is the intersection of all the intuitionistic ultra filters finer than $\mathcal{I_F}$.

Proof. Let $\mathcal{H} = \cap \{\mathcal{G} : \mathcal{G}$ is an intuitionistic ultra filter containing $\mathcal{I_F}\}$. As $\mathcal{I_F}$ is contained in $\mathcal{G}$ for all $\mathcal{G}$. Then $\mathcal{I_F} \subset \mathcal{H}$. Now if A = $\prec$ X, A^1, $A^2 \succ$ be any member of $\mathcal{H}$. Then A $\in \mathcal{G}$ for all $\mathcal{G}$. If possible, let A $\notin \mathcal{I_F}$, then every F $\in \mathcal{I_F}$ intersects $\tilde{X}$ - A. Hence there exists an intuitionistic filter [9] $\mathcal{I_F}^*$ which is finer than $\mathcal{I_F}$ and contains $\tilde{X}$ - A. By Theorem 2.5, every intuitionistic filter on a nonempty set X is contained in an intuitionistic ultra filter on X. That is there exists an intuitionistic ultra filter $\mathcal{G}^*$ finer than $\mathcal{I_F}$. But $\tilde{X}$ - A $\in \mathcal{I_F}^* \Rightarrow$ A $\notin \mathcal{I_F}^*$ and $\mathcal{G}^*$ is finer than $\mathcal{I_F}^*$. Therefore A $\notin \mathcal{G}^*$ which is a contradiction. Hence A $\in \mathcal{I_F}$. Also A $\in \mathcal{H} \Rightarrow$ A $\in \mathcal{I_F}$. Therefore $\mathcal{H} = \mathcal{I_F}$. ■

Theorem 3.5. Let $\mathcal{I_F}$ be an intuitionistic ultra filter on a set X. Then the intersection of all the members of $\mathcal{I_F}$ is either empty or a singleton intuitionistic subset of X.

Proof. Let $\{\prec X, {A_i}^1, {A_i}^2 \succ : \prec X, {A_i}^1, {A_i}^2 \succ \in \mathcal{I_F}\}$ be the collection of all intuitionistic sets in the intuitionistic ultra filter $\mathcal{I_F}$ on X and let $\mathcal{G} = \cap\{\prec X, {A_i}^1, {A_i}^2 \succ : \prec X, {A_i}^1, {A_i}^2 \succ \in \mathcal{I_F}\}$ denote the intersection of intuitionistic members of $\mathcal{I_F}$. If $\mathcal{G} = \tilde{\phi}$, then there is nothing to prove.

If $\mathcal{G} \neq \tilde{\phi}$,then it is to prove that $\mathcal{G}$ has a singleton intuitionistic subset of X. Since $\mathcal{G} \neq \tilde{\phi}$, there is atleast one element $\tilde{p} = \prec$ X, $\{p\}$, $\{p\}^c \succ \in$ X such that $\tilde{p} \in \mathcal{G}$. If possible, let $\tilde{q} \neq \tilde{p}$ be such that $\tilde{q} \in \mathcal{G}$. By Theorem 3.2, either $\tilde{p} \in \mathcal{I_F}$ (or) $\tilde{X} - \tilde{p} \in \mathcal{I_F}$ for some $\tilde{p}$. If $\tilde{p} \in \mathcal{I_F}$, then as $\tilde{q} \neq \tilde{p}$, $\tilde{q} \notin \tilde{p}$ and so $\tilde{q} \notin \mathcal{G}$ which is a contradiction to our assumption that $\tilde{q} \in \mathcal{G}$. Similarly if $\tilde{X} - \tilde{p} \in \mathcal{I_F}$ then $\tilde{p} \notin \tilde{X} - \tilde{p} \in \mathcal{I_F}$ and hence $\tilde{p} \notin \mathcal{G}$ which is again contradiction to our assumption that $\tilde{p} \in \mathcal{G}$. Hence $\tilde{q} = \tilde{p}$. Therefore $\mathcal{G}$ is either empty or a singleton intuitionistic subset of X. ■

4. Convergence of Intuitionistic filters

Definition 4.1. Let $\mathcal{I_F}$ be an intuitionistic filter on a nonempty set X and A be any intuitionistic subset of X. Then $\mathcal{I_F}$ is said to be eventually in the IS set A,where A = $\prec$ X, A^1, $A^2 \succ$ if and only if A $\in \mathcal{I_F}$.

