

Commuting Well bounded Operators on Banach Spaces

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Abstract

Well-bounded operators on Banach spaces are introduced by D. R. Smart [1]. They coincide with the operators having, in some weaker sense, a spectral decomposition that converges only conditionally. T.A.Gillespie[2] proved that the sum of two commuting well-bounded operators is not always well-bounded even on Hilbert space. In this paper we give a general procedure to construct well-bounded operators on Banach space and show that the sum and product of well-bounded operators on Banach space need not be well-bounded operators.

Keywords: Banach Space, Scalar-type Operators, Well-bounded operators.

Subject Code Classification: MSC(2010): 47B40,47B48.

INTRODUCTION:

Definition1:

A bounded linear operator T on a complex reflexive Banach space is said to be well-bounded if there is a compact interval $J = [a, b]$ and a positive constant M such that

$$\|p(T)\| \leq M \{ \|p\|_J + \text{var}_J p \} \quad \text{---(I)}$$

for every complex polynomial p , where $\|p\|_J$ denotes $\sup \{ |p(t)| : t \in J \}$ and

$$\text{var}_J p = \sum_{j=1}^n |p(\lambda_j) - p(\lambda_{j+1})| / \lambda_0 = a, \lambda_1, \dots, \lambda_n = b \text{ is any partition of } J$$

Theorem 2:

Suppose B is reflexive Banach space such that each closed linear subspace can be complemented then for every bounded linear operator T and any real number μ there is a bounded projection P_μ such that $R(P_\mu) = N(T - \mu I)$ and P_μ commutes with every bounded operator commuting with T .

Suppose X is a Banach space and $\{P_n\}$ is a sequence of bounded linear transformation on X satisfying the following:

- 1) Each P_n is a projection on X . i.e. $P_n^2 = P_n$.
- 2) $P_n P_m = 0$, $n \neq m$.
- 3) $\sum_{n=1}^{\infty} P_n$ converges strongly to I .
- 4) $\| \sum_{j=1}^n P_{2j-1} \| \rightarrow \infty$ as $n \rightarrow \infty$

Lemma 3:

Let $\{P_n\}$ is a sequence of bounded linear transformations on a Banach space X satisfying (1),(2) ,(3) & (4) then $\| \sum_{j=1}^n P_{2j} \| \rightarrow \infty$ as $n \rightarrow \infty$.

Proof: Put, $S_n(x) = \sum_{j=1}^n P_{2j} \sum_{j=1}^n P_j(x)$, $x \in X$, $n \geq 1$

Then we have by (3),

$$s\text{-} \lim_{n \rightarrow \infty} \sum_{j=1}^n P_j = I$$

i.e. $S_n(x) \rightarrow I(x) = x$, for every $x \in X$.

Hence, $S_n(x)$ is bounded for every $x \in X$.

Therefore, by uniform boundedness theorem, $\{S_n\}$ is uniformly bounded.

$$\text{i.e. } \|S_n\| \leq M, \quad \forall n \geq 1$$

Put, $A_n = \sum_{j=1}^n P_{2j-1}$ and $B_n = \sum_{j=1}^n P_{2j}$

$$\begin{aligned} \text{Then, } \|A_n\| &= \|A_n + B_n - B_n\| \\ &\leq \|S_{2n} - B_n\| \\ &\leq M + \|B_n\| \end{aligned}$$

As $\|A_n\| = \|\sum_{j=1}^n P_{2j-1}\| \rightarrow \infty$ as $n \rightarrow \infty$,

$$\|B_n\| = \|\sum_{j=1}^n P_{2j}\| \rightarrow \infty \text{ as } n \rightarrow \infty.$$

Lemma 4:

Let $\{\lambda_n\}$ be a sequence in \mathbf{R} such that $\sum_{j=1}^n P_j < \infty$. Then the series $\sum_{n=1}^{\infty} \lambda_n P_n$ converges strongly.

