International Journal of Mathematics Research.
ISSN 0976-5840 Volume 9, Number 1 (2017), pp. 45-51

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# A collocation method for singular integral equations with cosecant kernel via Semi-trigonometric interpolation

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**Abstract**

In the paper, Semi-trigonometric Lagrange interpolation method for solving singular integral equations (SIE) with cosecant kernel is proposed. For solving SIE, difficulties lie in its singular term. In order to remove singular term of SIE, we choose appropriate weights quadrature formula. Compared with known investigations, its advantages are that the present method requires the least computational effort and accuracy in numerical computation is higher. The final numerical experiment illustrate the method is efficient.

**Keywords:** singular integral equations; cosecant kernel; Semi-trigonometric; Lagrange interpolation; collocation method

**2010 AMS Subject Classifications:** 45E99; 30E20

## 1. INTRODUCTION

This paper is concerned with numerical evaluation for singular integral equations with cosecant kernel of the form

$$a(t)\varphi(t)+\frac{b(t)}{2\pi}\int_0^{2\pi}\csc\frac{\tau-t}{2}\varphi(\tau)d\tau+\lambda\int_0^{2\pi}K(t,\tau)\varphi(\tau)d\tau=f(t),\quad t\in[0,2\pi] \tag{1}$$

where $a,b\in H_{2\pi},\ K\in\bar{H}_{2\pi},\ \lambda$ is a given constant to find the solution $\varphi\in\bar{H}_{2\pi}$.

So far, the study on equation (1) is only limited to some discussions on its characteristic equation [3-6],there are few papers devoted to the numerical method of Equation (1). Du and Shen [1] applied the reproducing kernel method to solve this problem, while Wei and Zhong [2] adopted Chebyshev polynomial method. In this paper, a new collocation method is developed by using Semi-trigonometric Lagrange interpolation polynomial. Lagrange basis for the Semi-trigonometric interpolation are firstly given, meanwhile we give numerical quadrature rules for the improper integral with cosecant kernel using it. Through comparison, the numerical results show the accuracy and effectiveness of this method.

The work is organized as follows. Section 2 gives basic knowledge and an useful lemma for the following content. Section 3 is devoted to describing the numerical solution of the integral equation with cosecant kernel. In Section 4, the numerical results confirm that the method is accurate and efficient. Section 5 ends this paper with a brief conclusion.

## 2. SEMI-TRIGONOMETRIC INTERPOLATION

**Theorem 2.1** The system

$$\left\{\sin\frac{1}{2}t,\cos\frac{1}{2}t,\sin\frac{3}{2}t,\cos\frac{3}{2}t,\cdots,\sin\left(k+\frac{1}{2}\right)t,\cos\left(k+\frac{1}{2}\right)t,\cdots\right\} \tag{2}$$

is orthogonal in the space $L^2([-\pi,\pi])$, moreover , it is a maximal orthogonal set Ref.[8].

**Definition 2.1** For $n\in\mathbb{N}$, we denote by $T_{n+\frac{1}{2}}$ the linear space of Semi-trigonometric polynomial

$$q_n(t)=\sum_{k=0}^{n}\left\{a_k\sin\left(k+\frac{1}{2}\right)t+b_k\cos\left(k+\frac{1}{2}\right)t\right\} \tag{3}$$

with real (or complex) coefficients $a_0,\cdots,a_n$ and $b_0,\cdots,b_n$ . A Semi-trigonometric polynomial $q\in T_{n+\frac{1}{2}}(t)$ is said to be of degree $n$ if $a_n^2+b_n^2\neq 0$ .

If $f(t)$ satisfies $f(t+2\pi)=-f(t)$, $\forall t$, then we call it $2\pi$ anti-periodic function, we know $f(t)\in T_{n+\frac{1}{2}}$.