Definition 4.2. Let $\mathcal{I_F}$ be an intuitionistic filter on a nonempty set X and A be any intuitionistic subset of X. Then $\mathcal{I_F}$ is said to be frequently in the IS set A if and only if A intersects every member of $\mathcal{I_F}$ i.e. A $\cap$ F $\neq \tilde{\phi}$ for all F $\in \mathcal{I_F}$.

Remark 4.3. From above definitions it is clear that if $\mathcal{I_F}$ is eventually in A, then $\mathcal{I_F}$ is frequently in A because when $\mathcal{I_F}$ is eventually in A then, A $\in \mathcal{I_F}$ implies A $\cap$ F is the intersection of two members of intuitionistic filter $\mathcal{I_F}$ is again in $\mathcal{I_F}$.Since $\tilde{\phi} \notin \mathcal{I_F}$,it follows that A $\cap$ F $\neq \tilde{\phi}$ for all F $\in \mathcal{I_F}$.i.e A intersects every member of $\mathcal{I_F}$. Hence $\mathcal{I_F}$

is frequently in A .$\mathcal{I_F}$ is eventually in A implies $\mathcal{I_F}$ is frequently in A.

The converse of above is not true as will be clear from example given below.

$X = \{a, b, c\}, \mathcal{I_F} = \{\prec X, X, \phi \succ, \prec X, \{a, b\}, \phi \succ\}$.

Let $A = \prec X, \{a\}, \phi \succ$.

$\mathcal{I_F}$ is an intuitionistic filter on X and A intersects every member of $\mathcal{I_F}$, so $\mathcal{I_F}$ is frequently in A. But as A $\notin \mathcal{I_F}$, $\mathcal{I_F}$ is not eventually in A.

Remark 4.4. If all the intuitionistic sets A for which intuitionistic filter is frequently in A is also eventually in A, then the intuitionistic filter is an intuitionistic ultra filter. This is sustained by Example 2.3, in which the intuitionistic ultra filter $\mathcal{I_{UF}}$ is both frequently and eventually in every subset of $\mathcal{I_{UF}}$.

Definition 4.5. Let (X, τ) be an intuitionistic topological space and $\mathcal{N}_{\tilde{p}}$ be the collection of all τ-intuitionistic neighbourhoods of an intuitionistic point $\tilde{p} = \prec$ X, $\{p\}$, $\{p\}^c \succ \in$ X.

Theorem 4.6. Let (X, τ) be an intuitionistic topological space and $\tilde{p}$ be an intuitionistic point in X. Then the τ-intuitionistic neighbourhood of $\tilde{p}$ say $\mathcal{N}_{\tilde{p}\tau}$ is an intuitionistic filter on X.

Proof. Let $\tilde{p}$ be an intuitionistic point in X, $\prec$ X, X, $\phi \succ$ is an intuitionistic neighbourhood of $\tilde{p}$ and as such belongs to $\mathcal{N}_{\tilde{p}\tau}$ and so $\mathcal{N}_{\tilde{p}\tau}$ is nonempty. By Definition 4.5, each member of $\mathcal{N}_{\tilde{p}\tau}$ being an intuitionistic neighbourhood of $\tilde{p}$ must contain $\tilde{p}$ and as such no member of $\mathcal{N}_{\tilde{p}\tau}$ is empty and so $\prec$ X, ϕ, X $\succ \notin \mathcal{N}_{\tilde{p}\tau}$. If A is an intuitionistic neighbourhood of $\tilde{p}$, then a superset of A is also an intuitionistic neighbourhood of $\tilde{p}$. Hence A $\in \mathcal{N}_{\tilde{p}\tau}$ and B $\supset$ A then B $\in \mathcal{N}_{\tilde{p}\tau}$. Also it is known that if A and B are intuitionistic neighbourhoods of $\tilde{p}$, then A $\cap$ B is also a intuitionistic neighbourhood of $\tilde{p}$ and as belongs $\mathcal{N}_{\tilde{p}\tau}$. Therefore $\mathcal{N}_{\tilde{p}\tau}$ is an intuitionistic filter on X. ■

Remark 4.7. Let (X, τ) be an intuitionistic topological space and let $\tilde{p}$ be an intuitionistic point in X. Then the τ-intutionsitic neighbourhood of the IS point $\tilde{p}$, $\mathcal{N}_{\tilde{p}}$ is an intuitionistic filter in X and it is denoted by $\mathcal{N}_{\tilde{p}\tau}$. Here after $\mathcal{N}_{\tilde{p}\tau}$ is called the neighbourhood intuitionistic filter $\tilde{p}$ with respect to τ.

