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pg**- compact spaces

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Abstract

The concepts of pg**-compact, pg**-countably compact, sequentially pg**-compact, pg**-locally compact and pg**- paracompact are introduced, and several properties are investigated. Also the concept of pg**-compact modulo *I* and pg**-countably compact modulo *I* spaces are introduced and the relation between these concepts are discussed.

Keywords: pg**-compact, pg**-countably compact, sequentially pg**-compact, pg**-locally compact, pg**- paracompact, pg**-compact modulo *I*, pg**-countably compact modulo *I*.

1. Introduction

Levine [3] introduced the class of g-closed sets in 1970. Veerakumar [7] introduced g*-closed sets. P M Helen[5] introduced g**-closed sets. A.S.Mashhour, M.E Abd El. Monsef [4] introduced a new class of pre-open sets in 1982. Ideal topological spaces have been first introduced by K.Kuratowski [2] in 1930. In this paper we introduce pg**-compact, pg**-countably compact, sequentially pg**-compact, pg**-locally compact, pg**-paracompact, pg**-compact modulo *I* and pg**-countably compact modulo *I* spaces and investigate their properties.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) represent non-empty topological spaces of which no separation axioms are assumed unless otherwise stated.

Definition 2.1

A subset A of a topological space (X, τ) is called a pre-open set [4] if $A \subseteq int(cl(A)$ and a pre-closed set if $cl(int(A)) \subseteq A$.

Definition 2.2 A subset A of topological space (X, τ) is called

1. generalized closed set (g-closed) [3] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
2. g*-closed set [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ).
3. g**-closed set [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g*-open in (X, τ).
4. pg**- closed set[6] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is g*-open in (X, τ).

Definition 2.3 A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called

1. pg**-irresolute[6] if $f^{-1}(V)$ is a pg**-closed set of (X, τ) for every pg**-closed set V of (Y, σ).
2. pg**-continuous[6] if $f^{-1}(V)$ is a pg**-closed set of $(X, \tau)$for every closed set V of (Y, σ).
3. pg**-resolute[6] if $f(U)$ is pg**- open in Y whenever U is pg**- open in X.

Definition 2.4 An ideal[2] I on a nonempty set X is a collection of subsets of X which satisfies the following properties. (i) $A \in I$, $B \in I \Longrightarrow A \cup B \in I$ (ii) $A \in I, B \subset A \Longrightarrow B \in I$. A topological space (X, τ) with an ideal I on X is called an ideal topological space and is denoted by (X, τ, I).

Definition 2.5 [1] A collection $\mathcal{C}$ of subsets of X is said to have finite intersection property if for every finite subcolection $\{C_1, C_2, \dots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap C_2 \cap \dots \cap C_n$ is nonempty.

3. pg**- Compact Space

We introduce the following definitions

Definition 3.1 Let X be a topological space. A collection$\{U_\alpha\}_{\alpha\in\Delta}$ of pg**-open subsets of X is said to be *pg**-open cover* of X if each point in X belongs to at least one U_α that is, if $\bigcup_{\alpha\in\Delta} U_\alpha = X$.

Definition 3.2 The topological space(X,τ) is said to be *pg**- compact* if every pg**-open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X. A subset A of X is said to be pg**-compact if every pg**-open covering of A contains a finite subcollection that also covers$A$.

Remark 3.3

- A pg**- compact space is compact since every open set is pg**- open but not conversely.
- Any topological space having only finitely many points is necessarily pg**-compact, since in this case every pg**-open covering of X is finite.

Example 3.4 Let X be an infinite set with cofinite topology. Then,

$$PG^{**}O(X) = \{\varphi, X, A\ /A^c is\ finite\}.$$

Let $\{U_\alpha\}_{\alpha\in\Delta}$ be an arbitrary pg**-open cover for X. Let U_{α_0} be a pg**- open set in the pg**- open cover $\{U_\alpha\}_{\alpha\in\Delta}$. Then $X - U_{\alpha_0}$ is finite say $\{x_1, x_2, x_3, \dots, x_n\}$. Choose U_{α_i} such that $x_i \in U_{\alpha_i}$ for $i = 1,2,3,\dots,n$. Then $X = U_{\alpha_0} \cup U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_n}$. Hence the space X is pg**- compact and hence compact.

Example 3.5 The real line $\mathbb{R}$ with usual topology is not pg**- compact since the pg**- open cover $\mathfrak{G} = \{(n, n+2)/n \in \mathbb{Z}\}$ has no finite subcover.

Example 3.6 Let X be an infinite indiscrete topological space. Obviously it is compact. But $\{\{x\}_{x\in X}\}$ is a pg**- open cover which has no finite sub cover. Hence it is not pg**- compact.

Definition 3.7 A topological space (X,τ) is said to be *pg**-discrete* if every subset of X is pg**-open. Equivalently every subset is pg**-closed.

Theorem 3.8 A pg**-discrete space is pg**- compact if and only if the space is finite.

