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Relations of Centralizers on Semiprime Semirings

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Abstract

We define and study the relations of centralizers on semiprime semiring. In this paper, we prove that an additive mapping T of a 2-torsion free semiprime semiring S into itself such that $T(xyx) = xT(y)x$ for all $x, y \in S$, is a Centralizer. We also show that if S contains a multiplicative identity1, then T is a centralizer.

Keywords : Semiring, Semiprime Semiring, Centralizer, Jordan centralizer, left(right) centralizer.

Mathematical Subject Classification (2000): 16Y60

1. INTRODUCTION

Semirings arise erupt in various fields of mathematics. Indeed, the first mathematical structure we encounter- the set of natural numbers is a semiring. Historically semirings first appear implicitly in[6] and the algebraic structure of semiring was introduced by H.S. Vandiver in 1934. Later semirings were investigated by numerous researches in their own right, in order to broaden techniques and to generalize results from ring theory or semigroup theory or in connection with some applications (see [10]).

H.E.Bell and W.S.Martindale [1] established the study of Centralizing mapping on semiprime rings. After many mathematicians made good works on centralizers of semiprime rings (see [2], [11]).M.F. Hoque and A.C. Paul [8] studied the centralizers of semiprime Gamma rings. In [5] Chandramouleeswaran and Thiruveni, made good works on derivations of semirings. Joso Vukman[12] proved that if R is a 2-torsion free Semiprime Ring and $T: R \to R$ is an additive mapping such that $T(xyx) = xT(y)x$ for all $x, y \in R$, then T is a centralizer. Motivated by this D. Mary Florence and R. Murugesan [14] studied the notion of Semirings and proved that every Jordan Centralizer of a 2 - torsion free Semiprime Semiring is a centralizer. Also show that in general, every Jordan centralizer is not a centralizer.The purpose of this paper is to obtain the centralizers on semiprime semirings in the sense of Joso Vukman.

2. PRELIMINARIES

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1

A Semiring is a nonempty set S followed with two binary operation ' $+$ *' and '*$.$*'such that*

(1) $(S, +)$*is a commutative monoid with identity element* $'0'$.

(2) $(S, .)$*is a monoid with identity element 1*

(3) Multiplication distributes over addition from either side

That is $a.(b + c) = a.b + a.c$

$(b + c).a = b.a + c.a$

Definition 2.2

A Semiring S is said to be prime if $xSy = 0 \Rightarrow x = 0 \; or \; y = 0 \;\; for \; all \;\; x, y \in S$

Definition 2.3

A Semiring S is said to be semiprime if $xSx = 0 \;\Rightarrow\; x = 0 \; for \; all \;\; x \in S$

Definition 2.4

A Semiring S is said to be 2-torsion free if $2x = 0 \;\Rightarrow\; x = o \; for \; all \;\; x \in S$

Definition 2.5

A Semiring Sis said to be commutative Semiring, if $xy = yx \;\; for \; all \; x, y \in S$*, then the set*

$Z(S) = \{ x \in S,\ xy = yx\ for\ all\ y \in S\}$ *is called the center of the Semiring* S.

Definition 2.6

For any fixed $a \in S$, *the mapping* $T(x) = ax$ *is a left centralizer and* $T(x) = xa$ *is a right centralizer.*

Definition 2.7

An additive mapping $T: S \to S$ *is a left (Right) centralizer* $if\ T(xy) = T(x)y, \big(T(xy) = xT(y)\big)$ *for all* $x, y \in S$

A centralizer is an additive mapping which is both left and right centralizer.

Definition 2.8

An additive mapping $T: S \to S$*is Jordan left (Right) Centralizer if*

$$T(xx) = T(x)x, (T(xx) = xT(x))\ \textit{for all } x \in S$$

Every left centralizer is a Jordan left centralizer but the converse is not in general true.

Definition 2.9

An additive mapping $T: S \to S$ *is a Jordan centralizer if* $T(xy + yx) = T(x)y + yT(x)$ *for all* $x, y \in S$

Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Definition 2.10

An additive mapping $D: S \to S$ *is called a derivation if* $D(xy) = D(x)y + xD(y)$ *for all* $x, y \in S$ *and is called a Jordan derivation if* $D(xx) = D(x)x + xD(x)$ *for all* $x \in S$.

Definition 2.11

If S *is a semiring then* $[x, y] = xy - yx$ *is known as the commutator of* x *and* y

The following are the basic commutator identities:

$$[xy, z] = [x, z]y + x[y, z] and [x, yz] = [x, y]z + y[x, z]\ \textit{for all } x, y, z \in S$$

Lemma :2.12 (Lemma 2.1 in [13])

Let S be a Semiprime Semiring. Suppose that the relation $axb + bxc = 0$ *for all* $x \in S$ *and some* $a, b, c \in S$. *In this case* $(a + c)xb = 0$ *for all* $x \in S$.