Example 4.8. Let $X = \{a, b, c\}$ and $\tau = \{\prec X, \phi, X \succ, \prec X, \{a\}, \phi \succ, \prec X, \{a, b\}, \phi \succ, \prec X, \{b, c\}, \phi \succ, \prec X, \{b\}, \phi \succ, \prec X, \phi, \phi \succ \prec X, X, \phi \succ\}$ be the given intuitionistic topology on X.

Let $\tilde{a} = \prec X, \{a\}, \{b, c\} \succ$ and $\tilde{b} = \prec X, \{b\}, \{a, c\} \succ$ be two intuitionistic points in X. Then $\mathcal{N}_{\tilde{a}\tau} = \{\prec X, \{a\}, \phi \succ, \prec X, \{a, b\}, \phi \succ, \prec X, \{a, c\}, \phi \succ, \prec X, X, \phi \succ\}$ and $\mathcal{N}_{\tilde{b}\tau} = \{\prec X, \{b\}, \phi \succ, \prec X, \{a, b\}, \phi \succ, \prec X, \{b, c\}, \phi \succ, \prec X, X, \phi \succ\}$ are intuitionistic filters on X.

Definition 4.9. Let (X, τ) be an intuitionistic topological space and let $\mathcal{I_F}$ be an intuitionistic filter on X. Then $\mathcal{I_F}$ is said to τ-intuitionistic converge to an intuitionistic point $\tilde{p} = \prec$ X, $\{p\}$, $\{p\}^c \succ \in X$ if and only if $\mathcal{I_F}$ is eventually in every τ-intuitionistic neighbourhood of $\tilde{p}$. In that case, we write $\mathcal{I}_{\mathcal{F}} \to \tilde{p}$ and $\tilde{p}$ is an intuitionistic limit point of $\mathcal{I}_{\mathcal{F}}$. Therefore $\mathcal{I}_{\mathcal{F}} \to \tilde{p} \Leftrightarrow \mathcal{I}_{\mathcal{F}}$ is eventually in every τ-intuitionistic neighbourhood of $\tilde{p}$. The set of all intuitionistic limit points of an intuitionistic filter $\mathcal{I}_{\mathcal{F}}$ denoted by Lim($\mathcal{I}_{\mathcal{F}}$).

Example 4.10. Let $X = \{a, b, c\}$ and let $\tau = \{\prec X, \phi, X \succ, \prec X, \{a\}, \phi \succ, \prec X, X, \phi \succ, \prec X, \{a, b\}, \phi \succ\}$ be the given intuitionistic topology on X.

$$\mathcal{I}_{\mathcal{F}1} = \{\prec X, \{a\}, \phi \succ, \prec X, \{a, b\}, \phi \succ, \prec X, \{a, c\}, \phi \succ, \prec X, X, \phi \succ\},$$

$$\mathcal{I}_{\mathcal{F}2} = \{\prec X, \{b\}, \phi \succ, \prec X, \{a, b\}, \phi \succ, \prec X, \{b, c\}, \phi \succ, \prec X, X, \phi \succ\}$$

and

$$\mathcal{I}_{\mathcal{F}3} = \{\prec X, \{c\}, \phi \succ, \prec X, \{a, c\}, \phi \succ, \prec X, \{b, c\}, \phi \succ, \prec X, X, \phi \succ\}$$

are any three intuitionistic filters on X.