Proof Let (X, τ)be pg**- discrete space. Suppose X is a finite set, then obviously it is pg**- compact. Hence in particular every pg**- discrete finite space is pg**- compact. Conversely suppose X is pg**- compact and assume that X is an infinite set. Then the collection $\mathcal{A} = \{\{x\}: x \in X\}$ is a pg**- open cover of X. But $\mathcal{A}$ does not contain any finite subcover for X. Therefore X is not pg**- compact, contrary to our supposition. Thus X is a finite set.

Theorem 3.9 Every pg**- closed subset of a pg**- compact space is pg**- compact but not conversely.

Proof Let A be a pg**- closed subset of the pg**- compact space X. Given a pg**- open covering $\{U_\alpha\}_{\alpha\in\Delta}$ of A. Let us form an pg**- open covering of X by adjoining to $\{U_\alpha\}_{\alpha\in\Delta}$ the single pg**- open set $X - A$, that is $\{ \{U_\alpha\}_{\alpha\in\Delta} \cup (X - A) \}$. Some finite subcollection of $\{\{U_\alpha\}_{\alpha\in\Delta} \cup (X - A) \}$ covers X. If this subcollection contains the set $X - A$, discard $X - A$, otherwise leave the subcollection alone. The resulting collection is a finite subcollection of $\{U_\alpha\}_{\alpha\in\Delta}$ that covers A.

Example 3.10 Let $X = \{a, b, c\}, \tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$. Here $\mathrm{PG}^{**}O(X) = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$. X is pg**- compact. $Y = \{a, b\}$ is pg**- compact but not pg**- closed.

Theorem 3.11 Let X and Y be topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function. Then,

1. f is pg**- irresolute and A is a pg**- compact subset of $X \Longrightarrow f(A)$ is a pg**- compact subset of Y.
2. f is one-one pg**-resolute map and B is a pg**- compact subset of Y $\Longrightarrow f^{-1}(B)$ is a pg**-compact subset of X.

Proof (1) & (2) obvious from the definitions.

The following theorems give several equivalent forms of pg**-compactness which are often easier to apply.

Theorem 3.12 If $A_1, A_2, A_3, \ldots, A_n$ are pg**-compact subsets of a pg**- multiplicative space X then $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$ is also pg**-compact.

Proof Let $A = \overset{n}{\underset{i=1}{\cup}} A_i$. Suppose $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Delta}$ is an pg**-open cover for A. Then $\mathcal{A}$ is a pg**-open cover for A_1, A_2, A_3, ... , A_n separately also. Since each A_i is pg**-compact there exists finite subcovers $\mathcal{A}_i$'s of each A_i, then $\overset{n}{\underset{i=1}{\cup}} \mathcal{A}_i$ forms a finite subcover of $\overset{n}{\underset{i=1}{\cup}} A_i = A$. Therefore A is pg**-compact.

Theorem 3.13 A topological space is pg**-compact if and only if every collection of pg**-closed sets with empty intersection has a finite sub collection with empty intersection.

Proof Follows from the fact that a collection of pg**-open sets is a pg**-open cover if and only if the collection of all their complements has empty intersection.

Theorem 3.14 A topological space X is pg**-compact ⇔ for every collection $\mathcal{C}$ of pg**-closed sets in X having the finite intersection property, the intersection $\underset{c\in\mathcal{C}}{\cap} C$ of all the elements of $\mathcal{C}$ is nonempty.

Proof Let (X,τ)be pg**-compact and $\mathcal{C}$ be a collection of pg**-closed sets with finite intersection property. Suppose $\underset{c\in\mathcal{C}}{\cap} C = \varphi$ then $\{X - C\}_{C\in\mathcal{C}}$ is a pg**-open cover for X. Therefore there exists C_1 , C_2 , C_3 , ..., $C_n \in \mathcal{C}$ such that $\overset{n}{\underset{i=1}{\cup}} (X - C_i) = X$. This implies $\overset{n}{\underset{i=1}{\cap}} C_i = \varphi$ which is a contradiction. $\therefore \underset{c\in\mathcal{C}}{\cap} C \neq \varphi$. Conversely, assume the hypothesis given in the statement. To prove X is pg**-compact. Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**-open cover for X. Then $\underset{\alpha\in\Delta}{\cup} U_\alpha = X \Longrightarrow \underset{\alpha\in\Delta}{\cap} (X - U_\alpha) = \varphi$. By the hypothesis α_1, α_2, α_3, ... , α_n such that $\overset{n}{\underset{i=1}{\cap}} (X - U_{\alpha_i}) = \varphi$. Therefore $\overset{n}{\underset{i=1}{\cup}} U_{\alpha_i} = X$ and hence X is pg**-compact.

Corollary 3.15 Let (X,τ)be pg**-compact space and let $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$ be a nested sequence of non-empty pg**-closed sets in X. Then the intersection $\underset{n\in Z_+}{\cap} C_n$is nonempty.

Proof Obviously $\{C_n\}_{n\in Z_+}$has finite intersection property. Therefore by the previous theorem $\underset{n\in Z_+}{\cap} C_n$is nonempty.