Theorem : 2.13 (Theorem 4.1 in [14])

Every Jordon centralizer of a 2-torsion free semiprime semiring is a centralizer.

3. OUR MAIN RESULT

Theorem:3.1 *Let S be a 2-torsion free Semiprime Semiring and let* $T: S \to S$ *be an additive mapping.*

Such that $T(xyx) = xT(y)x$ *holds for all* $x, y \in S$*.Then T is a centralizer.*

Proof:

(1) we have $T(xyx) = xT(y)x$

After replacing x by $x + z$ in (1) we obtain

$$T[(x + z)y(x + z)] = (x + z)T(y)(x + z)$$

$$T[(xy + zy)(x + z)] = (xT(y) + zT(y))(x + z)$$

$$T(xyx + xyz + zyx + zyz) = xT(y)x + xT(y)z + zT(y)x + zT(y)z$$

$$T(xyx) + T(xyz + zyx) + T(zyz) = xT(y)x + xT(y)z + zT(y)x + zT(y)z$$

$$xT(y)x + T(xyz + zyx) + zT(y)z = xT(y)x + xT(y)z + zT(y)x + zT(y)z$$

(2) $T(xyz + zyx) = xT(y)z + zT(y)x$

For $y = x$ and $z = y$ in (2) we get

(3) $T(x^2y + yx^2) = xT(x)y + yT(x)x$

Putting $z = x^3$ in (2) which implies that

(4) $T(xyx^3 + x^3yx) = xT(y)x^3 + x^3T(y)x$

After replacing y *by* xyx in (3) we obtain

(5) $T(x^3yx + xyx^3) = xT(x)xyx + xyxT(x)x$

Substitution for $y = x^2y + yx^2$ in (1) yields

(6) $T(x^3yx + xyx^3) = x^2T(x)yx + xyT(x)x^2$

After comparing (5) with (6) we arrive at

$$xT(x)xyx + xyxT(x)x = x^2T(x)yx + xyT(x)x^2$$

(7) $x[T(x), x]yx + xy[x, T(x)]x = 0$

From the above we assume, $a = x[T(x), x)], x = y, b = x, c = [x, T(x)]x$

Now applying lemma2.12 follows that $(x[T(x),x]+[x,T(x)]x)yx=0$

$(x[T(x),x]-[T(x),x]x)yx=0$

$x[T(x),x]yx=[T(x),x]xyx$

$[T(x),x]xyx-x[T(x),x]yx=0$

(8) $[[T(x),x],x]yx \;=0$

Applying$y\;by\;\;y[T(x),x]$ in the above relation we get

(9) $[[T(x),x],x]y[T(x),x]x=0$

Right multiplying (8) by $[T(x),x]$ we obtain

(10) $[[T(x),x],x]yx[T(x),x]\;=0$

Subtracting (10) from (9) yields

$[[T(x),x],x]y\,([T(x),x]x-x[T(x),x])=0$

$[[T(x),x],x]y[[T(x),x],x]=0$

By the semiprimeness of S,

(11) $[[T(x),x],x]=0$

From the above relation replacing $x\;by\;\;x+y$ and using (7)and (11) we see that

$[[T(x+y),x+y],x+y]=0$

$[[T(x)+T(y),x+y],x+y]\;=0$

$[[T(x),x]+[T(x),y]+[T(y),x]+[T(y),y],x+y]=0$

$$\big[[T(x),x],x\big]+[[T(x),y],x]+[[T(y),x],x]+[[T(y),y],x]+[[T(x),x],y]+[[T(x),y],y]+[[T(y),x],y]+[[T(y),y],y]=0$$

$$[[T(x),y],x]+[[T(y),x],x]+[[T(x),y],y]+[[T(y),x],y]=0$$

$$[T(x),y]x-x[T(x),y]+[T(y),x]x-x[T(y),x]+[T(x),y]y-y[T(x),y]+[T(y),x]y-y[T(y),x]=0$$