Now take $\tilde{a} = \prec$ X, $\{a\}, \{b, c\} \succ$ and $\tilde{b} = \prec$ X, $\{b\}, \{a, c\} \succ$ and $\tilde{c} = \prec$ X, $\{c\}, \{a, b\}$ $\succ$ be the three intuitionistic points on X. Then

$$\mathcal{N}_{\tilde{a}\tau} = \{\prec X, \{a\}, \phi \succ, \prec X, \{a, b\}, \phi \succ, \prec X, \{a, c\}, \phi \succ, \prec X, X, \phi \succ\}$$

and

$$\mathcal{N}_{\tilde{b}\tau} = \{\prec X, \{a, b\}, \phi \succ, \prec X, X, \phi \succ\}$$

and

$$\mathcal{N}_{\tilde{c}\tau} = \{\prec X, X, \phi \succ\}$$

are the neighbourhood intuitionistic filters. $\mathcal{I}_{\mathcal{F}1}$ is eventually in every τ-intuitionistic neighbourhood of intuitionistic point $\tilde{a}$. That is every τ-intuitionistic neighbourhood of intuitionistic point $\tilde{a}$ is contained in $\mathcal{I}_{\mathcal{F}1}$ so that $\mathcal{I}_{\mathcal{F}1} \to \tilde{a}$. Similarly $\mathcal{I}_{\mathcal{F}1} \to \tilde{b}$, $\mathcal{I}_{\mathcal{F}1} \to \tilde{c}$ as $\mathcal{I}_{\mathcal{F}1}$ contains every τ-intuitionistic neighbourhood of $\tilde{b}$ and $\tilde{c}$. Hence Lim($\mathcal{I}_{\mathcal{F}1}$) = $\{\tilde{a}, \tilde{b}, \tilde{c}\}$. Similarly Lim($\mathcal{I}_{\mathcal{F}2}$) = $\{\tilde{b}, \tilde{c}\}$ and Lim($\mathcal{I}_{\mathcal{F}3}$) = $\tilde{c}$.

5. Properties of convergence of intuitionistic filter

In this section, we study some basic properties of the convergence of intuitionistic filter space.

Theorem 5.1. Let X be a given nonempty set and τ be an intuitionistic topological space on X Also let $\mathcal{N}_{\tilde{p}\tau}$ be the τ-intuitionistic neighbourhood of the intuitionistic point $\tilde{p}$. Then

(a) Every τ-nbd intuitionistic filter $\mathcal{N}_{\tilde{p}\tau}$ converges to a unique limit.

(b) If τ is an indiscrete intuitionistic topological space, then every intuitionistic filter on X,converges to every intuitionistic point of X.

(c) If $\mathcal{I_F} \to \tilde{p}$, then $\mathcal{I_F}^* \to \tilde{p}$ where $\mathcal{I_F}^*$ is finer than $\mathcal{I_F}$.

(d) If $\mathcal{I_F} \to \tilde{p}$ w.r.t τ, then $\mathcal{I_F} \to \tilde{p}$ w.r.t τ^* where τ^* is an intuitionistic topology on X, which is coarser than τ.

Proof.

(a) Let $\mathcal{N}_{\tilde{p}\tau}$ be a collection of all τ-intuitionistic neighbourhoods of $\tilde{p}$ in an intuitionistic topological space (X, τ). By Theorem 4.6, $\mathcal{N}_{\tilde{p}\tau}$ is an intuitionistic filter on X and $\mathcal{N}_{\tilde{p}\tau}$ is eventually in every τ-intuitionistic neighbourhood of intuitionistic point $\tilde{p}$ or every τ-intuitionistic neighbourhood of $\tilde{p}$ is contained in $\mathcal{N}_{\tilde{p}\tau}$ and hence $\mathcal{N}_{\tilde{p}\tau} \to \tilde{p}$. Further this $\tilde{p}$ is unique because if $\tilde{q}$ is any other intuitionistic point distinct from intuitionistic point $\tilde{p}$, then $\prec$ X, $\{q\}$, $\phi \succ$ is a intuitionistic neighbourhood of $\tilde{q}$ but it is not belong to $\mathcal{N}_{\tilde{p}\tau}$.

(b) Let $\mathcal{I_F}$ be a intuitionistic filter on X and $\tilde{p} = \prec$ X, $\{p\}$, $\{p\}^c \succ$ be any arbitrary intuitionistic point of X. Then only τ-neighbourhood intuitionistic filter of $\tilde{p}$ is $\{\prec X, X, \phi \succ\}$ in an indiscrete intuitionistic topological space and $\prec$ X, X, $\phi \succ \in \mathcal{I_F}$ so that $\mathcal{I_F} \to \tilde{p}$. Since $\tilde{p}$ was chosen arbitrarily, every intuitionistic filter on X converges to every intuitionistic point of X.