Theorem 3.16 Let (X, τ) and (Y, σ) be two topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function. Then,

1. f is onto, pg**- continuous and X is pg**- compact $\Longrightarrow Y$ is compact.
2. f is onto, continuous and X is pg**- compact $\Longrightarrow Y$ is compact.
3. f is strongly pg**- irresolute and X is compact $\Longrightarrow Y$ is pg**- compact.
4. f is bijection and pg**- resolute then Y is pg**- compact $\Longrightarrow X$ is compact.
5. f is a bijection and pg**- open then Y is pg**- compact $\Longrightarrow X$ is compact.
6. f is onto, pg**- irresolute and X is pg**- compact $\Longrightarrow Y$ is pg**- compact.
7. f is a bijection and pg**- resolute then Y is pg**- compact $\Longrightarrow X$ is pg**- compact.

Proof (1) Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**-open cover for Y. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\Delta}$ is a pg**-open cover for X. Since X is pg**-compact, there exists $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ such that $X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Then $Y = f(X) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Therefore Y is compact.
Proofs for (2) to (7) are similar to the above.

Remark 3.17 The property of being pg**- compact, is a pg**- topological property. This follows from (6) and (7) of the above theorem.

Definition 3.18 A sequence $\langle x_n\rangle$ in topological space(X, τ) is *pg**- congregates* to x in X ($\langle x_n\rangle \xrightarrow{pg^{**}} x$) is for every pg**-neighbourhood U of x there exists a positive integer N such that $x_n \in U, \ \forall\, n \geq N$. We say that x is the *pg**-limit* of the sequence $\langle x_n\rangle$.

Result 3.19 In a topological space (X, τ), if a sequence $\langle x_n\rangle$ *pg**- congregates* to x_0 then the set $A = \{x_0, x_1, x_2, \dots\}$ is pg**- compact in X.

Proof Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**-open cover $\mathcal{A}$ for A, then $x_0 \in U_{\alpha_0}$for some $\alpha_0 \in \Delta$. Since $\langle x_n\rangle \xrightarrow{pg^{**}} x$, there exists a positive integer N such that $x_n \in U_{\alpha_0}, \ \forall\, n \geq N$. Therefore the finite set $\{x_0, x_1, \dots, x_{N-1}\}$ can be covered by atmost a finite number of pg**-open sets $U_1, U_{2,} \dots, U_{N-1}$. Therefore the pg**-open cover $\mathcal{A}$ contains a finite subcover $\mathcal{C} = \{U_{\alpha_0}, U_1, U_{2,} \dots, U_{N-1}\}$ of A. Hence A pg**- compact in X.

Theorem 3.20 Let (X, τ) be a pg**- compact pg **- multiplicative space then every infinite subset of X has a pg**-cluster point.

Proof Let (X, τ) be a pg**- compact pg**- multiplicative space and A be a subset of X. We desire to prove that, if A is infinite then A has a pg**-cluster point. We prove it contrapositively, that is 'if A has no pg**-cluster point, then A must be finite'. Suppose A has no pg**-cluster point, then A is pg**-closed. Furthermore, for each $a \in A$ we can choose a pg**-neighbourhood U_a of a such that $U_a \cap A = \{a\}$. Then the space X can be covered by $\{(X - A), \{U_a\}_{a \in A}\}$, being pg**- compact, it can be covered by finitely many of these sets. Since $(X - A)$ does not intersect A and each U_a contains only one point of A, the set A must be finite.

Theorem 3.21 (Generalization of extreme value theorem): Let $f : X \to Y$ be pg**- continuous, where Y is an ordered set in the order topology. If X is pg**-compact, then there exist points c and d in X such that $f(c) \leq f(x) \leq f(d)\ \forall\ x \in X$.

Proof Since f is pg**- continuous and X is pg**- compact, The set $A = f(X)$ is compact. Suppose A has no largest element then the collection $\{(-\infty, a)/a \in A\}$ forms an open covering of A. Since A is compact it has some finite subcover $\{(-\infty, a_1), (-\infty, a_2), \ldots, (-\infty, a_n)\}$. Let a = max $\{a_1, a_2, a_3, \ldots, a_n\}$, then a belong to none of these sets, contrary to the fact that they cover A. $\therefore A$ has a largest element M. Similarly it can be proved that it has the smallest element m. Therefore there exists c and d in X such that $f(c) = m,\ f(d) = M$ and $f(c) \leq f(x) \leq f(d)\ \forall\ x \in X$.

Definition 3.22 A family $\mathcal{C}$ of subsets of a topological space X is said to be *pg**-short* if $\mathcal{C}$ is not a pg**-open cover of X. $\mathcal{C}$ is said to be *finitely pg**-short* if no finite subcollection of pg**-open covering $\mathcal{C}$ covers X.

Theorem A topological space (X, τ) is pg**- compact if and only if each finitely pg**-short family of pg**-open sets in X is pg**-short.

Proof Suppose X is pg**- compact. Let $\mathcal{C}$ be any finitely pg**-short family of pg**-open sets in X, then no finite subcollection of $\mathcal{C}$ covers X. We show that $\mathcal{C}$ does not cover X. Suppose $\mathcal{C}$ is a pg**-open cover of X. Then $\mathcal{C}$ is a pg**-open cover of X which has no finite subcover which is a contradiction to X is pg**- compact. Therefore $\mathcal{C}$ is pg**-short. Conversely assume each finitely pg**-short family of pg**-open sets in X is pg**-short. Suppose X is not pg**- compact, then there exist a pg**-open cover $\mathcal{A}$ of X which has no finite subcover. Therefore $\mathcal{A}$ is a finitely pg**-short family of pg**-open sets in X, then by hypothesis, $\mathcal{A}$ is pg**-short that is $\mathcal{A}$ does not cover X, which is a contradiction. Hence X is pg**- compact.