Proof: Put, $S_n = \sum_{j=1}^n P_j, \quad n \geq 1$

$$\sum_{j=1}^n \lambda_j P_j = \lambda_{n+1} S_n + \sum_{j=1}^n (\lambda_j - \lambda_{j+1}) S_j, \quad n \geq 1 \dots\dots\dots(A)$$

We show that $\{\lambda_n\}$ is convergent.

for $n > m$, we have

$$\begin{aligned} &|\lambda_n - \lambda_m| \\ &\leq |\lambda_n - \lambda_{n-1} + \lambda_{n-1} - \lambda_{n-2} + \dots + \lambda_{m+1} - \lambda_m| \\ &\leq \sum_{j=1}^n P_j \end{aligned}$$

Since, $\sum_{j=1}^n P_j < \infty$, $\sum_{j=m}^{n-1} |\lambda_{j+1} - \lambda_j| \rightarrow 0$ as $n, m \rightarrow \infty$.

Thus, $\{\lambda_n\}$ is a Cauchy sequence and hence $\{\lambda_n\}$ is convergent.

Since $\{\lambda_{n+1}\}$ converges and $S_n \rightarrow I$ strongly, the sequence $\{\lambda_{n+1} S_n(x)\}$ converges for every $x \in X$.

Since $\{S_n(x)\}$ converges every $x \in X$, By uniform bounded theorem,

$\{S_n(x)\}$ is bounded.

Let, M be a positive constant such that $\|S_n\| \leq M$, $n \geq 1$

Now for $n > m$, consider

$$\|\sum_{j=1}^n (\lambda_j - \lambda_{j+1}) Q_j - \sum_{j=1}^m (\lambda_j - \lambda_{j+1}) Q_j\|$$

$$= \|\sum_{j=m+1}^n (\lambda_j - \lambda_{j+1}) Q_j\|$$

$$\leq \sum_{j=m+1}^n \|(\lambda_j - \lambda_{j+1}) Q_j\|$$

$$\leq \sum_{j=m+1}^n |\lambda_j - \lambda_{j+1}| \|S_j\|$$

$$\leq M \sum_{j=m+1}^n |\lambda_j - \lambda_{j+1}|$$

$$= M \sum_{j=m+1}^n |\lambda_{j+1} - \lambda_j|$$

Since $\sum_{j=m+1}^n |\lambda_{j+1} - \lambda_j| \rightarrow 0$ as $n, m \rightarrow \infty$.

We get that $\sum_{j=1}^n (\lambda_j - \lambda_{j+1}) Q_j$ is a Cauchy sequence. Therefore it converges in norm.

Now by using (A), we get that $\sum_{n=1}^{\infty} \lambda_n P_n$ converges strongly.

Lemma 5:

Let $\{\lambda_n\}$ be a monotonic bounded sequence in \mathbb{R} . Then

$$(\sum_{j=1}^n \lambda_j P_j) = \sum_{j=1}^n p(\lambda_j) P_j + a_0 (I - S_n)$$

for every complex polynomial $p(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_k \lambda^k$, $a_k \neq 0$.

Proof: Let $p(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_k \lambda^k$, $a_k \neq 0$.

$$\begin{aligned}
 & p\left(\sum_{j=1}^n \lambda_j P_j\right) \\
 &= a_0 I + a_1 \sum_{j=1}^k \lambda_j P_j + a_2 \left(\sum_{j=1}^k \lambda_j P_j\right)^2 + \dots + a_n \left(\sum_{j=1}^k \lambda_j P_j\right)^k \\
 &= a_0 I + a_1 \sum_{j=1}^k \lambda_j P_j + a_2 \left(\sum_{j=1}^k \lambda_j\right)^2 P_j + \dots + a_n \left(\sum_{j=1}^k \lambda_j\right)^k P_j \\
 &= a_0 I + \sum_{j=1}^n (a_1 \lambda_j + a_2 \lambda_j^2 + \dots + a_k \lambda_j^k) P_j = a_0 I + \sum_{j=1}^n [p(\lambda_j) - a_0] P_j \\
 &= a_0 I + \sum_{j=1}^n p(\lambda_j) P_j - \sum_{j=1}^n a_0 P_j \\
 &= \sum_{j=1}^n p(\lambda_j) P_j + a_0 \left(I - \sum_{j=1}^n P_j\right)
 \end{aligned}$$