$$[T(x),y](x+y)-(x+y)[T(x),y]+[T(y),x](x+y)-(x+y)[T(y),x]=0$$

Replacing$x+y=x$ in the above relation reduces to

$[T(x),y]x-x[T(x),y]+[T(y),x]x-x[T(y),x]=0$

(12) $[[T(x),y],x]+[[T(y\,),x],x]=0$

Putting xyx for y in (12)and using (12) we obtain

$[[T(x), xyx], x] + [[T(xyx), x], x] = 0$

$[[T(x), x]yx + x[T(x), y]x + xy[T(x), x], x] + [[xT(y)x, x], x] = 0$

$[[T(x), x]yx, x] + [x[T(x), y]x, x] + [xy[T(x), x], x] + [[xT(y)x, x], x] = 0$

$$[[T(x), x], x]yx + [T(x), x][y, x]x + [T(x), x]y[x, x] + x[T(x), y][x, x] + x[[T(x), y], x]x + [x, x][T(x), y]x + xy[[T(x), x], x] + x[y, x][T(x), x] + [x, x]y[T(x), x] + [[x, x][T(y), x] + x[T(y), x]x + xT(y)[x, x], x] = 0$$

$[T(x), x][y, x]x + x[y, x][T(x), x] + x[[T(x), y], x]x + [x[T(y), x]x, x] = 0$

$$[T(x), x][y, x]x + x[y, x][T(x), x] + x[[T(x), y], x]x + x[T(y), x][, x, x] + x[[T(y), x], x]x + [x, x][T(y, x]x = 0$$

$[T(x), x][y, x]x + x[y, x][T(x), x] + x\{[[T(x), y], x] + [[T(y), x], x]\}x = 0$

$[T(x), x][y, x]x + x[y, x][T(x), x] = 0$

$[T(x), x](yx - xy)x + x(yx - xy)[T(x), x] = 0$

$[T(x), x]yx^2 - [T(x), x]xyx + xyx[T(x), x] - x^2y[T(x), x] = 0$. which reduces because of (7) we reach to

(13) $[T(x), x]yx^2 - x^2y[T(x), x] = 0$

(14) $y[T(x), x]x^2 - x^2y[T(x), x] = 0$

Left multiplication of the above relation by $T(x)x$

(15) $T(x)xy[T(x), x]x^2 - T(x)x^3y[T(x), x] = 0$

Replacing $y = xT(x)y$in (14) we reach

(16) $xT(x)y[T(x), x]x^2 - x^3T(x)y[T(x), x] = 0$

Subtracting (15) from (16)leads to

$$xT(x)y[T(x), x]x^2 - x^3T(x)y[T(x), x] - T(x)xy[T(x), x]x^2 + T(x)x^3y[T(x), x] = 0$$

$(xT(x) - T(x)x)y[T(x), x]x^2 + (T(x)x^3 - x^3T(x))y[T(x), x] = 0$

$[x, T(x)]y[T(x), x]x^2 + [T(x), x^3]y[T(x), x] = 0$

$[T(x), x^3]y[T(x), x] + [T(x), x]y[x, T(x)]x^2 = 0$

We now apply lemma 2.12,it follows that

$([T(x), x^3] + [x, T(x)]x^2)y[T(x), x] = 0$

$([T(x), xxx] + [x, T(x)]x^2)y[T(x), x] = 0$

$([T(x), x]x^2 + x[T(x), x]x + x^2[T(x), x] + [x, T(x)]x^2)y[T(x), x] = 0$

From relation (11) we can write $x[T(x), x]$ instead of $[T(x), x]x$

$(x[T(x), x]x + x[T(x), x]x + x^2[T(x), x] - [T(x), x]x^2)y[T(x), x] = 0$

Relation(7) makes it possible to write $[T(x), x]x^2 = x^2[T(x), x]$ the above relation becomes

$2x[T(x), x]xy[T(x), x] = 0$

Since S is 2-torsion free Semiprime Semiring, $x[T(x), x]xy[T(x), x] = 0$

Right multiplication of the above relation by x and replacing$y$ by yx gives

$x[T(x), x]xyx[T(x), x]x = 0$

(17) Since S is Semiprime, x$[T(x), x]x = 0$

From relation (7) we can write $x[T(x), x]yx = xy[T(x), x]x$

Putting y by yx and applying (17) in the above we arrive at

(18) $x[T(x), x]yx^2 = 0$

The substitution$yT(x)$for y in(18) yields

(19) $x[T(x), x]yT(x)x^2 = 0$

Right multiplication of (18) by $T(x)$ we get

(20) $x[T(x), x]yx^2T(x) = 0$

Subtracting (20) from (19) we obtain

$x[T(x), x]y[T(x), x^2] = 0$

$x[T(x), x]y[T(x), xx] = 0$

$x[T(x), x]y([T(x), x]x + x[T(x), x]) = 0$

According to (7) one can replace $x[T(x), x]$ by $[T(x), x]x$ which gives

$2x[T(x), x]yx[T(x), x] = 0$

Since we are in semiprime semirings we conclude that

(21) $x\,[T(x), x] = 0$

According to (11) it is easy to see that

(22) $[T(x), x]x = 0$

Linearizing the above result (see how to obtain (12) from(11)) we reach

$[T(x), y]x + [T(y), x]x + [T(y), y]x + [T(x), x]y = 0$

Right multiplication of the above relation by $[T(x), x]$and using (21) this reduces to