Definition 3.23 A function $f: X \to Y$ between topological spaces is said to be *pg**-proper* if $f^{-1}(C)$ is pg**- compact for each pg**- compact subset C of Y.

Definition 3.24 A function $f: (X, \tau) \to (Y, \sigma)$ is called pg**- resolve if $f(F)$ is pg**- closed in Y whenever F is pg**- closed in X.

Theorem 3.25 If $f: X \to Y$ is a pg**- resolve map between pg**- multiplicative spaces X and Y. Also $f^{-1}(y)$ is pg**- compact for each $y \in Y$, then f is pg**-proper.

Proof Let $C \subset Y$ be pg**- compact and let $\{U_\alpha / \alpha \in A\}$ be a collection of pg**-open sets whose union contains $f^{-1}(C)$. For any $y \in C$ there is a finite subset $A_y \subset A$ such that $f^{-1}(y) \subset \cup\{U_\alpha / \alpha \in A_y\}$. Take $W_y = \cup\{U_\alpha / \alpha \in A_y\}$ and $V_y = Y - f(X - W_y)$, which is pg**-open. Now $f^{-1}(V_y) \subset W_y$ and $y \in V_y$. Since C is pg**- compact and is covered by V_y, there are points $y_1, y_2, \dots, y_n$ such that $C \subset V_{y_1} \cup V_{y_2} \cup \dots \cup V_{y_n}$. Thus $f^{-1}(C) \subset f^{-1}(V_{y_1}) \cup f^{-1}(V_{y_2}) \cup \dots \cup f^{-1}(V_{y_n}) \subset W_{y_1} \cup W_{y_2} \cup \dots \cup W_{y_n}$. This implies $f^{-1}(C) \subset \cup\{U_\alpha / \alpha \in A_{y_i}; i = 1,2, \dots, n\}$ which is finite. Therefore $f^{-1}(C)$ is pg**- compact and hence f is pg**-proper.

4. pg**- compact modulo *I*

Definition 4.1 An ideal topological space (X, τ, I) is said to be *pg**- compact modulo I* if for every pg**-open covering $\{U_\alpha\}_{\alpha\in\Delta}$ of X, there exists a finite subset Δ_0 of Δ such that $X - \underset{\alpha\in\Delta_0}{\cup} U_\alpha \in I$.

Remark 4.2 pg**- compactness implies pg**- compact modulo *I* for any ideal *I* but not conversely.

Example 4.3 Let (X, τ, I) be an infinite indiscrete topological space where $I = \wp(X)$. Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**-open cover for X. Let $\alpha_0 \in \Delta$ then, $X - U_{\alpha_0} \in I$. Therefore (X, τ, I) is pg**- compact modulo I but not pg**- compact.

Note "pg**- compact modulo *I*" and "pg**- compact" happens together when $I = \{\varphi\}$.

Remark 4.4 pg**- compact modulo I implies compact modulo *I* for any ideal *I* but not conversely.
Proof is obvious, since $\tau \subseteq G^{**}O(X)$.

Example 4.5 An indiscrete space $(X, \tau, \{\varphi\})$ is compact modulo I but not pg**-compact modulo I.

Theorem 4.6 Let X be an ideal topological space. Then the following are equivalent.

1) X is pg**- compact modulo I.
2) For every family $\{F_\alpha / \alpha \in \Omega\}$ of pg**-closed sets such that $\underset{\alpha\in\Omega}{\cap} F_\alpha = \varphi$, then there exists a finite sub family $\{F_{\alpha_i}\}_{i=1}^{n}$ such that $\overset{n}{\underset{i=1}{\cap}} F_{\alpha_i} \in I$.
3) For every family $\{F_\alpha / \alpha \in \Omega\}$ of pg**-closed sets with I-FIP $\underset{\alpha\in\Omega}{\cap} F_\alpha \neq \varphi$

Proof

$(1) \Rightarrow (2)$ Let X be pg**- compact modulo I and $\{F_\alpha / \alpha \in \Omega\}$ be a family of pg**-closed sets such that $\underset{\alpha\in\Omega}{\cap} F_\alpha = \varphi$. Therefore $\cup F_\alpha^c = X$ where F_α^c is pg**-open. Hence $\{F_\alpha^c / \alpha \in \Omega\}$ is a pg**-open cover for X, also since X is pg**- compact modulo I there exists $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ such that $X - \overset{n}{\underset{i=1}{\cup}} F_{\alpha_i}^c \in I$. This implies $\overset{n}{\underset{i=1}{\cap}} F_{\alpha_i} \in I$.