**Theorem 2.2** There exists an unique Semi-trigonometric polynomial

$$q_n(t)=\sum_{k=0}^{n}\left\{a_k\sin\left(k+\frac{1}{2}\right)t+b_k\cos\left(k+\frac{1}{2}\right)t\right\} \tag{4}$$

satisfying the interpolation property $q_n(t_j)=y_j,\ j=0,\cdots,2n-1.$

Its coefficients are given by

$$a_k=\frac{1}{n}\sum_{j=0}^{2n-1}f(t_j)\sin\left(k+\frac{1}{2}\right)t_j,\qquad b_k=\frac{1}{n}\sum_{j=0}^{2n-1}f(t_j)\cos\left(k+\frac{1}{2}\right)t_j,$$

where $t_j=j\pi/n,\ j=0,\cdots,2n-1,$ are equidistant subdivision of the interval $[0,2\pi]$ with grid points of even number.

From this we deduce that the Lagrange basis for Semi-trigonometric interpolation have the form

$$L_j(t)=\frac{1}{n}\sum_{k=0}^{n}\cos\left(k+\frac{1}{2}\right)\left(t-t_j\right) \tag{5}$$

for $t\in[0,2\pi]$ and $j=0,\cdots,2n-1$.

**Lemma 2.3** For the Semi-trigonometric polynomial we have integrals

$$\int_0^{2\pi}\csc\frac{t}{2}\sin\left(k+\frac{1}{2}\right)tdt=2\pi,\qquad \int_0^{2\pi}\csc\frac{t}{2}\cos\left(k+\frac{1}{2}\right)tdt=$$

in the sense of Cauchy principal values.

Proof. Using the following Lagrange trigonometric identities, and then integrating it with respect to $t$ on the interval $[0,2\pi]$.

The singular part of the integral equation is given by

$$\left(T_0\varphi\right)(t) := \int_0^{2\pi} \csc\frac{\tau - t}{2}\varphi(\tau)d\tau, \quad 0 \le t \le 2\pi. \tag{6}$$

The Semi-trigonometric basis functions $f_k(t) := e^{i(k+\frac{1}{2})t}$, $k \in \mathbb{Z}$ are eigenfunctions, i.e. ,

$$T_0 f_k = i f_k \tag{7}$$

from the Riesz theory for compact operators we have that $T_0$ has a bounded inverse if and only if $T_0$ is injective.

We now construct numerical quadratures for the improper integral

$$\left(Q\varphi\right)(t) := \int_0^{2\pi} \csc\frac{\tau - t}{2}\varphi(\tau)d\tau, \quad 0 \le t \le 2\pi \tag{8}$$

by replacing the continuous $2\pi$ anti-periodic function $\varphi \in \bar{H}_{2\pi}$ by its semi-trigonometric Lagrange interpolation polynomial described in Theorem 2.1. Using the Lagrange basis we obtain

$$\left(Q_n\varphi\right)(t) = \sum_{j=0}^{2n-1} C_j^{(n)}(t)\varphi(t_j) \tag{9}$$

with the equidistant quadrature points $t_j = j\pi/n$ and the quadrature weights

$$C_j^{(n)}(t) = \int_0^{2\pi} \csc\frac{\tau - t}{2} L_j(\tau)d\tau, \qquad j = 0, \cdots, 2n-1 \tag{10}$$

Using Lemma 2.3, and from the form (4) of the Lagrange basis we derive

$$C_j^{(n)}(t) = -\frac{2\pi}{n}\sum_{k=0}^{n} \sin\left(k + \frac{1}{2}\right)\left(t - t_j\right), \qquad j = 0, \cdots, 2n-1 \tag{11}$$

This quadrature is uniformly convergent for all semi-trigonometric polynomials, since by construction $Q_n$ integrates Semi-trigonometric polynomial of degree less than or equal to $n$ exactly.

Since $K,\varphi \in \bar{H}_{2\pi}$, the $K(t,\tau)\varphi(\tau)$ is $2\pi$ periodic function for variable $\tau$, thus we adopt composite trapezoidal rule of $2\pi$ periodic function for $K(t,\tau)\varphi(\tau)$

$$\int_0^{2\pi} K(t,\tau)\varphi(\tau)d\tau \approx \frac{\pi}{n}\sum_{j=0}^{2n-1} K(t,t_j)\varphi(t_j) \tag{12}$$

with the equidistant quadrature points $t_j = j\pi/n,\ j = 0,\cdots,2n-1.$

## 3. NUMERICAL SOLUTION

For the numerical solution of the integral equations we describe the collocation method based on Semi-trigonometric interpolation that is fully discreted via appropriate quadrature approximation of integral operators. In the collocation method we approximate the integrals by the quadrature formula (6) and (9), that is, we replace the (SIE) (1) by

$$a(t)\varphi_n(t) + \frac{b(t)}{2\pi}\sum_{j=0}^{2n-1} C_j^{(n)}(t)\varphi_n(t_j) + \lambda\frac{\pi}{n}\sum_{j=0}^{2n-1} K(t,t_j)\varphi_n(t_j) = f(t), \tag{13}$$

for $t \in [0,2\pi)$. Now solving (10) reduces to solving a finite dimensional linear system.