$[T(x), x]y[T(x), x] = 0$

Which implies

(23) $[T(x), x] = 0$

Next we will intent to prove the result

(24) $\boldsymbol{T(xy + yx) = T(y)x + xT(y)}$

In order to prove the above we need to prove the following relation

(25) $[G(x,y), x] = 0x, y \in S$

Where G(x,y) stands for $T(xy + yx) - T(y)x - xT(y). With\ respect\ to\ this\ notation,$

First we replacing$y = xy + yx$ in equation(1) gives

(26) $T(x^2yx + xyx^2) = xT(xy + yx)x$

On the otherhand putting$z = x^2$in (2) we obtain

(27) $T(xyx^2 + x^2yx) = xT(y)x^2 + x^2T(y)x$

Comparing (26) and (27) it is clear that

(28) $xG(x,y)x = 0$

Now let us prove (25). Replacing x by $x + y$ in relation(23)

(29) $[T(x), y] + [T(y), x] = 0 \quad x, y \in S.$

After replacing $y\ by\ xy + yx$in the above and using relation (23) leads to

(30) $x[T(x), y] + [T(x), y]x + [T(xy + yx), x] = 0$

According to (29) we can replace in the above relation [T(x),y] by – [T(y),x]. We then have

$[T(xy + yx), x] - x[T(y), x] - [T(y), x]x = 0$

This can be written in the form $[T(xy + yx) - T(y)x - xT(y), x] = 0$

Therefore the proof of relation (25) is complete.

Substituting x by x+z in (28) and using (28) gives

$xG(x,y)z + zG(x,y)x + zG(x,y)z + xG(z,y)x + xG(z,y)z + zG(z,y)x =$ 0x,y,z$\in S$

$xG(x,y)z + zG(x,y)(x + z) + xG(z,y)(x + z) + zG(z,y)z = 0$

Replacing$x + z = x$ in the above we get

$xG(x,y)z + zG(x,y)x + xG(z,y)x + zG(z,y)x = 0$

Right multiplication of the above relation by $G(x,y)x$ and using (28) follows that

$xG(x,y)zG(x,y)x = 0$

Using (25) , the above relation can be written in the form

$xG(x,y)zxG(x,y) = 0$

By the semiprimeness of S we get

(31) $xG(x,y) = 0$

From (25) and (31) we also get

(32) $G(x,y)x = 0$

The linearization of (32) by putting $x = x + z$ we get

(33) $G(x,y)z + G(z,y)x = 0 \qquad x,y \in S$

Right Multiplication of the above by $G(x,y)$ and applying (31) we arrive at

$G(x,y)zG(x,y) = 0$

By the semiprimeness of S it follows that $G(x,y) = 0$.

So the proof of (24) is completed.

In particular when y=x in relation (24)reduces to

$$2T(x^2) = T(x)x + xT(x)x \in S$$

Using (23) and S is 2-torsion free Semiprime Semiring, the above relation yields

$T(x^2) = T(x)x, \quad x \in S$

And $T(x^2) = xT(x)$, $x \in S$

This means that T is a Left and Right Jordan centralizer

By Theorem 4.1 in [14] follows that $T$is a centralizer . Which completes the proof of the theorem.

Putting $y = x$ in relation (1) we obtain $T(xxx) = xT(x)x$, $x \in S$. The question arises whether in a 2-torsion free Semiprime Semiring the above relation implies that T is a centralizer. Unfortunately, we were unable to answer it affirmative because S has an identity element 1.

Theorem :3.2

Let S be a 2-torsion free Semiprime Semiring with identity element 1and let $T: S \to S$ *be an additive mapping. Suppose that* $T(xxx) = xT(x)x$ *holds for all* $x \in S$*.Then*T *is a centralizer.*

Proof: BY Hypothesis, we have

(34) $T(xxx) = xT(x)x$

Putting $x = x + 1$ in the above follows that

$$T(x^3)+3T(x^2)+3T(x)+T(1)$$
$$=xT(x)x+xT(1)x+T(x)x+T(1)x+xT(x)+xT(1)+T(x)+T(1)$$