$(2) \Rightarrow (3)$ Let $\{F_\alpha / \alpha \in \Omega\}$ be a family of pg**-closed sets with I-FIP. Suppose $\underset{\alpha\in\Omega}{\cap} F_\alpha = \varphi$, then there exists $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ such that $\overset{n}{\underset{i=1}{\cap}} F_{\alpha_i} \in I$. Which contradicts the hypothesis. Therefore $\underset{\alpha\in\Omega}{\cap} F_\alpha \neq \varphi$.

$(3) \Rightarrow (1)$ Let $\{U_\alpha / \alpha \in \Omega\}$ be a pg**-open cover for X. Then $\underset{\alpha\in\Omega}{\cap} U_\alpha^c = \varphi$. Therefore the family of pg**-closed sets $\{U_\alpha^c / \alpha \in \Omega\}$ does not satisfy I-FIP and therefore there exists $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ such that $\overset{n}{\underset{i=1}{\cap}} U_{\alpha_i}^c \in I$ (ie) $X - \overset{n}{\underset{i=1}{\cup}} U_{\alpha_i} \in I$. Therefore X is pg**-compact modulo I.

Theorem 4.7 If $I \subseteq J$, then (X, τ, I) is pg**- compact modulo I implies (X, τ, J) is pg**-compact modulo J.
Proof is obvious.

Theorem 4.8 Let I_F denote the ideal of all finite subsets of X. Then (X, τ) is pg**-compact if and only if (X, τ, I_F) is pg**- compact modulo I_F.

Proof

Necessity follows since $\{\varphi\} \in I_F$.

Sufficiency Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**-open cover for X, then there exists a finite subset Δ_0 of Δ such that $X - \bigcup_{\alpha\in\Delta_0} U_\alpha \in I_F$. $X - \bigcup_{\alpha\in\Delta_0} U_\alpha = \{x_1, x_2, x_3, \ldots, x_n\}$. Choose α_i such that $x_i \in U_{\alpha_i}$ for $i = 1, 2, 3, \ldots, n$. Then $X = \{\bigcup_{\alpha\in\Delta_0} U_\alpha\} \cup \left\{\bigcup_{i=1}^{n} U_{\alpha_i}\right\}$. Therefore X is pg**- compact.

5. pg**- countably compact space

Definition 5.1 A subset A of a topological space(X, τ) is said to be *pg**- countably compact* if every countable pg**-open covering of A has a finite sub cover.

Example 5.2 An infinite. cofinite topological space is pg**- countably compact.

Example 5.3 A countably infinite indiscrete topological space is not pg**- countably compact.

Remark 5.4 Every pg**- compact space is pg**- countably compact.

Theorem 5.5 Every pg**- closed subset of a pg**- countably compact space is pg**- countably compact.
Proof is similar to theorem (3.9)

Theorem 5.6 A topological space X is pg**- countably compact $\Leftrightarrow$ for every collection $\mathcal{C}$ of pg**-closed sets in X having the finite intersection property, the intersection $\bigcap_{c\in\mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.
Proof is similar to the proof of theorem (3.14)

Corollary 5.7 Let (X, τ) be pg**- countably compact space and let $C_1 \supset C_2 \supset C_3 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$ be a nested sequence of non-empty pg**-closed sets in X. Then the intersection $\bigcap_{n\in Z_+} C_n$ is nonempty.

Proof Obviously $\{C_n\}_{n\in Z_+}$has finite intersection property. Therefore by theorem (5.6) $\bigcap_{n\in Z_+} C_n$is nonempty.

Theorem 5.8 A topological space (X, τ) is pg**-countably compact if and only if every infinite subset has a pg**-cluster point.

Proof Suppose if every infinite subset of (X,τ) has a pg**-cluster point and $\mathfrak{F} = F_i$ countable collection of pg**-closed sets with finite intersection property. The intersection $H_n = \bigcap_{j=1}^{n} F_j$ is nonempty for all n. Choose $x_n \in H_n$ for each n. Then the set $E = \{x_n \,/\, x_n \in H_n\}$ has a pg**-cluster point x . But $x_n \in F_i$ for all $n \geq i$, since F_i is pg**-closed $x \in F_i$, $\forall\, i$. Therefore $x \in \cap\, F_i$, hence every countable collection $\mathfrak{F}$ of pg**-closed sets with finite intersection property has a nonempty intersection, then by theorem (5.6) X is pg**- countably compact.

Conversely let (X,τ)be pg**- countably compact and suppose that there exists an infinite subset A has no pg**-cluster point. Let $E = \{x_n \,/ n \in N\}$ be a countable subset of A . Since E has no pg**-cluster point of E , there exists a pg**-neighbourhood U_n of x_n such that $E \cap U_n = \{x_n\}$. Therefore $\{E^c, \{U_n\}_{n\in N}\}$ is a countable pg**-open cover for X. This pg**-open cover has no finite subcover, this is a contradiction for X is pg**- countably compact. Therefore every infinite subset of X has a pg**-cluster point.