In particular, for any solution of (10) the values $\varphi_{n,i} = \varphi_n(t_i), i = 0,1,\cdots,2n-1,$ trivially satisfy the linear system at the quadrature points,

$$a(t_i)\varphi_n(t_i) + \sum_{j=0}^{2n-1}\left\{\frac{b(t_i)}{2\pi}C_j^{(n)}(t_i) + \lambda\frac{\pi}{n}K(t_i,t_j)\right\}\varphi_n(t_j) = f(t_i), \tag{14}$$

for $i = 0,1,\cdots,2n-1.$ And conversely, given a solution $\varphi_{n,i}$, $i = 0,1,\cdots,2n-1$, of the system (11), then the function $\varphi_n$ is defined by its Semi-trigonometric Lagrange interpolation polynomial,

$$\varphi_n(t) := \sum_{j=0}^{2n-1} \varphi_{n,j} L_j(t), \quad t \in [0, 2\pi). \tag{15}$$

## 4. NUMERICAL EXAMPLES

Let us consider the SIE with cosecant kernel in Ref.[1]:

$$3\varphi(t) + \frac{1}{\pi}\int_0^{2\pi} \csc\frac{\tau - t}{2}\varphi(\tau)d\tau + 5\int_0^{2\pi} \frac{\tau + t}{2}\varphi(\tau)d\tau = f(t) \tag{16}$$

where right term $f(t) = 10(t+\pi) + 2\cos(t/2) + 3\sin(t/2)$ and the exact solution is $\varphi(t) = \sin(t/2)$. In order to imply the convergence of the method, we take different $n$, the Table 1 illustrates the convergence behaviour through the difference between the exact solution $\varphi(t)$ and the approximate solution $\varphi_n(t)$ at five points. Of course, the approximation $\varphi_n(t)$ is obtained from via (12) by the corresponding Semi-trigonometric Lagrange interpolation. From Table 1 we note that rapid convergence is clearly exhibited.

**Table 1.** Absolution error between the exact solution and approximate solution.

| $n$ | $t=\pi/10$ | $t=\pi/5$ | $t=\pi/2$ | $t=4\pi/5$ | $t=9\pi/10$ |
|---|---|---|---|---|---|
| 8 | 3.6311E-3 | 3.2768E-3 | 3.1554E-3 | 2.6393E-3 | 2.3314E-3 |
| 16 | 8.5054E-4 | 9.2932E-4 | 8.4496E-4 | 6.8569E-4 | 6.4205E-4 |
| 64 | 5.7050E-5 | 6.1219E-5 | 5.5816E-5 | 4.5248E-5 | 4.1813E-5 |
| 256 | 3.5870E-6 | 3.8509E-6 | 3.5388E-6 | 2.8745E-6 | 2.6475E-6 |
| 512 | 9.0099E-7 | 9.6455E-7 | 8.8684E-7 | 7.2034E-7 | 6.6280E-7 |
| 2048 | 5.6334E-8 | 6.0354E-8 | 5.5527E-8 | 4.5106E-8 | 4.1497E-8 |

For the error and convergence analysis of this method, we pointed out that as $n \to \infty$ the approximate solution converges uniformly to the exact solution of integral equation and the convergence order of quadrature errors for (6) and (9) carries over to the error $\varphi_n - \varphi$. For detailed description of the quadrature error, we refer to Kress [7] and involved literatures therein.

## 5. CONCLUSION

In this paper, we give the approximate solution of SIE with cosecant kernel by Semi-trigonometric interpolation polynomial. The final numerical results show that this method is effective. Moreover, this method can be extended to solve other equations with anti-periodic integral kernel.

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