Therefore,

$$p\left(\sum_{j=1}^n \lambda_j P_j\right) = \sum_{j=1}^n p(\lambda_j) P_j + a_0(I - S_n)$$

Lemma 6:

Let $\{\lambda_n\}$ be a monotonic bounded sequence in \mathbb{R} . Then the series $\sum_{n=1}^{\infty} \lambda_n P_n$ defines a well-bounded operator on X .

Proof: Let $\{\lambda_n\}$ be a monotonically increasing then

$$\begin{aligned}
 \sum_{j=1}^n |\lambda_{j+1} - \lambda_j| &= \sum_{j=1}^n (\lambda_{j+1} - \lambda_j) \\
 &= \lambda_2 - \lambda_1 + \lambda_3 - \lambda_2 + \dots + \lambda_{n+1} - \lambda_n \\
 &= \lambda_{n+1} - \lambda_1
 \end{aligned}$$

Since $\{\lambda_n\}$ is a monotonic and bounded, it is convergent. i.e. $\{\lambda_{n+1} - \lambda_1\}$ converges.

$$\text{Hence } \sum_{j=1}^{\infty} |\lambda_{j+1} - \lambda_j| < \infty$$

Similarly, if $\{\lambda_n\}$ be a monotonically decreasing then

$$\begin{aligned} \sum_{j=1}^n |\lambda_{j+1} - \lambda_j| &= \sum_{j=1}^n (\lambda_j - \lambda_{j+1}) \\ &= \lambda_1 - \lambda_2 + \lambda_2 - \lambda_3 + \dots + \lambda_n - \lambda_{n+1} \\ &= \lambda_1 - \lambda_{n+1} \end{aligned}$$

which is convergent by the same argument.

$$\text{i.e. } \sum_{j=1}^{\infty} |\lambda_{j+1} - \lambda_j| < \infty$$

Thus by lemma (4) $\sum_{n=1}^{\infty} \lambda_n P_n$ converges strongly.

$$\text{Define } T = s - \lim_{n \rightarrow \infty} \sum_{j=1}^n \lambda_j P_j$$

Since $\{\lambda_n\}$ is bounded, we can choose a compact interval $J=[a,b]$ such that $\{\lambda_n / n=1,2,\dots\} \subset J$

Further for any complex polynomial $p(\lambda) = \sum_{i=0}^k a_i \lambda^i$, we have by lemma 5,

$$p\left(\sum_{j=1}^n \lambda_j P_j\right) = \sum_{j=1}^n p(\lambda_j) P_j + a_0 (I - S_n)$$

Hence

$$\begin{aligned} &p\left(\sum_{j=1}^n \lambda_j P_j\right) \times \\ &= \sum_{j=1}^n p(\lambda_j) P_j \times + a_0 (I - S_n) \times \end{aligned}$$

$$\begin{aligned}
 &= p(\lambda_{n+1}) S_n x + \sum p(\lambda_j) - p(\lambda_{j+1}) S_j x + p(0) (I - S_n)x \\
 &\|p(\sum_{j=1}^n \lambda_j P_j)x\| \\
 &\leq |p(\lambda_{n+1})| \|S_n x\| + \sum_{j=1}^n |p(\lambda_j) - p(\lambda_{j+1})| \|S_j x\| + |p(0)| \|x - S_n x\| \dots \dots \dots (B) \\
 &\leq |p(\lambda_{n+1})| \cdot \|S_n\| \|x\| + \sum_{j=1}^n |p(\lambda_j) - p(\lambda_{j+1})| \cdot \|S_j\| \|x\| \\
 &\quad + |p(0)| \|x - S_n x\| \\
 &\leq \|p\|_J \cdot M \|x\| + \sum_{j=1}^n |p(\lambda_j) - p(\lambda_{j+1})| \cdot M \|x\| + |p(0)| \|x - S_n x\|
 \end{aligned}$$