Theorem 5.9 Let (X,τ) and (Y,σ) be two topological spaces and $f : (X,\tau) \to (Y,\sigma)$ be a bijective function. Then,

1. f is pg**- continuous and X is pg**- countably compact $\Longrightarrow$ Y is countably compact.
2. f is continuous and X is pg**- countably compact $\Longrightarrow$ Y is countably compact.
3. f is strongly pg**- irresolute and X is countably compact $\Longrightarrow$ Y is pg**- countably compact.
4. f is pg**- resolute and Y is pg**- countably compact $\Longrightarrow$ X is countably compact.
5. f is pg**-open and Y is pg**- countably compact $\Longrightarrow$ X is countably compact.
6. f is pg**- irresolute and X is pg**- countably compact $\Longrightarrow$ Y is pg**- countably compact.
7. f is pg**- resolute and Y is pg**- countably compact $\Longrightarrow$ X is pg**- countably compact.

Proof Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**- countable open cover for Y. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\Delta}$ is a pg**- countable open cover for X. Since X is pg**- countably compact, there exists $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ such that $X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Then $Y = f(X) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Therefore Y is countably compact.

Proofs for (2) to (7) are similar to the above.

Remark 5.10 The property of being pg**- countably compact, is a pg**- topological property. This follows from (6) and (7) of the above theorem.

6. pg**- countably compact modulo *I*

Definition 6.1 An ideal topological space $(X, \tau,\ I\ )$ is said to be *pg**-countably compact modulo I* if for every countable pg**-open covering $\{U_\alpha\}_{\alpha\in\Delta}$ of X, there exists a finite subset Δ_0 of Δ such that $X - \underset{\alpha\in\Delta_0}{\cup} U_\alpha \in I$.

All the results from remark (4.2) to theorem are true in the case when $(X, \tau,\ I\ )$ is pg**-countably compact modulo *I*.

7. Sequentially pg**- compact

Definition 7.1 A subset A of a topological space (X, τ) is said to be *sequentially pg**-compact* if every sequence in A contains a subsequence which pg**- congregates to some point in A.

Example 7.2 A finite topological space is sequentially pg**- compact.
Let $\langle x_n \rangle$ be an arbitrary sequence in a finite topological space X. Since X is finite $x_n = x_0$ except for finitely many n's, therefore the constant sequence $x_0, x_0, \ldots$ is pg**- congregates to x_0 in X. Hence X is sequentially pg**- compact.

Example 7.3 An infinite indiscrete topological space is not sequentially pg**- compact.
Let (X, τ) be an infinite indiscrete topological space and let $x_0 \in X$ be arbitrary. Since all the subsets are pg**-open in this space $\{x_0\}$ is a pg**-neighbourhood of x_0, therefore no sequence other than the constant sequence $\langle x_0 \rangle$ can pg**- congregates to x_0 in X. Since x_0 is arbitrary (X, τ) is not sequentially pg**- compact.

Remark 7.4 Sequentially pg**- compactness implies sequentially compactness, but the reverse implication is not true as seen in the following example.

Example 7.5 Every infinite indiscrete topological space is sequentially compact but not sequentially pg**- compact.

Theorem 7.6 A finite subset A of a topological space (X, τ) is sequentially pg**- compact.

Proof Let $\langle x_n \rangle$ be an arbitrary sequence in a finite subset A. At least one element of the sequence say x_0 must be repeated infinite number times, since A is finite. Hence the constant subsequence $x_0, x_0, \ldots$ is pg**- congregates to x_0 in X.

Theorem 7.7 If a pg**- congregate sequence in a topological space has infinitely many distinct points, then its pg**-limit is a pg**-limit point of the set of points of the sequence.

Proof Let (X, τ) be a topological space and let $\langle x_n \rangle$ be a pg**- congregate sequence in X with pg**-limit x. Assume that x is not a pg**-limit point of the set A of points of the sequence, and show that the sequence has only finitely many distinct points. Since x is not a pg**-limit point of A there exists a pg**-neighbourhood U of x contains no point of the sequence different from x. However, since x is the pg**-limit of the sequence, all x_n's must lie in U, hence must coincide with x. Therefore there are only finitely many distinct points in the sequence.

Theorem 7.8 A topological space (X, τ) is sequentially pg**- compact if and only if every infinite subset has a pg**-limit point.

Proof Assume that X is sequentially pg**- compact and let A be an infinite subset of X. Since A is infinite a sequence $\langle x_n \rangle$ of distinct points can be extracted from A. Since X is sequentially pg**- compact this sequence has a subsequence which pg**- congregates to a point x. Hence from theorem (7.7) x is a pg**-limit point of the set of points of the subsequence, since this set is a subset of A, x is also a pg**-limit point of A. Conversely assume that every infinite subset of X has a pg**-limit point. Let $\langle x_n \rangle$ be an arbitrary sequence in X. If $\langle x_n \rangle$ has a point which is infinitely repeated, then it has a constant subsequence and this subsequence is pg**- congregate. If no point of $\langle x_n \rangle$ is infinitely repeated, then the set A of points of this sequence is infinite, then we can extract the subsequence from $\langle x_n \rangle$ which is pg**- congregate to x since the set A has a pg**-limit point x.

Theorem 7.9 Let (X, τ) and (Y, σ) be two topological spaces, then X and Y are sequentially pg**- compact if and only if X × Y is sequentially pg**- compact.