(c) $\mathcal{I_F} \to \tilde{p}$ if and only if every τ-intuitionistic neighbourhood of $\tilde{p}$ is contained in $\mathcal{I_F}$ As $\mathcal{I_F}^*$ is finer than $\mathcal{I_F}$, $\mathcal{I_F}^*$ is eventually in every τ-intuitionistic neighbourhood of $\tilde{p}$. Hence $\mathcal{I_F}^* \to \tilde{p}$.

(d) $\mathcal{I_F} \to \tilde{p}$ w.r.t τ if and only if every τ-intuitionistic neighbourhood of $\tilde{p}$ is contained in $\mathcal{I_F}$. As τ^* is coarser than τ. $\mathcal{I_F}$ is eventually in every τ^*-intuitionistic neighbourhood of $\tilde{p}$. Then $\mathcal{I_F} \to \tilde{p}$ w.r.t τ^*. ■

Theorem 5.2. An intuitionistic filter $\mathcal{I_F}$ on an intuitionistic topological space (X, τ) converges to an intuitionistic point $\tilde{p} \in$ X if and only if every intuitionistic ultra filter on X containing $\mathcal{I_F}$ converges to $\tilde{p}$.

Proof. Let $\mathcal{I_F} \to \tilde{p}$. Then $\mathcal{I_{F*}}$ be an intuitionistic ultra filter containing $\mathcal{I_F}$. That is $\mathcal{I_{F*}}$ is finer than $\mathcal{I_F}$. So that $\mathcal{I_{F*}} \to \tilde{p}$ by Theorem 5.1.

Conversely, let every intuitionistic ultra filter on X containing $\mathcal{I_F}$ converges to $\tilde{p} \in$ X. Therefore every τ intuitionistic neighbourhood of $\tilde{p}$ is contained in every intuitionistic ultra filter on X, which contains $\mathcal{I_F}$. By Theorem 3.4, every τ intuitionistic neighbourhood of $\tilde{p}$ is contained in the intersection of all the intuitionistic ultra filter on X which contains $\mathcal{I_F}$. Thus every τ intuitionistic neighbourhood of $\tilde{p}$ is contained in $\mathcal{I_F}$. Hence $\mathcal{I_F} \to \tilde{p}$. ■

Theorem 5.3. In an intuitionistic topological space (X, τ) a nonempty intuitionistic subset $G = \prec$ X, G^1, $G^2 \succ$ of X is τ-intuitionistic open if and only if G is contained in every intuitionistic filter which converges to an intuitionistic point of G.

Proof. Let G be a τ-intuitionistic open set and $\mathcal{I}_{\mathcal{F}}$ be an arbitrary intuitionistic filter on X, which converges to $\tilde{p} = \prec X, \{p\}, \{p\}^c \succ \in G$. Let $\mathcal{I}_{\mathcal{F}} \to \tilde{p} \in G$. Then every τ-intuitionistic neighbourhood of $\tilde{p}$ is contained in $\mathcal{I}_{\mathcal{F}}$. Ultimately τ-intuitionistic open set G is contained in $\mathcal{I}_{\mathcal{F}}$. Since $\mathcal{I}_{\mathcal{F}}$ is an arbitrary intuitionistic filter which converges to $\tilde{p} \in G$, and G is contained in every intuitionistic filter $\mathcal{I}_{\mathcal{F}}$ on X which converges to an intuitionistic point of G.

Conversely, let G be contained in every intuitionistic filter which converges to an intuitionistic point of G. Choose $\tilde{p} = \prec X, \{p\}, \{p\}^c \succ$ to be any arbitrary intuitionistic point of G. So that $\mathcal{N}_{\tilde{p}\tau}$ is the neighbourhood intuitionistic filter of $\tilde{p}$ which converges to $\tilde{p}$ and by the given condition $G \subset \mathcal{N}_{\tilde{p}\tau}$. In other words G is an intuitionistic neighbourhood of $\tilde{p}$ and $\tilde{p}$ is an arbitrary intuitionistic point of G, we have G is a τ intuitionistic neighbourhood of each of its intuitionistic points. Hence G is τ-intuitionistic open. ■

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