Since $T = \sum_{j=1}^n \lambda_j P_j$ converges strongly to T and p is continuous,

$$p(T_n)x \rightarrow p(T)x, x \in X.$$

Hence by taking $n \rightarrow \infty$ in (B) we get

$$\|p(T)x\| \leq (\|p\|_J + \text{var } p) M \|x\|$$

This implies that

$$\|p(T)\| \leq M\{ \|p\|_J + \text{var } p \|x\| \}$$

Thus, T is well-bounded.

Theorem 7:

There exists well-bounded operators S, T with $ST = TS$, but $S+T$ is not well bounded.

Proof: Define the sequences $\{\lambda_n\}$ & $\{\mu_n\}$ are as follows:

$$\lambda_k = \frac{2n+1}{2n} \quad \& \quad \mu_{2n} = \frac{2n-1}{2n}, \quad \lambda_k = \frac{2n}{2n-1} \quad \& \quad \mu_{2n-1} = \frac{2n-1}{2n}.$$

We show that $\{\lambda_n\}$ is decreasing

$$\text{i.e. } \lambda_{k+1} \leq \lambda_k \quad \text{for all } k \geq 1.$$

If k is even, $k = 2n$, we have

$$\lambda_k = \frac{2(n+1)}{2(n+1)-1} = \frac{2n+2}{2n+1}$$

And

$$\begin{aligned} \lambda_k - \lambda_k &= \frac{2n+1}{2n} \frac{2n+1}{2n} - \frac{2(n+1)}{2n+1} \\ &= 1 - \frac{1}{2n} + \frac{1}{2n} - 1 - \frac{1}{2n+1} \\ &= \frac{1}{2n} - \frac{1}{2n+1} > 0. \end{aligned}$$

Similarly, If k is odd, $k = 2n-1$, then we have

$$\begin{aligned} \lambda_k - \lambda_k &= \frac{2n}{2n-1} - \frac{2n+1}{2n} \\ &= \frac{4n^2 - (4n^2 - 1)}{(2n-1)(2n)} \\ &= \frac{1}{(2n-1)(2n)} > 0.. \end{aligned}$$

Hence, $\{\lambda_k\}$ is decreasing for all $k \geq 1$.

Now we show that $\{\mu_k\}$ is increasing. i.e. $\mu_{k+1} \geq \mu_k$ for all $k \geq 1$.

If k is even, $k = 2n$, we have

$$\mu_{2n} - \mu_{2n+1} = \frac{2n-1}{2n} - \frac{2(n+1)-1}{2(n+1)}$$

$$\begin{aligned}
 &= 1 - \frac{1}{2n} - 1 + \frac{1}{2(n+1)} \\
 &= \frac{1}{2n+2} - \frac{1}{2n} < 0 \lambda_k
 \end{aligned}$$

Similarly , If k is odd , $k = 2n-1$, then we have

$$\begin{aligned}
 \mu_{2n-1} - \mu_{2n} &= \frac{2(n-1)-1}{2(n-1)} - \frac{2n-1}{2n} \\
 &= \frac{2n-2-1}{2n-2} - \frac{2n-1}{2n} \\
 &= 1 - \frac{1}{2n-2} - 1 + \frac{1}{2n} \\
 &= \frac{1}{2n} - \frac{1}{2n-2} < 0.
 \end{aligned}$$

Hence, $\{\mu_k\}$ is increasing.

$$\text{If } S = \sum_{n=1}^{\infty} \lambda_n P_n \sum_{j=1}^n \lambda_j P_j \quad \text{and} \quad T = \sum_{n=1}^{\infty} \mu_n P_n$$

Then by lemma(6) S, T are well bounded operators on Banach space X with $ST=TS$.