Proof Suppose that X and Y are sequentially pg**- compact. If (x_n, y_n) is a sequence in X × Y,then there exists a subsequence (x_{n_k}) of (x_n) such that $(x_{n_k}) \xrightarrow{pg^{**}} a$ in X.

Similarly there is a subsequence (y_{n_k}) of (y_n) such that $(y_{n_k}) \xrightarrow{pg^{**}} b$ in Y. Thus $(x_{n_k}, y_{n_k}) \xrightarrow{pg^{**}} (a, b)$. Therefore X × Y is sequentially pg**- compact. To prove the converse consider the projection maps π_x: X × Y → X defined by $\pi_x(x, y) = x$ and π_y: X × Y → Y defined by $\pi_y(x, y) = y$. Suppose X × Y is sequentially pg**- compact. If (x_n) and (y_n) are sequences of X and Y respectively then (x_n, y_n) is a sequence in X × Y. Then there exists a subsequence (x_{n_k}, y_{n_k}) of (x_n, y_n) such that $(x_{n_k}, y_{n_k}) \xrightarrow{pg^{**}} (a, b)$ since X × Y is sequentially pg**- compact. This implies $(x_{n_k}) \xrightarrow{pg^{**}} a$ in X and $(y_{n_k}) \xrightarrow{pg^{**}} b$ in Y. Therefore X and Y are sequentially pg**- compact.

Theorem 7.10 Let (X, τ) and (Y, σ) be two topological spaces and f : (X, τ) → (Y, σ) be a function. Then,

1. f is a bijection and pg**- resolute then Y is sequentially pg**- compact ⟹ X is sequentially pg**- compact.
2. f is onto, pg**- irresolute and X is sequentially pg**- compact ⟹ Y is sequentially pg**-compact.
3. f is onto, strongly pg**- irresolute and X is sequentially compact ⟹ Y is sequentially pg**- compact.
4. f is onto, continuous and X is sequentially pg**- compact ⟹ Y is sequentially compact.

Proof (1) If $\langle x_n \rangle$ is a sequence in X, then $\langle f(x_n) \rangle$ is a sequence in Y. Since Y is sequentially pg**- compact, $\langle f(x_n) \rangle$ has a pg**- congregate subsequence $\langle f(x_{n_k}) \rangle$ such that $f(x_{n_k}) \xrightarrow{pg^{**}} y_0$ in Y. Then there exists $x_0 \in X$ such that $f(x_0) = y_0$. Let U be a pg**-open set containing x_0, then $f(U)$ is a pg**-open set containing$f(x_0)$. Then there exists N such that $f(x_{n_k}) \in f(U)$, $\forall\, k \geq N$. Therefore X is sequentially pg**- compact.

Proofs for (2) to (4) are similar to the above.

Remark 7.11 The property of being sequentially pg**-compact, is a pg**- topological property. This follows from (1) and (2) of the above theorem.

Theorem 7.12 Every sequentially pg**-compact is pg**- countably compact.

Proof Let (X, τ) be sequentially pg**-compact. Suppose X is not pg**- countably compact, then there exists countable pg**-open covering $\{U_\alpha\}_{\alpha\in\Delta}$such that $\underset{\alpha\in\Delta}{\cup} U_\alpha = X$, which has no finite subcover. Choose $x_1 \in U_1, x_2 \in U_2 - U_1, x_3 \in U_3 - \underset{i=1,2}{\cup} U_i \ldots x_n \in U_n - \overset{n-1}{\underset{i=1}{\cup}} U_i$. Now $\{x_n\}$ is a sequence in X. Let $x \in X$ be arbitrary, then $x \in U_j$ for some j. But by our choice of $\{x_n\}, x_i \notin U_j\ \forall\, i > j$. Therefore there exists no subsequence of $\{x_n\}$ which can pg**- congregates to x, which is a contradiction to X is sequentially pg**-compact. Therefore X is pg**- countably compact.

8. pg**- locally compact

Definition 8.1 A subset A of a topological space(X, τ) is said to be *relatively pg**-compact* if $pg^{**}cl(A)$ is pg**-compact in X.

Definition 8.2 A topological space(X, τ) is said to be *pg**- locally compact* if for every point x of X there is some pg**-compact subset C of X that contains a pg**-neighbourhood of x .

Example 8.3 Let X has discrete topology. Then for every $x \in X, \{x\}$ is a pg**-neighbourhood of x and pg**-compact. Therefore every discrete space is pg**-locally compact.

Remark 8.4 A pg**-compact space is pg**- locally compact, but the converse is not true as seen in the following example.

Example 8.5 Let X be an infinite discrete topological space. In this space for every $x \in X, \{x\}$ is a pg**-neighbourhood of x. Also $\{x\}$ is pg**-compact. Therefore X is pg**-locally compact, but it is not pg**-compact.

Theorem 8.6 If $f: (X, \tau) \to (Y, \sigma)$ is pg**-irresolute, pg**-open map from a pg**-locally compact space X onto a topological space Y, then Y is pg**- locally compact.