Putting $T(1)=a$ and apply (34) in the above relation implies that

(35) $3T(xx)+2T(x)=xax+T(x)x+ax+xT(x)+xa$

Once again substituting $x=x+1$ and a stands for T(1) yields

$3T(x^2)+8T(x)+5a=xax+3xa+3ax+5a+T(x)x+xT(x)+2T(x)$

(36) $3T(x^2)+6T(x)=xax+3xa+3ax+T(x)x+xT(x)$

Subtracting (35) from (36) leads to

$4T(x)=2xa+2ax$

(37) $2T(x)=xa+ax$

Again subtracting (37) from (35) yields

$3T(xx)=xax+T(x)x+xT(x)$

(38) $6T(xx)=2xax+2T(x)x+2xT(x)$

Weshall prove that $a\ \in\ Z(S)$

According to (37) replacing $2T(x)$ on the right side of (38) we obtain

$6T(xx)=2xax+(ax+xa)x+x(ax+xa)$

(39) $6T(xx)=4xax+axx+xxa$

Replacing$x=y$ in the above relation gives

(40) $6T(yy)=4yay+ayy+yya$

Substituting x=y^2in (37) we get

$2T(yy)=yya+ayy$

(41) $6T(yy)=3yya+3ayy$

Comparing (41) and (40) we obtain that

$3yya+3ayy=4yay+ayy+yya$

$2yya+2ayy-4yay=0$

$2(yya+ayy-2yay)=0$

Since S is 2-torsion free Semiprime Semirings we get

$yya+ayy-2yay\ =0$

The above relation can be written in the form

(42) $[[a,y],y]=0$

The linearization of the above relation by putting $y=x+y$and using (42) gives

(43) $\left[[a,x],y\right]+\left[[a,y],x\right]=0$

putting $y=xy$ in (43) and using (42) in the above relation reduces to

$x[[a,x],y]+[a,x][y,x]+x[[a,y],x]=0$

Using (43) in the above we get

(44) $[a,x][y,x]=0$

Substitutingya for y in the above relation and using (44) gives

$[a,x]y[a,x]=0$

By the semiprimeness of S $[a,x]=0$

$ax=xa$

$a\,Z(S)$

Now (37) is reduced to $T(x)=ax$

$and\ \ T(x)=xa,\ \ x\in S$

Hence T is a centralizer.

REFERENCES

[1] BELL H.E. AND MARTINDALE III, W.S., *Centralizing mappings of Semiprime ring,* Canad. Math. Bull. 30(1) (1989), 92-101.

[2] BORUT ZALAR, *On centralizers of Semiprime rings,* Commentationes Mathematicae Universitatis Carolinae, Vol, 32(1991), N0.4,609-614.

[3] BRESAR M., VUKMAN J.,*On some additive mapping in rings with involution*, Aequationes Math. 38(1989), 178-185.

[4] BRESAR M.,*Jordan derivations on Semiprime rings*, Proc. Amer. Math.Soc., 104(1988), 1003-1006.

[5] CHANDRAMOULEESWARAN, THIRUVENI, *On Derivations of Semirings*, Advances in Algebra, 3(1) (2010), 123-131.

[6] DEDEKINT R – Über die Theorie der ganzenalgebraischen Zahlen, Supplement XI to P.G. Lejeune Dirichlet : vorlesungen Über Zahlentherorie, 4Aufl., Druck and Verlag Braunschweig, 11894.

[7] HERSTEIN I.N., *Topics in ring theory*, University of Chicago Press, 1969.

[8] HOQUE M.F., AND A.C. PAUL.,*Centralizers on Semiprime Gamma Rings,* Italian Journal of Pure and Applied Mathematics –N. 30-2013(289-302).

[9] HOQUE M.F., AND PAUL A.C., *On Centralizers of Semiprime Gamma Rings*, International Mathematical Forum, Vol.6,no.13(2011),627-638.

[10] JONATHAN S.GOLAN, *Semirings and their Applications*, Kluwer Academic

Press(1969).

[11] JOSO.VUKMAN, *An identity related to centralizers in Semiprime rings*, comment. Math. Univ. Carolinae 40,3(1999),447-456.

[12] JOSO.VUKMAN, *Centralizers on semiprime rings*, Comment, Math. Univ. Carolinae 42,2(2001), 237-245.

[13] MARY FLORENCE D., R. MURUGESAN, *Some Relations Related to Centralizers on Semiprime Semiring*, Annals of Pure and Applied Mathematics, Vol. 13, No. 1, 2017, 119-124.

[14] MARY FLORENCE D., R. MURUGESAN, *Centralizers on Semiprime Semiring*, IOSR Journal of Mathematics, Vol. 12, 2016, (86-93).