We show that $S+T$ is not well bounded.

Suppose that $S+T$ is well bounded.

$$\begin{aligned}
 \text{Since, } (S+T)x &= \sum_{n=1}^{\infty} (\lambda_n + \mu_n) P_n x, \quad \text{for } x \in X, \\
 (S+T)P_2x &= \sum_{n=1}^{\infty} (\lambda_n + \mu_n) P_n P_2x \\
 &= (\lambda_2 + \mu_2) P_2x
 \end{aligned}$$

But

$$\lambda_{2n} + \mu_{2n} = \frac{2n+1}{2n} + \frac{2n-1}{2n} = 2, \quad n \geq 1.$$

Therefore, 2 is an eigenvalue of $S+T$.

By Theorem(2) there is a bounded projection Q which commutes with every operator commuting with $S+T$ and $R(Q) = N((S+T) - 2I)$

Thus, $QP_n = P_nQ$ for all $n \geq 1$.

Also

$$\begin{aligned}(S+T)P_{2n}x &= (\lambda_{2n} + \mu_{2n})P_{2n}x \\ &= 2P_{2n}x\end{aligned}$$

$$P_{2n}x \in N((S+T) - 2I) = R(Q), \quad QP_{2n}x = P_{2n}x, \quad x \in X.$$

We also have

$$\begin{aligned}(S+T)P_{2n-1}x &= (\lambda_{2n-1} + \mu_{2n-1})P_{2n-1}x \\ &= \alpha_n P_{2n-1}x\end{aligned}$$

Where,

$$\begin{aligned}\alpha_n &= \lambda_{2n-1} \\ &= 1 + \frac{1}{2n-1} + 1 - \frac{1}{2n} \\ &= 2 + \frac{1}{2n-1} - \frac{1}{2n}\end{aligned}$$

$$\alpha_n = 2 + \frac{1}{2n(2n-1)} > 2.$$

But $QP_{2n-1}x \in R(Q) = N((S+T) - 2I)$

$$((S+T) - 2I)QP_{2n-1}x = 0$$

$$\text{i.e. } (S+T)P_{2n-1}x = 2QP_{2n-1}x$$

Also, $(S+T)QP_{2n-1}x = Q(S+T)P_{2n-1}x$

$$= Q(\lambda_{2n-1} + \mu_{2n-1})P_{2n-1}x$$

$$\begin{aligned}
 &= (\lambda_{2n-1} + \mu_{2n-1}) QP_{2n-1} x \\
 &= \alpha_n Q P_{2n-1} x
 \end{aligned}$$

Thus, $2QP_{2n-1} = \alpha_n Q P_{2n-1} x$,

where $\alpha_n > 2$. Thus, $QP_{2n-1} = 0$

Thus, $QP_{2n}x = P_{2n} x$ & $QP_{2n-1} x = 0$, $x \in X$

i.e. $QP_{2n} = P_{2n}$, $QP_{2n-1} = 0$,

$$\begin{aligned}
 \text{Now } I &= s\text{-} \lim_{n \rightarrow \infty} \sum_{k=1}^n P_k \\
 Q &= s\text{-} \lim_{n \rightarrow \infty} \sum_{k=1}^n QP_k \\
 &= s\text{-} \lim_{n \rightarrow \infty} \sum_{k=1}^n QP_{2k} \\
 &= s\text{-} \lim_{n \rightarrow \infty} \sum_{k=1}^n P_{2k}
 \end{aligned}$$

Thus, the partial sums of the series $\sum_{n=1}^n P_{2n}$ converges pointwise to Qx .