Proof Let $y \in Y$ be arbitrary. Since f is surjective we can find $x \in X$ such that $f(x) = y$. But X is pg**- locally compact, therefore there exists a pg**-compact subset C of X such that $C \supseteq U$ where U is a pg**-neighbourhood of x. Therefore $x \in U \subseteq C$, this implies $y = f(x) \in f(U) \subseteq f(C)$ since f is a pg**-open map, $f(U)$ is a pg**-

neighbourhood of $f(x)$. Also $f(C)$ is pg**-compact subset of Y since f is pg**-irresolute. Thus Y is pg**- locally compact.

9. pg**- paracompact

Definition 9.1 A collection $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Delta}$ of subsets of a topological space X is said to be *pg**- locally finite* if each point $x \in X$ has a pg**- neighbourhood having nonempty intersection with atmost finitely many members of $\mathcal{A}$.

Example 9.2 Consider $\mathbb{R}$ with usual topology, then the collection of intervals $\mathcal{A} = \{(n, n+2)/\ n \in \mathbb{Z}\}$ is pg**- locally finite.

Theorem 9.3 Let (X, τ) be a topological space and let $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Delta}$ be a pg**- locally finite family of subsets of X. Then $\mathcal{C} = \{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta}$ is also pg**- locally finite.

Proof Choose a point $x \in X$ and a pg**- neighbourhood G of x such that $U_\alpha \cap G = \varphi$ for all except for finitely many $\alpha \in \Delta$, then $U_\alpha \subset X - G$ this implies $pg^{**}cl(U_\alpha) \subset X - G$. Consequently $pg^{**}cl(U_\alpha) \cap G = \varphi$ for all except for finitely many $\alpha \in \Delta$. Therefore $\mathcal{C}$ is pg**- locally finite.

Theorem 9.4 In a topological space, if the collection $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Delta}$ is pg**- locally finite then $\cup\{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta} = pg^{**}cl(\cup\{U_\alpha\}_{\alpha\in\Delta})$.

Proof For every $\alpha \in \Delta$, we have $U_\alpha \subset \cup\{U_\alpha\}_{\alpha\in\Delta}$, then $pg^{**}cl(U_\alpha) \subset pg^{**}cl(\cup\{U_\alpha\}_{\alpha\in\Delta})$, this implies $\cup\{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta} \subset pg^{**}cl(\cup\{U_\alpha\}_{\alpha\in\Delta})$. On the other hand, let $x \in pg^{**}cl(\cup\{U_\alpha\}_{\alpha\in\Delta})$. Since $\mathcal{A}$ is pg**- locally finite, there exists a pg**- neighbourhood G_x of x such that G_x has nonempty intersection with atmost finitely many members of $\mathcal{A}$. Let these finite members of $\mathcal{A}$ be $U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \ldots, U_{\alpha_n}$. Therefore $G_x \cap U_\alpha \neq \varphi$ for $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$. Let H_x be any pg**- neighbourhood of x and let $I_x = G_x \cap H_x$. Now $I_x \cap (U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n}) \neq \varphi$, since $x \in pg^{**}cl(\cup\{U_\alpha\}_{\alpha\in\Delta})$. But $I_x \subset H_x$, and H_x is an arbitrary pg**- neighbourhood of x and $x \in pg^{**}cl(U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n}) = pg^{**}cl(U_{\alpha_1}) \cup pg^{**}cl(U_{\alpha_2}) \cup \ldots \cup pg^{**}cl(U_{\alpha_n})$. Then $x \in pg^{**}cl(U_{\alpha_i})$ for some i where $1 \leq i \leq n$. Therefore $x \in \cup\{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta}$. Therefore $\cup\{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta} = pg^{**}cl(\cup\{U_\alpha\}_{\alpha\in\Delta})$. Hence the theorem.

Definition 9.5 Let $\mathcal{A}$ be a pg**-open covering of a topological space X. A collection $\mathcal{P}$ is called a *pg**-refinement* of $\mathcal{A}$ if $\mathcal{P}$ is also pg**-open covering of X and each member of $\mathcal{P}$ is contained in some member of $\mathcal{A}$.

Definition 9.6 A topological space X is said to be *pg**- paracompact* if every pg**-open covering of X has a pg**- locally finite pg**-refinement which is also a pg**-open covering of X.

Example 9.7 Every pg**- compact space is pg**- paracompact. Since every finite subcover $\mathcal{P}$ of the pg**-open cover $\mathcal{A}$ is a pg**- locally finite pg**-refinement of $\mathcal{A}$.

Theorem 9.8 Every pg**- closed subset of a pg**- paracompact space is pg**-paracompact.

Proof Let A be a pg**- closed subset of the pg**- paracompact space X. Given a pg**- open covering $\{U_\alpha\}_{\alpha\in\Delta}$ of A. Let us form an pg**- open covering of X by adjoining to $\{U_\alpha\}_{\alpha\in\Delta}$ the single pg**- open set $X - A$, that is $\{ \{U_\alpha\}_{\alpha\in\Delta} \cup (X - A) \}$. Take a pg**- locally finite pg**-refinement of this pg**- open covering of X and intersect it with A. This gives a pg**- locally finite pg**-refinement of pg**- open covering $\{U_\alpha\}_{\alpha\in\Delta}$ of A.

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