This implies that the sequence $\sum_{j=1}^n P_{2j}$ is pointwise bounded and by uniform boundedness theorem the sequence is uniformly bounded. This leads to contradiction Since by lemma 1,

$$\left\| \sum_{j=1}^n P_{2j} \right\| \rightarrow \infty \text{ as } n \rightarrow \infty.$$

This proves that $S+T$ is not well-bounded.

Theorem 8:

There exists well-bounded operators S, T with $ST=TS$, but ST is not well bounded.

Proof:

Let S, T are well bounded operators defined as in theorem 7.

Then, $S = \sum_{n=1}^{\infty} \lambda_n P_n \sum_{j=1}^n \lambda_j P_j$ and $T = \sum_{n=1}^{\infty} \mu_n P_n$

and we have

$$\begin{aligned} (ST)(x) &= S(T(x)) \\ &= S\left(\sum_{n=1}^{\infty} \mu_n P_n x\right) \\ &= \sum_{n=1}^{\infty} \mu_n \sum_{m=1}^{\infty} \lambda_m P_m P_n x \\ &= \sum_{n=1}^{\infty} \mu_n \lambda_n P_n x \end{aligned}$$

Furthermore, for, $n \geq 1$

$$\mu_{2n-1} \cdot \lambda_{2n-1} = \frac{2n}{2n-1} \cdot \frac{2n-1}{2n} = 1$$

Hence, for any $x \in X$,

$$\begin{aligned} (ST)P_{2n-1} x &= \mu_{2n-1} \cdot \lambda_{2n-1} P_{2n-1} x \\ &= P_{2n-1} x \end{aligned}$$

Since $\mu_{2n-1} \cdot \lambda_{2n-1} = 1$

Thus, 1 is an eigenvalue of ST, Hence by theorem(A) there is a bounded projection U such that

$R(U) = N(ST-I)$ and U commutes with every bounded operator commuting with ST.

$(ST)P_{2n-1} x = 0, x \in X,$

$$P_{2n-1}x \in N((ST) - I) = R(U),$$

Thus

$$U.P_{2n-1}x = P_{2n-1}x$$

$$\text{Also for any } x \in X, U.P_{2n}x \in R(U) = N((ST) - I)$$

Therefore,

$$((ST) - I)UP_{2n}x = 0$$

$$\text{i.e. } (ST)UP_{2n}x = UP_{2n}x$$

$$\text{But, } (ST)UP_{2n}x = U(ST)P_{2n}x$$

$$= U(\lambda_{2n} \cdot \mu_{2n})P_{2n}x$$

$$= (\lambda_{2n} \mu_{2n})UP_{2n}x$$

$$= \beta_n UP_{2n}x$$

$$\text{where, } \beta_n = \lambda_{2n} \mu_{2n} = \frac{4n^2 - 1}{4n^2} < 1$$

$$\text{Hence, } U.P_{2n}x = \beta_n UP_{2n}x, \quad x \in X.$$

$\beta_n < 1$, implies that $U.P_{2n}x = 0$, for every $x \in X$.

$$\text{Thus, } U.P_{2n-1}x = P_{2n-1}x \quad \text{and } U.P_{2n}x = 0, \quad \text{for every } x \in X$$

$$\text{Now } I = s - \lim_{n \rightarrow \infty} \sum_{k=1}^n P_k$$

$$\text{Thus, } U = s - \lim_{n \rightarrow \infty} \sum_{k=1}^n U.P_k$$

$$= s - \lim_{n \rightarrow \infty} \sum_{k=1}^n U.P_{2k-1}$$

$$= s - \lim_{n \rightarrow \infty} \sum_{k=1}^n P_{2k-1}$$

This again gives contradiction to (4) that,

$$\| \sum_{j=1}^n P_{2j-1} \| \rightarrow \infty \text{ as } n \rightarrow \infty .$$

This proves that ST is not well-bounded.

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